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THE METHOD OF FINITE DIFFERENCES FOR AN INVERSE PROBLEM RELATED TO A ONE-DIMENSIONAL VISCOELASTIC EQUATION OF MOTION

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Abstract. An inverse problem related to a one-dimensional linear viscoelastic equation of motion is transformed to a system of hyperbolic and second kind Volterra equations. The obtained system is discretized by the use of the method of finite differences. The convergence of the method is proved.

Key words: hyperbolic equation, inverse problem, difference scheme.

1. INTRODUCTION AND PROBLEM FORMULATION

In paper [1] a method based on the technique of finite differences was applied to an inverse problem for the reconstruction of two relaxation kernels of one-dimensional quasilinear viscoelastic media. In the present work we shall prove the convergence of this method in a simpler, linear case.

We consider the oscillation of the linear homogeneous viscoelastic rod, which is governed by the following equation of motion (cf. [1]):

$$\begin{gathered} U_{xx}(x,t) - \int_0^t R(t-s)U_{xx}(x,s)\,ds = a^2 U_{tt}(x,t) + F(x,t), \\ (x,t) \in D = [0,X]\times[0,T]. \end{gathered} \tag{1.1}$$

Here R is the relaxation kernel, U – displacement, and F – density of external forces. We add the initial conditions:

$$U(x,0) = A(x), \quad U_t(x,0) = B(x), \quad 0 \le x \le X, \tag{1.2}$$

and the homogeneous boundary conditions:

$$U(0,t) = U(X,t) = 0, \quad 0 \le t \le T. \tag{1.3}$$

Also we introduce the stress–strain relation at the endpoint $x = 0$ of the rod:

$$U_x(0,t) - \int_0^t R(t-s)U_x(0,s)\,ds = G(t), \quad 0 \le t \le T. \tag{1.4}$$

Here G and U_x stand for the stress and the strain, respectively.

Now we formulate the following inverse problem: on the ground of the given functions F, A, B, G and the scalar $a > 0$, determine the pair of unknown functions (R, U) from the conditions (1.1)–(1.4).

The problem (1.1)–(1.4) was theoretically studied in [2]. Assuming the functions F, A, B, G to be smooth enough and $A'(0)$ to be nonzero, it was proved that (1.1)–(1.4) admits a unique (local in time) solution which is locally stable with respect to perturbations of the given data in certain spaces involving derivatives.

The plan of our paper is as follows. Differentiating the problem (1.1)–(1.4), we derive a system containing a hyperbolic equation and Volterra equations of the second kind. Thereupon we apply the method of finite differences to this system and prove a stability theorem for the discrete problem. Particularly, the convergence of the method of finite differences follows from this theorem. Finally, we discuss some questions related to the ill-posedness and the regularization of the problem under consideration.

2. DIFFERENTIATED PROBLEM

Theorem 1. *Let $F \in C^3(D)$, $A, B \in C^3[0,X]$, $G \in W^{2,1}(0,T)$ and let the problem (1.1)–(1.4) have a solution $(R,U) \in W^{1,1}(0,T) \times C^5(D)$. Assume that $A'(0) \neq 0$ and F is vanishing in neighbourhoods of the endpoints $x = 0$, $x = X$. Define*

$$u = U_{xxx}, \quad \phi = U_{xtt}(0,\cdot), \quad r = R'. \tag{2.1}$$

Then the quadruple (u, ϕ, R, r) is a solution of the following differentiated problem:

$$u_{xx}(x,t) - \int_0^t R(t-s)u_{xx}(x,s)ds = a^2 u_{tt}(x,t) + f(x,t),$$

$$(x,t) \in D, \tag{2.2}$$

$$u(x,0) = \alpha(x), \quad u_t(x,0) = \beta(x), \quad 0 \le x \le X, \tag{2.3}$$

$$u_x(0,t) = u_x(X,t) = 0, \quad 0 \le t \le T, \tag{2.4}$$

$$\phi(t) = \frac{1}{a^2}\left[u(0,t) - \int_0^t R(t-s)u(0,s)ds\right], \quad 0 \le t \le T, \tag{2.5}$$

$$R(t) = \int_0^t r(s)ds + \rho, \qquad 0 \le t \le T, \tag{2.6}$$

$$r(t) = \frac{1}{\kappa_0}\left[-g(t) + \phi(t) - \kappa_1 R(t) - \int_0^t \phi(t-s)R(s)ds \right], \qquad 0 \le t \le T. \tag{2.7}$$

Here

$$f = F_{xxx}, \quad \alpha = A''', \quad \beta = B''', \quad g = G'', \\ \kappa_0 = A'(0), \quad \kappa_1 = B'(0), \quad \rho = \frac{1}{\kappa_0}(G'(0) - \kappa_1). \tag{2.8}$$

Proof. Equation (2.2) and initial conditions (2.3) immediately follow from (1.1) and (1.2). The boundary conditions (1.3) yield $U_{tt}(0,t) = U_{tt}(X,t) = 0, \quad 0 \le t \le T$. This equality together with the vanishing conditions about F imply that the right-hand side of (1.1) is equal to zero if $x = 0$ or $x = X$. Thus, Eq. (1.1) turns out to be a homogeneous Volterra equation of the second kind with respect to $U_{xx}(x,\cdot)$ if $x = 0$ or $x = X$. Consequently, we have

$$U_{xx}(0,t) = U_{xx}(X,t) = 0, \qquad 0 \le t \le T, \tag{2.9}$$

which in turn yields $U_{xxtt}(0,t) = U_{xxtt}(X,t) = 0$. Now we see that Eq. (1.1) differentiated two times by x has also the vanishing right-hand side if $x = 0$ or $x = X$. Hence, we have homogeneous Volterra equations of the second kind for the functions $U_{xxxx}(0,t)$ and $U_{xxxx}(X,t)$, too. This implies

$$U_{xxxx}(0,t) = U_{xxxx}(X,t) = 0, \qquad 0 \le t \le T.$$

Since $u = U_{xxx}$, we obtain the boundary conditions (2.4).

Differentiating the formula (1.1) with respect to x and setting x equal to zero, we immediately obtain (2.5). Moreover, computing a derivative from the expression (1.4) and setting $t = 0$, we deduce the formula $R(0) = \rho$ for the initial value of R. Consequently, the formula (2.6) holds as well. Finally, differentiating the condition (1.4) two times, we immediately derive Eq. (2.7). Theorem is proved. □

As we see, the obtained system (2.2)–(2.7) contains the hyperbolic Eq. (2.2) for u and the Volterra equations of the second kind (2.5), (2.6), and (2.7) for ϕ, R, and r, respectively. Solving the problem (2.2)–(2.7), we automatically determine the relaxation kernel R together with its derivative. If an evaluation of the second component of the solution (R, U) for (1.1)–(1.4) is also necessary, then we must implement some additional computations. For instance, U is the solution of the following family of boundary value problems for ODE:

$$U_{xxx}(x,t) = u(x,t), \qquad 0 \le x \le X, \\ U(0,t) = U(X,t) = U_{xx}(0,t) = 0; \quad 0 \le t \le T \tag{2.10}$$

(cf. (1.3), (2.1), (2.9)).

In paper [1] we used a one step lower differentiation to deduce a system of hyperbolic and second kind Volterra equations from a problem which is quite similar to (1.1)–(1.4). This system includes an equation for R and not an equation for the derivative $r = R'$. But it turns out that it is very difficult to analyse such systems because even an estimation of U on the basis of (1.1) brings along the derivative of R. For that reason we have applied the higher-order differentiation in the present paper and derived an equation for the derivative r, too.

