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MULTIVARIATE MINIMAL DISTRIBUTIONS

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Abstract. In this paper the multivariate minimal distribution (lower Fréchet bound) is discussed. Using different definitions of the multivariate minimal distribution, given by Dall'Aglio (*Inst. Stat. Univ. Paris*, 1960, **9**), and by Kotz and Tiit (*Acta et Comment. Univ. Tartuensis*, 1992, **942**), some useful properties of this distribution are deduced. The other purpose of this paper is to construct an example of a discrete k-variate minimal distribution for any $k > 2$. The distribution constructed has a support consisting of k equiprobable points and is unique (up to the scaling constant a and some special transformation of coordinates). The distribution has the minimal possible correlation for a k-variate distribution with equal marginals and hence is in some sense *the globally minimal k-variate distribution.*

Key words: distributions with fixed marginals, Fréchet bounds, multivariate dependency.

1. INTRODUCTION

For a bivariate random vector (X, Y) it is possible to measure not only the strength but also the direction of dependence (correlation). If the univariate distributions P_X and P_Y (having the distribution functions $F_1(x)$ and $F_2(y)$, respectively) are fixed, then there exists a set $\Pi(P_X, P_Y)$ of bivariate distributions with marginals P_X and P_Y. This set has two so-called Fréchet bounds (see [1]). *The upper Fréchet bound* or *maximal distribution* is defined by its distribution function $F^+(x, y)$,

$$F^+(x, y) = \min(F_1(x), F_2(y)) \tag{1}$$

and *the lower Fréchet bound* or *minimal distribution* has the following bivariate distribution function

$$F^-(x, y) = \max(0, (F_1(x) + F_2(y) - 1)) \tag{2}$$

(see [2]). The distributions P^- and P^+ defined via $F^-(x,y)$ and $F^+(x,y)$ have respectively the minimal and the maximal correlation coefficient r^- and r^+ for given marginals,

$$r^- \leq r(P_X, P_Y) \leq r^+,$$

where always the following natural condition is satisfied

$$-1 \leq r^- < 0 < r^+ \leq 1.$$

The problem, how to generalize the concepts of minimal and maximal distributions when they do exist, and which are their properties in the multivariate case, has excited statisticians during the last fifty years. Below we will regard several properties of maximal and minimal multivariate distributions and construct one simple example of a k-variate minimal distribution, $k > 2$.

2. MAXIMAL DISTRIBUTION

Let $P_1, \ldots, P_k$ be given univariate distributions having the distribution functions $F_i(x)$, $i = 1, \ldots, k$, and let $\Pi(P_1, \ldots, P_k)$ be the class of all k-variate distributions with marginal distributions $P_1, \ldots, P_k$. The concept of maximal distribution can easily be generalized for any dimensionality. Its distribution function $F^+(x_1, \ldots, x_k)$ or the upper Fréchet bound of the set Π can be defined by the following formula, generalizing the formula (1):

$$F^+(x_1, \ldots, x_k) = \min[F_1(x_1), \ldots, F_k(x_k)]. \tag{3}$$

The k-variate maximal distribution always exists as demonstrated in [3, 4]. Let P^+ be a k-variate maximal distribution defined by its distribution function $F^+(\cdot)$ (see [3]). From the same formula it is easy to deduce several important properties of the maximal distribution.

Property 1. *For every h $(1 < h < k)$ all h-variate marginal distributions of P^+ are h-variate maximal distributions corresponding to their marginals $P_{i_1}, \ldots, P_{i_h}$.*

As this fact holds in the case $h = 2$, too, then from here the second property of the maximal distribution follows.

Property 2. *All correlations of the maximal distribution are maximal r^+_{ij}, $i = 1, \ldots, k-1$, $j = i+1, \ldots, k$.*

From the formulae (1) and (3) also the third property of the maximal distribution follows.

Property 3. *The maximal distribution is uniquely defined by its bivariate marginals.*

3. MINIMAL DISTRIBUTION

It is tempting to generalize the concept of minimal distribution for the k-dimensional case as well but here some serious difficulties arise.

Dall'Aglio (see [3]) has defined the minimal distribution using a generalization of the formula (2),

$$F^{-}(x_1,\ldots,x_k) = \max[0, (F_1(x_1) + \ldots + F_k(x_k) - (k-1))]. \quad (4)$$

The problem is that the formula (4) does not always define a distribution function (see [5–7]). Dall'Aglio in [3, 8] has found the necessary and sufficient conditions for marginals P_i so that $F^{-}(\cdot)$ defined by (4) would be a distribution function. Let us give his result in the following.

