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NONHYDROSTATIC ACOUSTICALLY FILTERED EQUATIONS OF ATMOSPHERIC DYNAMICS IN PRESSURE COORDINATES

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Abstract. Nonhydrostatic acoustically filtered equations of motion of nonviscid fluid are derived in pressure coordinates. A complete set of nonhydrostatic nonlinear equations for ideal fluid in pressure coordinates serves as the starting basis. These equations are linearized and transformed to a convenient for filtering form. Acoustic filtering is achieved in the limit of the infinitely high sound speed, $c_a \to \infty$. The filtered model lacks acoustic wave solutions but maintains without loss of accuracy all slow processes, including buoyancy waves. The obtained in this way linear model is complemented to a nonlinear set by the inclusion of incompressible advection terms in the pressure space. The final equations may describe slow processes from local turbulence to planetary-scale waves. Still, the main domain of the application of the model is mesoscale dynamics.

Key words: atmospheric dynamics, Lagrangian function, Hamilton's principle, pressure coordinates, nonhydrostatic equations, optimum acoustic filtering.

1. INTRODUCTION

The idea to use pressure related coordinates for nonhydrostatic (NH) dynamics is not new. A pressure coordinate (p-coordinate) acoustically filtered NH model was first proposed by Miller [1] and Miller and Pearce [2]. The Miller–Pearce model (MPM) has been widely used in numerical modelling [3–5]. A variant of the numerical package, developed in [5], is presently in use at Tartu Observatory. Though at first sight the p-coordinate presentation in combination with the NH assumption looks exotic and sophisticated, its incontestable advantage consists in ability to treat NH processes of shorter mesoscale ($l_x \sim 10^2$–10^3 m), and hydrostatic processes of longer mesoscale ($l_x \sim 10^4$–10^5 m) and synoptic scale ($l_x \sim 10^6$–10^7 m) in the framework of unified formalism.

The main assumption of the MPM is the approximation of incompressibility of motion in p-space, which filters sound waves:

$$\nabla \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0 \,. \tag{I1}$$

Here $\mathbf{v}$ and $\omega = \mathrm{d}p/\mathrm{d}t$ are the horizontal wind vector and the vertical speed of the air-particle in the p-space. The assumption (I1) enables filtering of acoustic waves, unwanted in slow dynamics, in a most straightforward manner. As the same assumption is exact in the hydrostatic limit, (I1) presents an extrapolation of the main characteristic feature of primitive equations to the mesoscale.

Here we introduce a different acoustically filtered NH model which rejects the hypothesis (I1). Though the approximation (I1) will be used at the final stage of the model development for the introduction of nonlinear momentum advection, it is not used for wave filtering.

The model we develop is based on the complete set of NH equations in p-coordinates [6]. The filtering proceeds as follows. At first the initial set of NH p-coordinate equations is linearized. The filtering is carried out in the linear model. Finally, the model obtained is complemented to a nonlinear filtered model with the maintenance of energy conservation.

2. LINEAR NH MODEL IN p-COORDINATES

2.1. Linearization of equations of paper [6] according to the hydrostatic equilibrium state, characterized by the mean temperature, $T_0(p)$, yields equations

$$\frac{\partial z'}{\partial t} = v_z + H_0 \frac{\omega}{p} \,, \tag{1a}$$

$$\frac{\partial v_z}{\partial t} = -g^2 \rho' \,, \tag{1b}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\, g \nabla z' \,, \tag{1c}$$

$$\frac{\partial T'}{\partial t} = \frac{T_i \omega}{p} + Q \,, \tag{1d}$$

$$\frac{\partial \rho'}{\partial t} = -\frac{1}{g} (\nabla \cdot \mathbf{v} + \partial \omega / \partial p) \,, \tag{1e}$$

$$\frac{\partial p_s'}{\partial t} = \omega|_{p_0} \,, \tag{1f}$$

$$\rho' = -\frac{1}{g} \left(\frac{p}{H_0} \frac{\partial z'}{\partial p} + \frac{T'}{T_0} \right) . \tag{1g}$$