3. DISCRETIZATION PARAMETERS AND AUXILIARY RESULTS

Let N and M be positive integers and

$$h = \frac{X}{N}, \quad \tau = \frac{T}{N}. \tag{3.1}$$

We define the following uniform meshes at the intervals $[0, X]$ and $[0, T]$:

$$\begin{aligned} \omega_h &= \{x_i = ih \,:\, i = 0, \ldots, N\}, \\ \omega_\tau &= \{t_j = j\tau \,:\, j = 0, \ldots, M\}. \end{aligned} \tag{3.2}$$

For values of functions $y \in (\omega_h \to \mathbb{R})$ we shall use the simplified notation:

$$y_i = y(x_i), \quad x_i \in \omega_h, \quad 0 \le i \le N. \tag{3.3}$$

Let $y, z \in (\omega_h \to \mathbb{R})$. We introduce the following discrete operations being analogues of derivatives:

$$\begin{aligned} \partial_x y \,&:\, (\partial_x y)_i \equiv \partial_x y_i = \frac{y_{i+1} - y_i}{h}, \quad i = 0, \ldots, N-1, \\ \hat{\partial}_x y \,&:\, (\hat{\partial}_x y)_i \equiv \hat{\partial}_x y_i = \frac{y_i - y_{i-1}}{h}, \quad i = 1, \ldots, N, \\ \Lambda y \,&:\, (\Lambda y)_i \equiv \Lambda y_i = \partial_x \hat{\partial}_x y_i, \quad i = 1, \ldots, N-1. \end{aligned} \tag{3.4}$$

Moreover, we define the scalar products:

$$\begin{aligned} (y, z) &:= h \sum_{i=1}^{N-1} y_i z_i \,, \\ (y, z] &:= h \sum_{i=1}^{N} y_i z_i \,, \\ [y, z] &:= h \sum_{i=0}^{N} y_i z_i \end{aligned} \tag{3.5}$$

and norms:

$$\begin{aligned} \| y \|_2 &:= \sqrt{(y, y)}, \quad \| y \,]\|_2 := \sqrt{(y, y]}, \\ \|[\, y \,]\|_2 &:= \sqrt{[y, y]}, \quad \| y \|_\infty := \max_{0 \le i \le N} |\, y_i \,| \,. \end{aligned} \tag{3.6}$$

For the operator Λ an analogue of the Green's first formula is valid (see [3]):

$$(\Lambda y, z) = -(\hat{\partial}_x y, \hat{\partial}_x z] + \partial_x y_N z_N - \partial_x y_0 z_0 . \tag{3.7}$$

Let now $y \in (\omega_\tau \to \mathbb{R})$ or $y \in (\omega_\tau \to (\omega_h \to \mathbb{R}))$. We denote

$$y^j = y(t_j), \quad t_j \in \omega_\tau , \quad 0 \leq j \leq M \tag{3.8}$$

and define

$$\diamond y : (\diamond y)^j \equiv \diamond y^j = \frac{y^j + y^{j-1}}{2}, \quad 1 \leq j \leq M , \tag{3.9}$$

$$\begin{aligned} \partial_t y \, &: \, (\partial_t y)^j \equiv \partial_t y^j = \frac{y^{j+1} - y^j}{\tau}, \quad 0 \leq j \leq M - 1 , \\ \hat{\partial}_t y \, &: \, (\hat{\partial}_t y)^j \equiv \hat{\partial}_t y^j = \frac{y^j - y^{j-1}}{\tau}, \quad 1 \leq j \leq M . \end{aligned} \tag{3.10}$$

The operator ∂_t satisfies the following analogues of the formula for the differentiation of the product (see [3]):

$$\partial_t (y^j z^j) = y^j \partial_t z^j + \partial_t y^j z^{j+1} , \tag{3.11}$$

and integration by parts:

$$\tau \sum_{j=l_1}^{l_2-1} y^j \partial_t z^j = y^{l_2} z^{l_2} - y^{l_1} z^{l_1} - \tau \sum_{j=l_1}^{l_2-1} \partial_t y^j z^{j+1} . \tag{3.12}$$

If y^j, z^j are vectors, i.e. $y^j, z^j \in (\omega_h \to \mathbb{R})$, then the formulae (3.11), (3.12) hold componentwise.

Let us prove four lemmas that are necessary in the sequel.

Lemma 1. *For* $y \in (\omega_\tau \to (\omega_h \to \mathbb{R}))$ *the following estimates are fulfilled*:

$$\| \hat{\partial}_x \diamond y^j]|_2^2 \geq \frac{1}{2} \| \hat{\partial}_x y^j]|_2^2 - \frac{\tau^2}{4} \| \hat{\partial}_x \hat{\partial}_t y^j]|_2^2 , \qquad 1 \leq j \leq M , \tag{3.13}$$

$$\begin{aligned} \| y^k \|_\infty \leq \max \left\{ \frac{T}{\sqrt{X - h}}, \sqrt{X} \right\} \Bigg[&\max_{1 \leq j \leq k} \| \hat{\partial}_x y^j]|_2 \\ &+ \max_{1 \leq j \leq k} \| \hat{\partial}_t y^j \|_2 \Bigg] + \| y^0 \|_\infty , \qquad 1 \leq k \leq M. \end{aligned} \tag{3.14}$$

Proof. Since

$$\hat{\partial}_x y_i^j = \hat{\partial}_x \diamond y_i^j + \frac{\tau}{2} \hat{\partial}_x \hat{\partial}_t y_i^j , \qquad 1 \leq j \leq N ,$$

and

$$(d_1 + d_2)^2 \leq 2(d_1^2 + d_2^2), \quad \forall d_1, d_2 \in \mathbb{R} , \tag{3.15}$$

we have

$$(\hat{\partial}_x y_i^j)^2 \leq 2(\hat{\partial}_x \diamond y_i^j)^2 + \frac{\tau^2}{2} (\hat{\partial}_x \hat{\partial}_t y_i^j)^2 , \quad 1 \leq i \leq N .$$

Summing over $i = 1, \dots, N$ in view of (3.5), (3.6), we deduce (3.13).

Let us prove (3.14). Define

$$i_*(k) : \mid y^k_{i_*(k)} \mid = \min_{0 \le i \le N} \mid y^k_i \mid . \tag{3.16}$$

We can rewrite y^k_l in the following form:

$$\begin{aligned} y^k_l &= y^k_{i_*(k)} + h \sum_{i=i_*(k)+1}^{l} \hat{\partial}_x y^k_i \quad \text{if} \quad l \ge i_*(k) + 1 , \\ y^k_l &= y^k_{i_*(k)} - h \sum_{i=l+1}^{i_*(k)} \hat{\partial}_x y^k_i \quad \text{if} \quad l \le i_*(k) - 1 , \\ y^k_l &= y^k_{i_*(k)} \quad \text{if} \quad l = i_*(k) . \end{aligned}$$

Thus,

$$\mid y^k_l \mid \le \mid y^k_{i_*(k)} \mid + h \sum_{i=1}^{N} \mid \hat{\partial}_x y^k_i \mid .$$

Further, using (3.16), we obtain

$$\begin{aligned} \mid y^k_l \mid &\le \frac{1}{(N-1)h} h \sum_{i=1}^{N-1} \mid y^k_i \mid + h \sum_{i=1}^{N} \mid \hat{\partial}_x y^k_i \mid \\ &= \frac{1}{X-h} h \sum_{i=1}^{N-1} \left| \tau \sum_{j=1}^{k} \hat{\partial}_t y^j_i + y^0_i \right| + h \sum_{i=1}^{N} \mid \hat{\partial}_x y^k_i \mid \\ &\le \frac{1}{X-h} \tau \sum_{j=1}^{k} h \sum_{i=1}^{N-1} \mid \hat{\partial}_t y^j_i \mid \\ &\quad + \frac{1}{X-h} h \sum_{i=1}^{N-1} \mid y^0_i \mid + h \sum_{i=1}^{N} \mid \hat{\partial}_x y^k_i \mid \\ &\le \frac{T}{X-h} \max_{1 \le j \le k} h \sum_{i=1}^{N-1} \mid \hat{\partial}_t y^j_i \mid + \parallel y^0 \parallel_\infty + h \sum_{i=1}^{N} \mid \hat{\partial}_x y^k_i \mid . \end{aligned} \tag{3.17}$$

On the ground of the Cauchy–Bunjakowski inequality we have

$$h \sum_{i=l_1+1}^{l_2} \mid z_i \mid \le \sqrt{(l_2 - l_1)h} \left[h \sum_{i=l_1+1}^{l_2} (z_i)^2 \right]^{\frac{1}{2}} , \quad z \in (\omega_h \to \mathbb{R}), \ l_2 > l_1 . \tag{3.18}$$

Now (3.14) follows from (3.5), (3.6), (3.17), and (3.18). $\square$

Lemma 2. *For $y \in C^2[0,X]$ the following estimate is fulfilled:*

$$\| \hat{\partial}_x y \,]|_2 \leq c(X) \cdot \{ \| y' \|_{L^2(0,X)} + \| y' \|^{\frac{1}{2}}_{L^2(0,X)}$$
$$+ (\| y'' \|_{C[0,X]} + \| y'' \|^2_{C[0,X]}) h \} \,, \qquad (3.19)$$

where c is a certain constant.