Theorem 1. *The minimal distribution defined by the function* (4) *as its distribution function exists if and only if one of the following conditions is satisfied*:

$$F_1(a_1+) + \ldots + F_k(a_k+) \geq k-1 \quad (5)$$

or

$$F_1(b_1) + \ldots + F_k(b_k) \leq 1, \quad (6)$$

where

$$a_j = \inf[x : F_j(x) > 0]$$

and

$$b_j = \sup[x : F_j < 1].$$

From the formula (4) it follows that if the minimal distribution exists, then it is unique. It is easy to see that in this case the properties similar to those of maximal distributions hold also for minimal distributions.

Property 4. *For every h $(1 < h < k)$ all h-variate marginal distributions of P^{-} are h-variate minimal distributions corresponding to their marginals $P_{i_1},\ldots,P_{i_h}$.*

Property 5. *All correlations of the minimal distribution are minimal r^{-}_{ij}, $i = 1,\ldots,k-1$, $j = i+1,\ldots,k$.*

Obviously, the minimal distribution should be defined as a distribution having the properties 4 and 5 deduced above. Kotz and Tiit (see [9]) used this definition of the minimal distribution (the lower Fréchet bound) and regarded specially the case of equal marginals. In this case the minimal distribution has some additional properties.

Property 6. *The multivariate minimal distribution having equal marginals is exchangeable* (see [10]).

Property 7. *In the case of equal marginals of the k-variate exchangeable distribution the minimal possible value of the correlation coefficient is*

$$r^{-} = \frac{-1}{k-1}. \quad (7)$$

From this result of Kotz and Tiit (see [9]) we can deduce the following.

Theorem 2. *A necessary condition for the existence of the k-variate minimal distribution having equal marginals $P_i = P_0,\ i = 1, \ldots, k$, is that the minimal correlation coefficient $r^-(P_0, P_0)$ satisfies the condition*

$$r^-(P_0, P_0) \geq \frac{-1}{k-1}. \tag{8}$$

From Theorem 2 some corollaries follow.

Corollary 1. *The minimal distribution is degenerated in the $(k-1)$-variate space R^{k-1} that is orthogonal to the principal diagonal of the k-variate unit cube.*

The proof of this property is given in [9] for the case of the normal distribution. The proof does not depend essentially on the underlying distribution, it does depend on the properties of the correlation matrix and corresponding random vectors only.

Corollary 2. *For every univariate distribution P_0 from the condition* (8) *the highest dimensionality k can be calculated so that there exists the k-variate minimal distribution having marginals P_0, but the $(k+1)$-variate minimal distribution with the same marginals does not exist.*

Corollary 3. *For equal and symmetric marginals only bivariate minimal distributions exist.*

From the conditions of Dall'Aglio (see [3, 8]) and (8) it follows that for the majority of commonly used univariate marginal distributions the multivariate minimal distribution does not exist at all. If the minimal multivariate distribution exists, it should have quite a special shape. In the following part of the paper an attempt is made to find a simple construction for a family of multivariate minimal distributions.

4. EXAMPLE OF A k-VARIATE MINIMAL DISTRIBUTION

We are going to construct a simple k-variate minimal distribution. Let k be arbitrary but fixed. First of all we have to construct the suitable marginal distributions. The easiest way is to use the equal marginals, hence we have to construct at first the common univariate marginal distribution P_0. Let us build the distribution P_0 as a discrete one. Then the simplest way is to use the Bernoulli distribution. Let the support of the distribution consist of a pair of points $[0, a]$, where $a > 0$ is an arbitrary number, $P(a) = p$ and $P(0) = 1 - p$ are probabilities.

In this case the univariate marginal distribution function is the following:

$$F_0(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1-p, & \text{if } 0 \leq x < a \\ 1, & \text{if } x > a. \end{cases} \tag{9}$$

In addition, this distribution must satisfy the necessary and sufficient conditions of Dall'Aglio (5) or (6). Let us choose the second one. From this condition it follows that the probability p must satisfy the condition

$$0 \leq p \leq \frac{1}{k} < 1. \tag{10}$$

Now let us define a discrete k-variate distribution P^- using the marginal distributions defined via (9). Let the support of the distribution P^- consist of $k+1$ points in the k-variate space so that every point is situated on a different (ith) coordinate axis ($i = 1, ..., k$) at the distance a from the zero point and has the probability p. The remaining probability mass $q = 1 - kp$ corresponds to the zero point.