Here $v_z = \mathrm{d}z/\mathrm{d}t$ is vertical velocity, z' and T' represent isobaric height and temperature fluctuations, $\rho' = \rho - 1/g$ is the p-space density fluctuation from its equilibrium constant value $\overline{\rho} = 1/g$, and p'_s is the ground surface pressure fluctuation from the mean pressure distribution at the ground p_0. The parameter $H_0 = RT_0/g$ presents the height scale for hydrostatic pressure,

$$T_i = \frac{R}{c_p} T_0 - p \frac{\partial T_0}{\partial p}$$

is the stability parameter ("stability temperature") of the background state, and Q represents the given thermal forcing.

2.2. Boundary conditions. Conditions at the lateral boundaries are the same as in Cartesian coordinate models and do not present special interest in the context of the present study. The main differences from the ordinary model occur in the "horizontal" conditions at the top and at the bottom. The domain occupied by the atmosphere in the p-space is

$$0 < p < p_0(\mathbf{x}, t), \quad -\infty < x, y < \infty . \tag{2}$$

The boundary conditions at the bottom and top are

$$z'|_{p_0} = H_0(p_0)\frac{p'_s}{p_0}, \quad v_z|_{p_0} = \mathbf{v}|_{p_0} \cdot \nabla h, \quad \omega|_0 = 0 , \tag{3}$$

where $h(\mathbf{x})$ is the ground surface height above sea level.

2.3. Diagnostic equation for ω. Model (1) presents a closed system consisting of eight equations for eight fields z', v_x, v_y, v_z, T', ρ', p'_s, and ω. All quantities here, except ω, are prognostic fields, and system (1) includes a single diagnostic Eq. (1f). This equation should be used for the determination of the diagnostic field ω. As (1f) does not include ω explicitly, the only way to proceed is to differentiate (1f) by t and eliminate time derivatives by the help of other equations. The result is an explicit equation for ω

$$\alpha \frac{\omega}{p} = \frac{p}{H_0}\frac{\partial v_z}{\partial p} - \nabla \cdot \mathbf{v} + \frac{Q}{T_0}, \tag{4}$$

where

$$\alpha = c_v/c_p .$$

This relation presents the linearized pressure tendency equation in p-coordinate representation.

2.4. The reduced linear system. The obtained diagnostic Eq. (4) along with Eq. (1g) enables us to get from (1) a reduced set of equations which is closed according to z', v_z, $\mathbf{v}$, T', and p'_s, and does not include ρ' and ω (though equations for these fields, (1e) and (4), remain valid)

$$\frac{1}{c_a^2}\frac{\partial \zeta}{\partial t} = \frac{1}{gH_0}\left[\frac{1}{H_0}\left(\alpha + p\frac{\partial}{\partial p}\right) v_z - \nabla \cdot \mathbf{v} + \frac{Q}{T_0}\right], \tag{5a}$$

$$\frac{\partial \eta}{\partial t} = -\frac{T_i v_z}{T_0 H_0} + \frac{Q}{T_0}\,, \tag{5b}$$

$$\frac{\partial v_z}{\partial t} = g\left[\left(\frac{\partial}{\partial p}p - \alpha\right)\zeta + \eta\right], \tag{5c}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\,gH_0 \nabla\,\zeta\,, \tag{5d}$$

$$\frac{\partial p'_s}{\partial t} = \left[\frac{p}{\alpha}\left(\frac{p}{H_0}\frac{\partial v_z}{\partial p} - \nabla\cdot\mathbf{v} + \frac{Q}{T_0}\right)\right]_{p=p_0}. \tag{5e}$$

We have introduced nondimensional fluctuative fields in place of z' and T':

$$\zeta = \frac{z'}{H_0}\,, \quad \eta = \frac{T'}{T_0} - \frac{T_i z'}{T_0 H_0}\,. \tag{6}$$