Proof. Note that for arbitrary $z \in C^1[0,X]$ there holds the estimate

$$\left| h \sum_{i=1}^{N} (z_i)^2 - \int_0^X z^2(x) dx \right|$$
$$= \left| \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \int_x^{x_i} \frac{d}{ds} [z(s)]^2 ds dx \right|$$
$$= \left| 2 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \int_x^{x_i} z(s) z'(s)\, ds\, dx \right|$$
$$\leq 2 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (x_i - x)^{\frac{1}{2}}\, dx \, \| z \|_{L^2(0,X)} \| z' \|_{C[0,X]}$$
$$= \frac{4X}{3} \sqrt{h} \, \| z \|_{L^2(0,X)} \| z' \|_{C[0,X]} \, . \qquad (3.20)$$

Making use of the well-known relation

$$| \hat{\partial}_x y_i | \leq | y'(x_i) | + \frac{1}{2} \| y'' \|_{C[0,X]} \, h$$

and the inequality (3.20), we obtain

$$\| \hat{\partial}_x y_i \,]|_2 = \left\{ h \sum_{i=1}^{N} (\hat{\partial}_x y_i)^2 \right\}^{\frac{1}{2}}$$
$$\leq \left\{ \int_0^X [y'(x)]^2 dx \right\}^{\frac{1}{2}} + \left\{ h \sum_{i=1}^{N} [y'(x_i)]^2 \right.$$
$$\left. - \int_0^X [y'(x)]^2 dx \right\}^{\frac{1}{2}} + \frac{\sqrt{X}}{2} \| y'' \|_{C[0,X]} \, h$$
$$\leq \| y' \|_{L^2(0,X)} + \frac{2\sqrt{X}}{\sqrt{3}} \| y' \|^{\frac{1}{2}}_{L^2(0,T)} \| y'' \|^{\frac{1}{2}}_{C[0,T]} \, h^{\frac{1}{4}}$$
$$+ \frac{\sqrt{X}}{2} \| y'' \|_{C[0,X]} \, h \, . \qquad (3.21)$$

Estimating the middle term in (3.21) by means of the elementary formula

$$d_1 d_2 = \sqrt{d_1} \sqrt{d_1} d_2 \leq \frac{1}{2} d_1 + \frac{1}{2} d_1 d_2^2 \leq \frac{1}{2} d_1 + \frac{1}{4} d_1^2 + \frac{1}{4} d_2^4, \quad d_1, d_2 \geq 0,$$

we deduce (3.19). $\square$

Lemma 3. *(Analogue of the Gronwall inequality). Let* $y, z \in (\omega_\tau \to \mathbb{R})$, $y^j \geq 0, \quad z^j \geq 0, \quad d \geq 0$, *and*

$$y^k \leq d\Theta(k-1)\tau \sum_{j=0}^{k-1} y^j + z^k, \quad 0 \leq k \leq k_0\,, \tag{3.22}$$

where Θ *is the Heaviside function*:

$$\Theta(s) = 1, \quad s \geq 0\,, \quad \Theta(s) = 0, \quad s < 0\,. \tag{3.23}$$

Then

$$\max_{0\leq j\leq k} y^j \;\leq\; \bar{c}(dt_k) \max_{0\leq j\leq k} z^j, \quad 0 \leq k \leq k_0\,, \tag{3.24}$$

where $\bar{c}$ *is a certain constant.*

Proof. In case $d = 0$, the assertion is trivial. Let $d \neq 0$. From (3.22) we immediately derive

$$\max_{0\leq j\leq k} e^{-2dt_l} y^j \leq d \max_{0\leq j\leq k} \Bigg[\Theta(j-1) e^{-2dt_j} \tau \sum_{l=0}^{j-1} e^{2dt_l} \times \max_{0\leq l\leq j-1} e^{-2dt_l} y^l \Bigg] + \max_{0\leq j\leq k} z^j, \quad 0 \leq k \leq k_0\,.$$

Since

$$\Theta(j-1)e^{-2dt_j}\tau \sum_{l=0}^{j-1} e^{2dt_l} \;\leq\; \Theta(j-1)e^{-2dt_j} \int_0^{t_j} e^{2d\cdot s}\,ds \leq \frac{1}{2d}\,,$$

we have

$$\max_{0\leq j\leq k} e^{-2dt_j} y^j \leq \frac{1}{2} \max_{0\leq j\leq k} e^{-2dt_j} y^j + \max_{0\leq j\leq k} z^j\,, \quad 0 \leq k \leq k_0\,.$$

Thus

$$\max_{0\leq j\leq k} e^{-2dt_j} y^j \;\leq\; 2 \max_{0\leq j\leq k} z^j, \quad 0 \leq k \leq k_0\,,$$

and due to the estimate

$$e^{-2dt_k} \max_{0\leq j\leq k} y^j \;\leq\; \max_{0\leq j\leq k} e^{-2dt_j} y^j$$

we obtain (3.24). $\square$

Lemma 4. *If*

$$(y^k)^2 + (z^k)^2 \;\leq\; \xi^k, \quad 1 \leq k \leq M, \tag{3.25}$$

where $y^k, z^k, \xi^k \geq 0$ *and* $\xi^{k+1} \geq \xi^k$, *then*

$$\left(\max_{1\leq j\leq k} y^j + \max_{1\leq j\leq k} z^j \right)^2 \;\leq\; 4\xi^k, \quad 1 \leq k \leq M\,. \tag{3.26}$$

Proof. Due to (3.25) we have

$$(y^k)^2 \le \xi^k, \quad (z^k)^2 \le \xi^k, \quad 1 \le k \le M,$$

and

$$y^k \le \sqrt{\xi^k}, \quad z^k \le \sqrt{\xi^k}, \quad 1 \le k \le M. \tag{3.27}$$

Since ξ^k is monotonically increasing, from (3.27) we derive

$$\max_{1\le j\le k} y^k \le \sqrt{\xi^k}, \quad \max_{1\le j\le k} z^k \le \sqrt{\xi^k}, \quad 1 \le k \le M. \tag{3.28}$$

Summing the inequalities (3.28) and squaring, we get (3.26). □

4. DIFFERENCE SCHEME

Let us return to the system (2.2)–(2.7). We suppose that instead of the exact data $\alpha, \beta, f, g, \rho, \kappa_0, \kappa_1$ we know certain approximations $\tilde{\alpha} \approx \alpha$, $\tilde{\beta} \approx \beta$, $\tilde{f} \approx f$, $\tilde{g} \approx g$, $\tilde{\rho} \approx \rho$, $\tilde{\kappa}_0 \approx \kappa_0$, $\tilde{\kappa}_1 \approx \kappa_1$. According to the notation (3.3), (3.8), we write

$$\tilde{f}_i^j = \tilde{f}(x_i, t_j), \quad \tilde{\alpha}_i = \tilde{\alpha}(x_i), \quad \tilde{\beta}_i = \tilde{\beta}(x_i), \quad \tilde{g}^j = \tilde{g}(t_j). \tag{4.1}$$

We discretize the problem (2.2)–(2.7) making use of the method of finite differences. We replace the derivatives by the formulae (3.4), (3.10) and the integrals by the quadrangle rule. Assuming that $\tilde{\kappa}_0 \ne 0$, we obtain the following system:

$$\Lambda v_i^j - \tau \sum_{l=1}^{j} Q^{j-l} \Lambda v_i^l = a^2 \hat{\partial}_t \partial_t v_i^j + \tilde{f}_i^j,$$

$$1 \le i \le N-1, \quad 1 \le j \le M-1, \tag{4.2}$$

$$v_i^0 = \tilde{\alpha}_i, \quad \partial_t v_i^0 = \tilde{\beta}_i, \quad 0 \le i \le N, \tag{4.3}$$

$$\partial_x v_0^j = \hat{\partial}_x v_N^j = 0, \quad 1 \le j \le M, \tag{4.4}$$

$$\psi^j = \frac{1}{a^2}\left[v_0^j - \Theta(j-1)\tau \sum_{l=1}^{j} Q^{j-l} v_0^l \right], \quad 0 \le j \le M, \tag{4.5}$$

$$Q^j = \Theta(j-1)\tau \sum_{l=0}^{j-1} q^l + \tilde{\rho}, \quad 0 \le j \le M, \tag{4.6}$$

$$q^j = \frac{1}{\tilde{\kappa}_0}\left[-\tilde{g}^j + \psi^j - \tilde{\kappa}_1 Q^j - \Theta(j-1)\tau \sum_{l=1}^{j} \psi_{j-l} Q^l \right], \quad 0 \le j \le M. \tag{4.7}$$