The values p and q should be chosen in such way that all bivariate marginals were minimal, or, what is the same, so that the correlation coefficients would have their minimal value (8), if possible. Using the definition of the correlation coefficient

$$r(X_i, X_j) = \frac{\operatorname{cov}(X_i, X_j)}{\sqrt{DX_i DX_j}} = \frac{-a^2p^2}{a^2p(1-p)} = -\frac{p}{1-p}, \tag{11}$$

$(i = 1, \ldots, k-1,\ j = i+1, \ldots, k)$, we see that the correlation coefficient $r(P_0, P_0)$ will have the minimal possible value when

$$p = \frac{1}{k}, \tag{12}$$

and hence $q = 0$. In this case all correlation coefficients of the distribution P^- have the minimal possible value (7) and the distribution P^- defined via Eqs. (9) and (10) satisfies the following properties.

Property 8. *All correlation coefficients defined by the bivariate marginals of the distribution P^- equal to $\frac{-1}{k-1}$ and hence are the minimal possible ones for a k-variate distribution with equal marginals.*

Property 9. *The support of the distribution P^- consists of k points, one on every coordinate axis, and the distribution is degenerated on the $(k-1)$-variate hyperspace R^{k-1} of the k-variate space. This hyperspace is orthogonal to the main diagonal of the unit cube in the k-variate space.*

Property 10. *The distribution is exchangeable.*

The distribution P^- constructed is the *globally minimal k-variate distribution* since there does not exist any minimal k-variate distribution having all correlations less than the correlations of the distribution P^-.

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REFERENCES

1. Fréchet, M. Sur les tableaux de correlation dont les marges sont données. *Ann. Univ. Lyon Sci.*, 1951, **9**, 53–84.
2. Hoeffding, W. Maßstabinvariante Korrelationtheorie. *Schriften Math. Inst. Univ. Berlin*, 1940, **5**, 181–233.
3. Dall'Aglio, G. Les fonctions extrêmes de la classe de Fréchet à trois dimensions. *Inst. Stat. Univ. Paris*, 1960, **9**, 175–188.
4. Kemp, J. F. *Advanced Problem 5894. Amer. Math. Monthly.* Washington, DC, 1973, **80**.
5. Cuadras, C. M. *Methods de analisis multivariante*. University of Barcelona, Barcelona, 1981.
6. Rüschendorf, L. Construction of multivariate distribution with given marginals. *Ann. Inst. Stat. Math.*, 1983, **37**, 225–233.
7. Tiit, E.-M. Definition of random vectors with given marginal distributions and given correlation matrix. *Acta et Comment. Univ. Tartuensis*, 1984, **685**, 21–36.
8. Dall'Aglio, G. Fréchet classes. The beginnings. In *Advances in Probability Distributions with Given Marginals* (Dall'Aglio, G. et al., eds.). Kluwer, Dordrecht, 1991, 1–12.
9. Kotz, S. and Tiit, E.-M. Bounds in multivariate dependence. *Acta et Comment. Univ. Tartuensis*, 1992, **942**, 35–45.
10. Shaked, M. and Tong, Y. L. Positive dependence of random variables with a common marginal distribution. *Comm. Statist. Theory Methods*, 1991, **20**, 4299–4313.

MITMEMÕÕTMELINE MINIMAALJAOTUS

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On käsitletud mitmemõõtmelist maksimaal- ja minimaaljaotust ning eriti viimase olemasoluga seotud küsimusi. Esimene selle mõiste määratlus pärineb Dall'Agliolt (1960), kes esitas minimaaljaotuse definitsiooni (4) jaotusfunktsiooni kaudu, kuid tema enese esitatud tarvilikest tingimustest selgub, et niisugune mitmemõõtmeline minimaaljaotus eksisteerib vaid küllalt kitsendavate tingimuste täidetuse korral.

Alternatiivse minimaaljaotuse definitsiooni on andnud Kotz ja Tiit (1992), kasutades minimaaljaotuse määratlemiseks selle kahemõõtmelisi marginaaljaotusi. Käesolevas artiklis on esitatud rida mõlemast definitsioonist tulenevaid minimaaljaotuse omadusi.

Samuti on toodud maksimaalselt lihtne diskreetse k-mõõtmelise minimaaljaotuse konstruktsioon, mis on rakendatav suvalise antud k korral. Esitatud jaotus omab ühtlasi minimaalset võimalikku korrelatsiooni k-dimensionaalse juhu jaoks, olles sellega n.-ö. globaalselt minimaalne.