2.5. Wave equations. It is easy to get two second-order equations for ζ and η, differentiating (5a) and (5b) according to the time and eliminating the first-order time derivatives with the help of (5c) and (5d):

$$\left[H_0^2\left(\frac{1}{c_a^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) - \left(p\frac{\partial}{\partial p} + \alpha\right)\left(\frac{\partial}{\partial p}p - \alpha\right)\right]\zeta$$

$$-\left(p\frac{\partial}{\partial p} + \alpha\right)\eta = \frac{R}{g^2}\frac{\partial Q}{\partial t}\,, \tag{7a}$$

$$\left(\frac{1}{N^2}\frac{\partial^2}{\partial t^2} + 1\right)\eta + \left(\frac{\partial}{\partial p}p - \alpha\right)\zeta = \frac{RT_0}{g^2 T_i}\frac{\partial Q}{\partial t}\,, \tag{7b}$$

where $c_a = \sqrt{RT_0/\alpha}$ is the sound speed and $N = \sqrt{RT_i}/H_0$ represents the Väisälä frequency. These equations can be employed for modelling linear wave processes in p-coordinate presentation in a general, nonfiltered case.

2.6. The Lagrangian function and energy. For the present study the significance of wave equations is that they have a Lagrangian function

$$\mathcal{L} = \frac{1}{2}\left\{-\frac{H_0^2}{c_a^2}(\zeta_{,t})^2 - \frac{1}{N^2}(\eta_{,t})^2 + H_0^2(\nabla\zeta)^2\right.$$

$$\left. + \left[\left(\frac{\partial}{\partial p}p - \alpha\right)\zeta + \eta\right]^2\right\}. \tag{8}$$

On the one hand, the existence of the Lagrangian guarantees energy conservation. On the other hand, with the help of Lagrangian formalism it is easy to get filtered versions of the model which are still energy-conserving. Explicitly the energy density can be presented for model (5) as

$$e = \frac{1}{2}\left(\frac{g^2 H_0^2}{c_a^2}\zeta^2 + \frac{RT_0^2}{T_i}\eta^2 + \mathbf{v}^2 + v_z^2\right). \tag{9}$$

The first two terms present potential energy which the air particle has due to the isobaric height and temperature fluctuations, the remaining two terms are kinetic energy.

3. ACOUSTIC FILTERING

For slow atmospheric movements with a small Mach number,

$$\mathcal{M} \equiv U^2/c_a^2 << 1\,, \tag{10}$$

where U is the characteristic amplitude of the velocity, it is reasonable to simplify model equations in the way they do not include acoustic-wave solutions any more. This procedure is called filtering.

3.1. Filtering of linearized equations (5). The physical basis and proof for acoustic filtering can be received from the scale analysis of the Lagrangian (8). It is easy to verify that the first term in (8) is small in comparison with others in all scales if the Mach number is small and, thus, it can be neglected. Formally filtering is straightforward with the help of the limiting process

$$c_a \to \infty \tag{11}$$

in Eq. (5a), which yields

$$\frac{1}{H_0}\left(\alpha + p\frac{\partial}{\partial p}\right)v_z - \nabla\cdot\mathbf{v} + \frac{Q}{T_0} = 0 \tag{12}$$

This equation along with other Eqs. (5b)–(5e) (which did not change at filtering) presents the basic acoustically filtered set of linear equations in p-coordinates. The wave equations for the filtered model can be obtained from (7) with the help of the same formal filtering procedure (11) and they possess the Lagrangian function

$$\mathcal{L} = \frac{1}{2}\left\{-\frac{1}{N^2}(\eta_{,t})^2 + H_0^2(\nabla\zeta)^2 + \left[\left(\frac{\partial}{\partial p}p - \alpha\right)\zeta + \eta\right]^2\right\}. \tag{8'}$$

As a consequence, filtering does not harm energy conservation. The energy density can be deduced from (9), using passage (11):

$$e = \frac{1}{2}\left(\frac{RT_0^2}{T_i}\eta^2 + \mathbf{v}^2 + v_z^2\right). \tag{9'}$$

Equation (12) presents the main diagnostic relation in the filtered model. In nonstationary problems it presents the basic equation for ζ determination, simultaneously it can be used for the calculation of the vertical speed v_z. After filtering is carried out, it is impossible to return from the field η back to the ordinary temperature fluctuation T' and the hydrodynamic content of η alters. In the filtered model it has the content of the relative density fluctuation. To prove this feature, we note that from Eqs. (4) and (12) a relationship follows

$$\nabla \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = -\frac{T_i v_z}{T_0 H_0} + \frac{Q}{T} .$$

The comparison of this equation with (5b) exhibits that η satisfies the condition

$$\frac{\partial \eta}{\partial t} - \nabla \cdot \mathbf{v} - \frac{\partial \omega}{\partial p} = 0 .$$

That means, η evolves according to the same equation which is valid for $-g\rho'$ in the nonfiltered model.