In the forthcoming sections we will show that the solution of (4.2)–(4.7) v_i^j,ψ^j,Q^j,q^j, $0 \le i \le N$, $0 \le j \le M$ approximates the solution of (2.2)–(2.7) in the nodes (x_i, t_j) :

$$v_i^j \approx u(x_i, t_j), \quad \psi^j \approx \phi(t_j), \quad Q^j \approx R(t_j), \quad q^j \approx r(t_j).$$

The difference scheme (4.2)–(4.7) is uniquely solvable because it is explicit. Indeed, from (4.3), (4.5)–(4.7) we easily derive formulae for the first two levels of j :

$$v_i^0 = \tilde{\alpha}_i, \quad 0 \le i \le N; \quad \psi^0 = \frac{1}{a^2}\tilde{\alpha}_0;$$

$$Q^0 = \tilde{\rho}; \quad q^0 = \frac{1}{\tilde{\kappa}_0}\Big[- \tilde{g}^0 + \frac{1}{a^2}\tilde{\alpha}_0 - \tilde{\kappa}_1\tilde{\rho}\Big];$$

$$v_i^1 = \tilde{\alpha}_i + \tau\tilde{\beta}_i, \quad 0 \le i \le N; \quad \psi^1 = \frac{1}{a^2}(1 - \tau\tilde{\rho})v_0^1;$$

$$Q^1 = \tilde{\rho} + \tau q^0; \quad q^1 = \frac{1}{\tilde{\kappa}_0}\big[- \tilde{g}^1 + \psi^1 - (\tilde{\kappa}_1 + \tau\psi^0)Q^1\big].$$

Suppose that we have computed the solution up to the level $j-1$, i.e. we know

$$v_i^l,\ 0 \le i \le N,\ \psi^l,\ Q^l,\ q^l, \quad \text{where} \quad l = 0, \dots, j-1.$$

Owing to this information the expression (4.2) with j replaced by $j-1$ turns out to be an explicit formula for the values v_i^j, $1 \le i \le N-1$. Moreover, from (4.4) we obtain v_0^j, v_N^j, too. Using the computed quantities v_0^j, $1 \le l \le j$, Q^l, $0 \le l \le j-1$ and q^l, $0 \le l \le j-1$, from (4.5) and (4.6) we get ψ^j and Q^j. Finally, using ψ^l, $0 \le l \le j$, and Q^l, $1 \le l \le j$, from (2.7) we determine q^j. Thus, the level j :

$$v_i^j,\ 0 \le i \le N,\ \psi^j,\ Q^j,\ q^j$$

is completed as well.

5. STABILITY ESTIMATE

In this section we shall deduce a stability estimate for solutions of schemes of the type (4.2)–(4.7). Let

$$_k\mu_i^j,\ _k\nu_1^j,\ _k\nu_2^j,\ _k\nu_3^j,\ _k\gamma_0,\ _k\gamma_1, \quad 0 \le i \le N, \quad 0 \le j \le M, \quad k = 1,2 \tag{5.1}$$

be certain prescribed quantities. Suppose that the functions

$$_kw_i^j,\ _k\chi^j,\ _kP^j,\ _kp^j, \quad 0 \le i \le N, \quad 0 \le j \le M, \quad k = 1,2$$

satisfy the following system with $k = 1,2$:

$$\Lambda({}_k w_i^j) - \tau \sum_{l=1}^{j} {}_k P^{j-l} \Lambda({}_k w_i^l) = a^2 \hat{\partial}_t \partial_t ({}_k w_i^j) + {}_k \mu_i^j ,$$

$$1 \le i \le N-1, \quad 1 \le j \le M-1, \tag{5.2}$$

$${}_k \chi^j = \frac{1}{a^2} \left[{}_k w_0^j - \Theta(j-1)\tau \sum_{l=1}^{j} {}_k P^{j-l} \cdot_k w_0^l \right] + {}_k \nu_1^j, \quad 0 \le j \le M, \tag{5.3}$$

$${}_k P^j = \Theta(j-1)\tau \sum_{l=0}^{j-1} {}_k p^l + {}_k \nu_2^j, \quad 0 \le j \le M, \tag{5.4}$$

$${}_k \gamma_0 \cdot_k p^j = {}_k \chi^j - {}_k \gamma_1 \cdot_k P^j$$

$$- \Theta(j-1)\tau \sum_{l=1}^{j} {}_k \chi^{j-l} \cdot_k P^l + {}_k \nu_3^j, \quad 0 \le j \le M. \tag{5.5}$$

Theorem 2. *Let us denote the differences of the data and the solutions of* (5.2)–(5.5) *as follows:*

$$\mu_i^j = {}_2\mu_i^j - {}_1\mu_i^j, \quad \nu_l^j = {}_2\nu_l^j - {}_1\nu_l^j, \quad \gamma_0 = {}_2\gamma_0 - {}_1\gamma_0, \quad \gamma_1 = {}_2\gamma_1 - {}_1\gamma_1,$$
$$w_i^j = {}_2 w_i^j - {}_1 w_i^j, \quad \chi^j = {}_2\chi^j - {}_1\chi^j, \quad P^j = {}_2P^j - {}_1P^j,$$
$$p^j = {}_2p^j - {}_1p^j, \quad 0 \le i \le N, \quad 0 \le j \le M, \quad l = 1,2,3. \tag{5.6}$$

If the steps τ and h satisfy the inequality

$$\tau \le \left(\frac{a}{\sqrt{2}} - \sigma \right) h, \quad \textit{where} \quad 0 < \sigma < \frac{a}{\sqrt{2}}, \tag{5.7}$$

and ${}_2\gamma_0 \ne 0$, then for the functions w_i^j, χ^j, P^j, p^j the following estimate holds:

$$\max_{0 \le j \le M} \| w^j \|_\infty + \max_{1 \le j \le M} \| \hat{\partial}_t w^j \|_2 + \max_{1 \le j \le M} \| \hat{\partial}_x w^j]|_2$$
$$+ \max_{0 \le j \le M} | \chi^j | + \max_{0 \le j \le M} | P^j | + \max_{0 \le j \le M} | p^j |$$
$$\le c_0(X, T, B_1, B_2, a, \sigma, {}_2\gamma_0, {}_1\gamma_1) E . \tag{5.8}$$

Here

$$B_1 = \max_{0 \le j \le M} | {}_1 w_0^j | + \tau \sum_{j=1}^{M-1} \| \Lambda({}_1 w^j) \|_2$$
$$+ \max_{0 \le j \le M} | {}_1\chi^j | + \max_{0 \le j \le M} | {}_1 p^j |, \tag{5.9}$$

$$B_2 = \max_{0 \le j \le M} | {}_2 P^j | + \tau \sum_{j=0}^{M-1} | \partial_t ({}_2 P^j) | ,$$

$$
\begin{aligned}
E = & \| w^0 \|_\infty + \| \hat{\partial}_x w^0]|_2 + |[\partial_t w^0]|_2 \\
& + \max_{1 \le j \le M} \{ | \partial_x w_0^j | + | \hat{\partial}_x w_N^j | \} \\
& + \tau \sum_{j=1}^{M-1} \{ | \partial_t \partial_x w_0^j | + | \partial_t \partial_x w_N^j | \} + \tau \sum_{j=1}^{M-1} \| \mu^j \|_2 \\
& + \sum_{l=1}^{3} \max_{0 \le j \le M} | \nu_l^j | + | \gamma_0 | + | \gamma_1 | ,
\end{aligned} \tag{5.10}
$$

and c_0 is a certain constant.