3.2. The inclusion of advective processes into the filtered model can be achieved in the simplest manner by modelling them as quasi-incompressibles in the p-space. For that in the adjusted model the local time derivatives $\partial/\partial t$ should be replaced by the individual derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \omega_s \frac{\partial}{\partial p} , \tag{13}$$

in momentum Eqs. (5c) and (5d). Here ω_s is the vertical p-velocity of the incompressible flow:

$$\frac{\partial \omega_s}{\partial p} + \nabla \cdot \mathbf{v} = 0 . \tag{14}$$

At the same time, (5b) should be maintained in its initial linear form, as η, having the content of small relative density fluctuations, is not redistributed in space advectively. The resulting filtered equations are

$$\frac{1}{H_0}\left(\alpha + p\frac{\partial}{\partial p}\right) v_z - \nabla \cdot \mathbf{v} + \frac{Q}{T_0} = 0 , \tag{15a}$$

$$\frac{\partial \eta}{\partial t} = -\frac{T_i v_z}{T_0 H_0} + Q/T_0 , \tag{15b}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = g\left[\left(\frac{\partial}{\partial p}p - \alpha\right)\zeta + \eta\right] , \tag{15c}$$

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\,gH_0 \nabla \zeta , \tag{15d}$$

$$\frac{\mathrm{d}p'_s}{\mathrm{d}t} = \left[\frac{p}{\alpha}\left(\frac{p}{H_0}\frac{\partial v_z}{\partial p} - \nabla \cdot \mathbf{v} + \frac{Q}{T_0}\right)\right]_{p=p_0} . \tag{15e}$$

Thus, our model treats linear processes as compressible in the p-space, while nonlinear processes are approximated as incompressible. The energy density for this nonlinear model coincides with the energy density of the linear acoustically filtered case (9′).

4. CONCLUSIONS

We have developed a version of an acoustically filtered set of model equations for atmospheric dynamics. Unlike common models, such as the MPM in p-space or anelastic models in ordinary Cartesian coordinates, our model does not use incompressibility for wave filtering. Certainly, the quality of the model should be tested in further experiments. Nevertheless, it is optimal at least in one respect: in linear case it presents the best approximation to the nonfiltered equations.

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ATMOSFÄÄRIDÜNAAMIKA MITTEHÜDROSTAATILISED AKUSTILISELT FILTREERITUD VÕRRANDID RÕHUKOORDINAATIDES

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On kirjeldatud atmosfääridünaamika akustiliselt filtreeritud mudelvõrrandite optimaalset tuletusalgoritmi rõhukoordinaatides. On lähtutud töös [6] tuletatud täielikest mittelineaarsetest hüdrodünaamika võrranditest rõhukoordinaatides. Need võrrandid on lineariseeritud ja toodud kujule (5), kus akustiline filtreerimine on lihtsaimal viisil teostatav

piirüleminekuga lõpmata suurele häälekiirusele, $c_a \to \infty$. Tulemusena väheneb võrrandite ajaline järk kahe võrra ning kaovad häälelained. Saadud akustiliselt filtreeritud lineaarsed võrrandid on täiendatud advektiivsete liikmete lisamisega tagasi mittelineaarseteks. Seejuures on lähendatud advektsiooni mudeliga, mis vastab kokkusurumatule voolamisele rõhukoordinaatide ruumis. Nii saadavas filtreeritud mittelineaarses mudelis (15) säilib energia. Mudel annab parima lähendi filtreerimata dünaamikale lineaarsete protsesside piirjuhul.