Proof. Subtracting the systems (5.2)–(5.5) with $k = 2$ and $k = 3$, respectively, we obtain

$$
\begin{aligned}
& \Lambda w_i^j - \tau \sum_{l=1}^{j} {}_2P^{j-l} \Lambda w_i^l \\
& \quad = a^2 \hat{\partial}_t \partial_t w_i^j + \mu_i^j + \tau \sum_{l=1}^{j} P^{j-l} \Lambda({}_1 w_i^l), \\
& \quad 1 \le i \le N-1, \quad 1 \le j \le M-1 ,
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
\chi^j = & \frac{1}{a^2} \left[w_0^j - \Theta(j-1) \tau \sum_{l=1}^{j} {}_2P^{j-l} w_0^l \right. \\
& \left. - \Theta(j-1) \tau \sum_{l=1}^{j} P^{j-l} \cdot {}_1 w_0^l \right] + \nu_1^j, \quad 0 \le j \le M ,
\end{aligned} \tag{5.12}
$$

$$
P^j = \Theta(j-1) \tau \sum_{l=0}^{j-1} p^l + \nu_2^j, \quad 0 \le j \le M , \tag{5.13}
$$

$$
\begin{aligned}
p^j = & \frac{1}{2\gamma_0} \left[\chi^j - \gamma_1 \cdot {}_2P^j - {}_1\gamma_1 \cdot P^j - \Theta(j-1) \tau \sum_{l=1}^{j} \chi^{j-l} \cdot {}_2P^l \right. \\
& \left. - \Theta(j-1) \tau \sum_{l=1}^{j} {}_1\chi^{j-l} \cdot P^l - \gamma_{0 \cdot 1} p^j + \nu_3^j \right], \quad 0 \le j \le M.
\end{aligned} \tag{5.14}
$$

The rest of the proof will consist of three parts: **(1)** making use of the method of discrete energy estimates (cf. [3]) for Eq. (5.11), we derive an estimate for w_i^j in terms P^j, E; **(2)** from (5.12), (5.14) we infer estimates for χ^j, p^j in terms w_0^j, P^j and from (5.13) an estimate for P^j in terms p^j, E; **(3)** combining the obtained results and applying Lemma 3, we derive an estimate for p^j in terms E, which in turn enables to prove the statement (5.8).

(**1**) Let us introduce the following discrete energy norms:

$$J^k = \max_{1\le j\le k} \| \hat{\partial}_t w^j \|_2 + \max_{1\le j\le k} \| \hat{\partial}_x w^j]|_2, \quad 1 \le k \le M. \tag{5.15}$$

We are going to derive an estimate for J^k. To this end we multiply Eq. (5.11) by the quantity $\partial_t \diamond w_i^j$, compute the scalar product (,) (cf. (3.5), (3.9)), and sum over j from 1 to $k-1$, where $2 \le k \le M$. Let us perform the mentioned operations separately for each addend in (5.11).

At first, on the ground of the formula (3.7) we have

$$\tau \sum_{j=1}^{k-1} (\Lambda w^j, \partial_t \diamond w^j)$$

$$= -\tau \sum_{j=1}^{k-1} (\hat{\partial}_x w^j, \partial_t \hat{\partial}_x \diamond w^j] + \tau \sum_{j=1}^{k-1} (\partial_t \diamond w_N^j \hat{\partial}_x w_N^j - \partial_t \diamond w_0^j \partial_x w_0^j),$$

$$2 \le k \le M. \tag{5.16}$$

Since

$$y_i^j \partial_t \diamond y_i^j = \frac{1}{2} \partial_t \left[(\diamond y_i^j)^2 - \frac{\tau^2}{4} (\hat{\partial}_t y_i^j)^2 \right],$$

with $y = \hat{\partial}_x w$ from (5.16) we obtain

$$\tau \sum_{j=1}^{k-1} (\Lambda w^j, \partial_t \diamond w^j)$$

$$= -\frac{1}{2} \tau \sum_{j=1}^{k-1} \partial_t \left(\| \hat{\partial}_x \diamond w^j]|_2^2 - \frac{\tau^2}{4} \| \hat{\partial}_t \hat{\partial}_x w^j]|_2^2 \right)$$

$$+ \tau \sum_{j=1}^{k-1} (\partial_t \diamond w_N^j \hat{\partial}_x w_N^j - \partial_t \diamond w_0^j \partial_x w_0^j),$$

$$2 \le k \le M.$$

Using here the formula (3.12), we have

$$\tau \sum_{j=1}^{k-1} (\Lambda w^j, \partial_t \diamond w^j) = -\frac{1}{2} \| \hat{\partial}_x \diamond w^k]|_2^2 + \frac{\tau^2}{8} \| \hat{\partial}_t \hat{\partial}_x w^k]|_2^2 + I_1^k,$$

$$2 \le k \le M, \tag{5.17}$$

where

$$I_1^k = \frac{1}{2}\left(\| \hat{\partial}_x \diamond w^1]|_2^2 - \frac{\tau^2}{4} \| \hat{\partial}_t \hat{\partial}_x w^1]|_2^2 \right) + \hat{\partial}_x w_N^k \diamond w_N^k$$

$$- \partial_x w_0^k \diamond w_0^k \ - \hat{\partial}_x w_N^1 \diamond w_N^1 + \partial_x w_0^1 \diamond w_0^1$$

$$- \tau \sum_{j=1}^{k-1} (\diamond w_N^{j+1} \partial_t \hat{\partial}_x w_N^j - \diamond w_0^{j+1} \partial_t \partial_x w_0^j). \tag{5.18}$$

On the ground of the assumption (5.7) we have

$$\frac{1}{h} \le \left(\frac{a}{\sqrt{2}} - \sigma \right) \frac{1}{\tau}.$$

Thus

$$| (\hat{\partial}_x \diamond w_i^1)^2 - \frac{\tau^2}{4} (\hat{\partial}_t \hat{\partial}_x w_i^1)^2 \ | = | \ \hat{\partial}_x w_i^0 \hat{\partial}_x w_i^1 \ |$$

$$\le | \ \hat{\partial}_x w_i^0 \ | \left(\left| \frac{w_i^1 - w_i^0}{h} \right| + \left| \frac{w_i^0 - w_{i-1}^0}{h} \right| + \left| \frac{w_{i-1}^1 - w_{i-1}^0}{h} \right| \right)$$

$$\le | \ \hat{\partial}_x w_i^0 \ | \left(\left(\frac{a}{\sqrt{2}} - \sigma \right) \left[\left| \frac{w_i^1 - w_i^0}{\tau} \right| \right. \right.$$

$$\left. \left. + \left| \frac{w_{i-1}^1 - w_{i-1}^0}{\tau} \right| \right] + \left| \frac{w_i^0 - w_{i-1}^0}{h} \right| \right). \tag{5.19}$$

Taking into account (3.14), (5.10), (5.15), (5.19), from (5.18) we infer the following estimate:

$$| \ I_1^k \ | \ \le \ c_1(a, \sigma) h \sum_{i=1}^{N} | \ \hat{\partial}_x w_i^0 \ | \ (| \ \partial_t w_i^0 \ | + | \ \partial_t w_{i-1}^0 \ |)$$

$$+ \max_{0 \le j \le k} \| \ w^j \ \|_\infty \left(| \ \partial_x w_0^1 \ | + | \ \hat{\partial}_x w_N^1 \ | + | \ \partial_x w_0^k \ | \right.$$

$$\left. + | \ \hat{\partial}_x w_N^k \ | + \tau \sum_{j=1}^{k-1} (| \ \partial_t \partial_x w_0^j \ | + | \ \partial_t \hat{\partial}_x w_N^j \ |) \right)$$

$$\le c_2(X, T) E J^k + c_3(X, T, a, \sigma) E^2, \quad 2 \le k \le M. \tag{5.20}$$

Let us perform the same operations with respect to the second addend in (5.11). Denoting

$$I_2^k = \tau \sum_{j=1}^{k-1} \tau \sum_{l=1}^{j} {}_2P^{j-l}(\Lambda w^l, \partial_t \diamond w^j), \quad 2 \le k \le M, \tag{5.21}$$

and using the formulas (3.7), (3.12), we obtain

$$I_2^k = \tau \sum_{j=1}^{k-1} \tau \sum_{l=1}^{j} {}_2P^{j-l} \partial_t \{ -(\hat{\partial}_x w^l, \hat{\partial}_x \diamond w^j] + \hat{\partial}_x w_N^l \diamond w_N^j - \partial_x w_0^l \diamond w_0^j \}$$

$$= \tau \sum_{j=1}^{k-1} \Big\{ {}_2P^{j-l} [-(\hat{\partial}_x w^l, \hat{\partial}_x \diamond w^k] + \hat{\partial}_x w_N^l \diamond w_N^k - \partial_x w_0^l \diamond w_0^k]$$

$$- {}_2P^0 [-(\hat{\partial}_x w^l, \hat{\partial}_x \diamond w^l] + \hat{\partial}_x w_N^l \diamond w_N^l - \partial_x w_0^l \diamond w_0^l]$$

$$- \sum_{j=l+1}^{k} \partial_t ({}_2P^{j-1-l}) [-(\hat{\partial}_x w^l, \hat{\partial}_x \diamond w^j] + \hat{\partial}_x w_N^l \diamond w_N^j - \partial_x w_0^l \diamond w_0^j] \Big\}$$

According to the inequality of Cauchy–Bunjakowski, the formulas (3.14), (5.9), (5.10), (5.15) and the relation $\diamond w^j = \frac{1}{2}(w^j + w^{j-1})$, we can estimate as follows:

$$| I_2^k | \leq \Big\{ 2 \max_{0 \leq j \leq k-1} | {}_2P^j | + \tau \sum_{j=0}^{k-2} | \partial_t ({}_2P^j) | \Big\}$$

$$\times \Big\{ \max_{1 \leq j \leq k} \| \hat{\partial}_x \diamond w^j]|_2 \; \tau \sum_{j=1}^{k-1} \| \hat{\partial}_x w^j]|_2 + \max_{1 \leq j \leq k} \| \diamond w^j \|_\infty$$

$$\times \tau \sum_{j=1}^{k-1} (| \hat{\partial}_x w_N^j | + | \partial_x w_0^j |) \Big\}$$

$$\leq c_4(X, T, B_2) \left(\tau \sum_{j=1}^{k-1} J^j + E \right) J^k + c_5(B_2) E^2,$$

$$2 \leq k \leq M \,. \tag{5.22}$$

For the addend $a^2 \hat{\partial}_t \partial_t w_i^j$ in (5.11) we obtain

$$a^2 \tau \sum_{j=1}^{k-1} (\hat{\partial}_t \partial_t w^j, \partial_t \diamond w^j)$$

$$= \frac{a^2}{2} \tau \sum_{j=1}^{k-1} \hat{\partial}_t (\| \partial_t w^j \|_2^2)$$

$$= \frac{a^2}{2} \| \hat{\partial}_t w^k \|_2^2 - \frac{a^2}{2} \| \partial_t w^0 \|_2^2, \quad 2 \leq k \leq M \,.$$

Thus

$$a^2 \tau \sum_{j=1}^{k-1} (\hat{\partial}_t \partial_t w^j, \partial_t \diamond w^j) = \frac{a^2}{2} \| \hat{\partial}_t w^k \|_2^2 + I_3^k, \quad 2 \leq k \leq M, \tag{5.23}$$

where

$$| I_3^k | \leq \frac{a^2}{2} E^2 . \tag{5.24}$$

Finally, for the quantity

$$I_4^k = \tau \sum_{j=1}^{k-1} \left(\mu^j + \tau \sum_{l=1}^{j} P^{j-l} \Lambda({}_1w^l), \, \partial_t \diamond w^j \right), \quad 2 \leq k \leq M, \tag{5.25}$$

we derive the estimate

$$\begin{aligned} | I_4^k | &\leq \tau \sum_{j=1}^{k-1} \left(\| \mu^j \|_2 + \tau \sum_{l=1}^{j} | P^{j-l} | \, \| \Lambda({}_1w^l) \|_2 \right) \\ &\quad \times \frac{1}{2} \| \hat{\partial}_t (w^j + w^{j-1}) \|_2 \\ &\leq c_6(T, B_1) \left(E + \max_{0 \leq j \leq k} | P^j | \right) J^k . \end{aligned} \tag{5.26}$$

Summing up, from Eq. (5.11) in view of (5.17), (5.21), (5.23), and (5.25) we obtain

$$\begin{aligned} &\frac{1}{2} \| \hat{\partial}_x \diamond w^k]|_2^2 - \frac{\tau^2}{8} \| \hat{\partial}_x \hat{\partial}_t w^k]|_2^2 + \frac{a^2}{2} \| \hat{\partial}_t w^k \|_2^2 \\ &\qquad = -I_1^k + I_2^k - I_3^k + I_4^k, \quad 2 \leq k \leq M , \end{aligned}$$

and due to (5.20), (5.22), (5.24), (5.26) we have

$$\begin{aligned} &\frac{1}{2} \| \hat{\partial}_x \diamond w^k]|_2^2 - \frac{\tau^2}{8} \| \hat{\partial}_t \hat{\partial}_x w^k]|_2^2 + \frac{a^2}{2} \| \hat{\partial}_t w^k \|_2^2 \\ &\qquad \leq c_7(X,T,B_1,B_2) \left(\tau \sum_{j=1}^{k-1} J^j + \max_{0 \leq j \leq k} | P^j | + E \right) J^k \\ &\qquad\quad + c_8(X,T,B_2,a,\sigma) E^2, \quad 2 \leq k \leq M . \end{aligned} \tag{5.27}$$

Making use of the inequality

$$\begin{aligned} &\tau^2 \| \hat{\partial}_x \hat{\partial}_t w^k]|_2^2 \\ &\qquad = \tau^2 h \sum_{i=2}^{N-1} (\hat{\partial}_x \hat{\partial}_t w_i^k)^2 + \tau^2 h (\hat{\partial}_t \hat{\partial}_x w_N^k)^2 + \tau^2 h (\hat{\partial}_t \partial_x w_0^k)^2 \\ &\qquad \leq \frac{\tau^2}{h^2} h \sum_{i=2}^{N-1} 2[(\hat{\partial}_t w_i^k)^2 + (\hat{\partial}_t w_{i-1}^k)^2] \\ &\qquad\quad + 2h[(\hat{\partial}_x w_N^k)^2 + (\hat{\partial}_x w_N^{k-1})^2 + (\partial_x w_0^k)^2 + (\partial_x w_0^{k-1})^2] \\ &\qquad \leq 4 \frac{\tau^2}{h^2} \| \partial_t w^k \|_2^2 + 4hE^2 \end{aligned}$$

and the formulae (3.13), (5.7), we can estimate the left-hand side of (5.27) from below:

$$\frac{1}{2}\,\|\hat\partial_x \diamond w^k\,]|_2^2 \;-\; \frac{\tau^2}{8}\,\|\,\hat\partial_x\hat\partial_t w^k\,]|_2^2 \;+\frac{a^2}{2}\,\|\,\hat\partial_t w^k\,\|_2^2$$

$$\geq \frac{1}{4}\,\|\,\hat\partial_x w^k\,]|_2^2 \;-\frac{\tau^2}{4}\,\|\,\hat\partial_x\hat\partial_t w^k\,]|_2^2 \;+\frac{a^2}{2}\,\|\,\hat\partial_t w^k\,\|_2^2$$

$$\geq \frac{1}{4}\,\|\,\hat\partial_x w^k\,]|_2^2 \;+\sigma(\sqrt{2}a-\sigma)\,\|\,\hat\partial_t w^k\,\|_2^2 \;-hE^2\,. \qquad (5.28)$$

Thus (5.27), (5.28) yield

$$\|\,\hat\partial_x w^k\,]|_2^2 + \|\,\hat\partial_t w^k\,\|_2^2$$

$$\leq c_9(X,T,B_1,B_2,a,\sigma)\left(\tau\sum_{j=1}^{k-1} J^j + \max_{0\leq j\leq k}|\,P^j\,|+E\right)J^k$$

$$+\;c_{10}(X,T,B_2,a,\sigma)E^2, \qquad (5.29)$$

where $2 \leq k \leq M$. Let us derive an analogue of (5.29) for $k = 1$, too. Since

$$(d_1+d_2+d_3)^2 \;\leq\; 3(d_1^2+d_2^2+d_3^2), \quad \forall d_1, d_2, d_3 \in \mathbb{R},$$

due to (5.7) we obtain

$$\|\,\hat\partial_x w^1\,]|_2^2 \;+\; \|\,\hat\partial_t w^1\,\|_2^2$$

$$= h\sum_{i=1}^{N}\left(\frac{w_i^1-w_{i-1}^1}{h}\right)^2 \;+\; \|\,\partial_t w^0\,\|_2^2$$

$$\leq 3h\sum_{i=1}^{N}\left[\left(\frac{w_i^1-w_i^0}{h}\right)^2+\left(\frac{w_i^0-w_{i-1}^0}{h}\right)^2+\left(\frac{w_{i-1}^1-w_{i-1}^0}{h}\right)^2\right]$$

$$+\,\|\,\partial_t w^0\,\|_2^2$$

$$\leq 3h\sum_{i=1}^{N}\left\{\left(\frac{a}{\sqrt{2}}-\sigma\right)^2\left[\left(\frac{w_i^1-w_i^0}{\tau}\right)^2+\left(\frac{w_{i-1}^1-w_{i-1}^0}{\tau}\right)^2\right]\right.$$

$$\left.+\left(\frac{w_i^0-w_{i-1}^0}{h}\right)^2\right\}+\|\,\partial_t w^0\,\|_2^2 \leq c_{11}(a,\sigma)E^2\,. \qquad (5.30)$$

The estimates (5.29), (5.30) imply

$$\|\,\hat\partial_x w^k\,]|_2^2 + \|\,\hat\partial_t w^k\,\|_2^2 \leq c_{12}(X,T,B_1,B_2,a,\sigma)$$

$$\times\left(\Theta(k-2)\tau\sum_{j=1}^{k-1} J^j + \max_{0\leq j\leq k}|\,P^j\,|+E\right)J^k$$

$$+\;c_{13}(X,T,B_2,a,\sigma)E^2, \quad 1\leq k\leq M.$$

According to Lemma 4 and the formula (5.15) we get

$$(J^k)^2 \le c_{14}(X,T,B_1,B_2,a,\sigma)\Bigg(\Theta(k-2)\tau\sum_{j=1}^{k-1} J^j + \max_{0\le j\le k} \mid P^j \mid + E\Bigg) J^k + c_{15}(X,T,B_2,a,\sigma)E^2, \quad 1 \le k \le M. \tag{5.31}$$

Solving the quadratic inequality (5.31) with respect to J^k, we derive the formula:

$$J^k \le c_{16}(X,T,B_1,B_2,a,\sigma)\Bigg(\Theta(k-2)\tau\sum_{j=1}^{k-1} J^j + \max_{0\le j\le k} \mid P^j \mid + E\Bigg), \quad 1 \le k \le M.$$

Making use of Lemma 3, we obtain the inequality

$$J^k \le c_{17}(X,T,B_1,B_2,a,\sigma)\big(\max_{0\le j\le k} \mid P^j \mid + E\big), \quad 1 \le k \le M, \tag{5.32}$$

which in view of (5.15) represents an estimate of w_i^j in terms P^j and E. Moreover, Lemma 1 together with (5.32) implies

$$\max_{0\le j\le k} \parallel w^j \parallel_\infty \le c_{18}(X,T,B_1,B_2,a,\sigma)\big(\max_{0\le j\le k} \mid P^j \mid + E\big), \quad 0 \le k \le M. \tag{5.33}$$

(2) Let us go on by estimating Eqs. (5.12)–(5.14). We immediately obtain

$$\mid \chi^j \mid \le \frac{1}{a^2}\Bigg[\mid w_0^j \mid + \Theta(j-1)T \max_{0\le l\le j-1} \mid {}_2P^l \mid \max_{1\le l\le j} \mid w_0^l \mid + \Theta(j-1)T \max_{0\le l\le j-1} \mid P^l \mid \max_{1\le l\le j} \mid {}_1w_0^l \mid\Bigg] + \mid \nu_1^j \mid, \quad 0 \le j \le M, \tag{5.34}$$

$$\mid P^j \mid \le \Theta(j-1)\tau\sum_{l=0}^{j-1} \mid p^l \mid + \mid \nu_2^j \mid, \quad 0 \le j \le M, \tag{5.35}$$

$$
\begin{aligned}
| p^j | \le \frac{1}{| {}_2\gamma_0 |} \Big[& | \chi^j | + | \gamma_1 | \cdot | {}_2P^j | + | {}_1\gamma_1 | \cdot | P^j | \\
& + \Theta(j-1)T \max_{0 \le l \le j-1} | \chi^l | \max_{1 \le l \le j} | {}_2P^l | \\
& + \Theta(j-1)T \max_{0 \le l \le j-1} | {}_1\chi^l | \max_{1 \le l \le j} | P^l | \\
& + | {}_1p^j | \cdot | \gamma_0 | + | \nu_3^j | \Big], \qquad 0 \le j \le M . \qquad (5.36)
\end{aligned}
$$

In view of (5.9), (5.10), from (5.34)–(5.36) we derive

$$
\max_{0 \le j \le k} | \chi^j | \le c_{19}(T, B_1, B_2, a) \big(\max_{0 \le j \le k} | w_0^j | + E \big), \qquad 0 \le k \le M, \qquad (5.37)
$$

$$
\max_{0 \le j \le k} | P^j | \le \Theta(k-1)\tau \sum_{j=0}^{k-1} | p^j | + E, \quad 0 \le k \le M, \qquad (5.38)
$$

Making use of Lemma 3, we obtain the inequality

$$
\begin{aligned}
| p^k | \le c_{20}(T, B_1, B_2, {}_2\gamma_0, {}_1\gamma_1) \big[& \max_{0 \le j \le k} | \chi^j | \\
& + \max_{0 \le j \le k} | P^j | + E \big], \quad 0 \le k \le M . \qquad (5.39)
\end{aligned}
$$

The latter inequality due to (5.37) yields

$$
\begin{aligned}
| p^k | \le c_{21}(T, B_1, B_2, {}_2\gamma_0, {}_1\gamma_1) \big[& \max_{0 \le j \le k} | w_0^j | \\
& + \max_{0 \le j \le k} | P^j | + E \big], \quad 0 \le k \le M . \qquad (5.40)
\end{aligned}
$$

(3) Finishing our proof we use the estimate (5.33) for w_0^j and (5.38) for P^j in (5.40). We get

$$
| p^k | \le c_{22}(T, B_1, B_2, a, \sigma, {}_2\gamma_0, {}_1\gamma_1) \Big(\Theta(k-1)\tau \sum_{j=0}^{k-1} | p^j | + E \Big), \qquad 0 \le k \le M .
$$

Thus, due to Lemma 3

$$
\max_{0 \le j \le M} | p^j | \le c_{23}(T, B_1, B_2, a, \sigma, {}_2\gamma_0, {}_1\gamma_1) E. \qquad (5.41)
$$

Now the statement of Theorem 2 follows from the estimates (5.41), (5.38), (5.37) and (5.33), (5.32). $\square$

On the ground of the formulae (3.3) and (3.8) we can write:

$$\begin{aligned} f_i^j &= f(x_i,t_j), \quad \alpha_i = \alpha(x_i), \quad \beta_i = \beta(x_i), \quad g^j = g(t_j), \\ u_i^j &= u(x_i,t_j), \quad \phi^j = \phi(t_j), \quad R^j = R(t_j), \quad r^j = r(t_j). \end{aligned} \tag{6.1}$$

Due to (2.2)–(2.7) and (4.1), the functions u_i^j, ϕ^j, R^j, r^j satisfy the following system:

$$\Lambda u_i^j - \tau \sum_{l=1}^{j} R^{j-l} \Lambda u_i^l = a^2 \hat{\partial}_t \partial_t u_i^j + f_i^j + \epsilon_i^j, \quad 1 \le i \le N-1, \quad 1 \le j \le M-1, \tag{6.2}$$

$$u_i^0 = \alpha_i, \quad \partial_t u_i^0 = \beta_i + \zeta_i, \quad 0 \le i \le N, \tag{6.3}$$

$$\partial_x u_0^j = \eta_0^j, \quad \hat{\partial}_x u_N^j = \eta_N^j, \quad 1 \le j \le M, \tag{6.4}$$

$$\phi^j = \frac{1}{a^2}\left[u_0^j - \Theta(j-1)\tau \sum_{l=1}^{j} R^{j-l} u_0^l\right] + \vartheta_1^j, \quad 0 \le j \le M, \tag{6.5}$$

$$R^j = \Theta(j-1)\tau \sum_{l=0}^{j-1} r^l + \rho + \vartheta_2^j, \quad 0 \le j \le M, \tag{6.6}$$

$$r^j = \frac{1}{\kappa_0}\left[-g^j + \phi^j - \kappa_1 R^j - \Theta(j-1)\tau \sum_{l=1}^{j} \phi^{j-l} R^l\right] + \frac{\vartheta_3^j}{\kappa_0}, \quad 0 \le j \le M, \tag{6.7}$$

where

$$\epsilon_i^j = \Lambda u_i^j - u_{xx}(x_i,t_j) - \tau \sum_{l=1}^{j} R^{j-l} \Lambda u_i^l + \int_0^{t_j} R(t_j - s) u_{xx}(x_i,s)ds + a^2 u_{tt}(x_i,t_j) - a^2 \hat{\partial}_t \partial_t u_i^j, \tag{6.8}$$

$$\zeta_i = \partial_t u_i^0 - u_t(x_i,0), \tag{6.9}$$

$$\eta_0^j = \partial_x u_0^j - u_x(0,t_j), \quad \eta_N^j = \hat{\partial}_x u_N^j - u_x(X,t_j), \tag{6.10}$$

$$\vartheta_1^j = -\frac{1}{a^2}\left(\int_0^{t_j} R(t_j - s)u(0,s)ds - \Theta(j-1)\tau \sum_{l=1}^{j} R^{j-l} u_0^l\right), \tag{6.11}$$

$$\vartheta_2^j = \int_0^{t_j} r(s)ds - \Theta(j-1)\tau \sum_{l=0}^{j-1} r^l\,, \tag{6.12}$$

$$\vartheta_3^j = \Theta(j-1)\tau \sum_{l=1}^{j} \phi^{j-l} R^l - \int_0^{t_j} \phi(t_j - s)R(s)ds\,. \tag{6.13}$$

Lemma 5. *Let* $u \in C^3(D)$, $\phi, R \in C^1[0,T]$. *Then*

$$|\,\epsilon_i^j\,| \le c_{24}(T, B_u, B_R)(h+\tau), \quad |\,\zeta_i\,| \le c_{25}(B_u)\tau, \tag{6.14}$$

$$|\,\eta_0^j\,| \le c_{26}(B_u)h, \quad |\,\eta_N^j\,| \le c_{27}(B_u)h, \tag{6.15}$$

$$|\,\partial_t \eta_0^j\,| \le c_{28}(B_u)(h+\tau), \quad |\,\partial_t \eta_N^j\,| \le c_{29}(B_u)(h+\tau), \tag{6.16}$$

$$\begin{aligned} &|\,\vartheta_1^j\,| \le c_{30}(T, B_u, B_R, a)\tau, \quad |\,\vartheta_2^j\,| \le c_{31}(T, B_R)\tau, \\ &|\,\vartheta_3^j\,| \le c_{32}(T, B_\phi, B_R)\tau, \end{aligned} \tag{6.17}$$

where

$$B_u = \|\, u \,\|_{C^3(D)}, \quad B_R = \|\, R \,\|_{C^1[0,T]}, \quad B_\phi = \|\, \phi \,\|_{C^1[0,T]}\,. \tag{6.18}$$

Proof. It is well known that

$$|\,\partial_x y_i - y_x(x_i)\,| \le \frac{1}{2} \,\|\, y_{xx} \,\|_{C[0,X]}\, h \quad \text{if} \quad y \in C^2[0,X]\,, \tag{6.19}$$

$$|\,\hat{\partial}_x y_i - y_x(x_i)\,| \le \frac{1}{2} \,\|\, y_{xx} \,\|_{C[0,X]}\, h \quad \text{if} \quad y \in C^2[0,X]\,, \tag{6.20}$$

$$|\,\Lambda y_i - y_{xx}(x_i)\,| \le \frac{1}{3} \,\|\, y_{xxx} \,\|_{C[0,X]}\, h \quad \text{if} \quad y \in C^3[0,X]\,, \tag{6.21}$$

$$|\,\partial_t y^j - y_t(t_j)\,| \le \frac{1}{2} \,\|\, y_{tt} \,\|_{C[0,T]}\, \tau \quad \text{if} \quad y \in C^2[0,T]\,, \tag{6.22}$$

$$|\,\hat{\partial}_t \partial_t y^j - y_{tt}(t_j)\,| \le \frac{1}{3} \,\|\, y_{ttt} \,\|_{C[0,T]}\, \tau \quad \text{if} \quad y \in C^3[0,T], \tag{6.23}$$

and

$$\begin{aligned} &\left| \int_0^{t_j} y(s)ds - \Theta(j-1)\tau \sum_{l=1}^{j} y^l \right| \\ &\qquad\qquad \le \frac{T}{2} \,\|\, y_t \,\|_{C[0,T]}\, \tau \quad \text{if} \quad y \in C^1[0,T]\,. \end{aligned} \tag{6.24}$$

The assertions (6.14), (6.15), and (6.17) simply follow from (6.19)–(6.24) and the relation $r = R'$. Let us prove (6.16). In view of (6.10) we have

$$\partial_t \eta_0^j = \partial_t \partial_x u_0^j - \partial_t[u_x(0,t_j)] = \frac{1}{h\tau}\int_0^h \int_{t_j}^{t_{j+1}} [u_{xt}(s,\hat{s}) - u_{xt}(0,t_j)]d\hat{s}ds + u_{xt}(0,t_j) - \partial_t[u_x(0,t_j)].$$

Since

$$\begin{aligned} u_{xt}(s,\hat{s}) - u_{xt}(0,t_j) &= \int_0^1 \frac{d}{d\xi} u_{xt}(s\xi, t_j + (\hat{s}-t_j)\xi)d\xi \\ &= \int_0^1 [s u_{xtx}(s\xi, t_j + (\hat{s}-t_j)\xi) \\ &\quad + (\hat{s}-t_j)u_{xtt}(s\xi, t_j + (\hat{s}-t_j)\xi)]d\xi \end{aligned}$$

and $0 \le s \le h, \ \ 0 \le \hat{s} - t_j \le \tau$, we obtain the inequality

$$|\,\partial_t \eta_0^j\,| \le \|\, u_{xxt}\,\|_{C(D)}\, h + \|\, u_{xtt}\,\|_{C(D)}\, \tau + \frac{1}{2}\,\|\, u_{xtt}\,\|_{C(D)}\, \tau\,,$$

which due to (6.18) yields the first estimate in (6.16). The second estimate in (6.16) can be proved in a similar manner. The proof is complete. □

Next we are going to compare the solutions of the systems (6.2)–(6.7) and (4.2)–(4.7) by means of Theorem 2 and Lemma 5. The relations (6.2), (6.5)–(6.7) and (4.2), (4.5)–(4.7) take the form (5.2)–(5.5) if we denote

$$\begin{aligned} &{}_1w_i^j = u_i^j, \quad {}_2w_i^j = v_i^j, \quad {}_1\chi^j = \phi^j, \quad {}_2\chi^j = \psi^j, \\ &{}_1P^j = R^j, \quad {}_2P^j = Q^j, \quad {}_1p^j = r^j, \quad {}_2p^j = q^j, \\ &{}_1\mu_i^j = f_i^j + \epsilon_i^j, \quad {}_2\mu_i^j = \tilde{f}_i^j, \quad {}_1\nu_1^j = \vartheta_1^j, \quad {}_2\nu_1^j = 0, \\ &{}_1\nu_2^j = \rho + \vartheta_2^j, \quad {}_2\nu_2^j = \tilde{\rho}, \quad {}_1\nu_3^j = -g^j + \vartheta_3^j, \quad {}_2\nu_3^j = -\tilde{g}^j, \\ &{}_1\gamma_0 = \kappa_0, \quad {}_2\gamma_0 = \tilde{\kappa}_0, \quad {}_1\gamma_1 = \kappa_1, \quad {}_2\gamma_1 = \tilde{\kappa}_1\,. \end{aligned} \tag{6.25}$$

In accordance with the definitions (5.6), (6.25), the initial and boundary conditions (4.3), (4.4), (6.3), (6.4), we have

$$\begin{aligned} &w_i^0 = \alpha_i - \tilde{\alpha}_i, \quad \partial_t w_i^0 = \beta_i - \tilde{\beta}_i + \zeta_i, \quad \partial_x w_0^j = \eta_0^j, \\ &\partial_x w_N^j = \eta_N^j, \quad \mu_i^j = f_i^j - \tilde{f}_i^j + \epsilon_i^j, \quad \nu_1^j = \vartheta_1^j, \\ &\nu_2^j = \rho - \tilde{\rho} + \vartheta_2^j, \quad \nu_3^j = \tilde{g}^j - g^j + \vartheta_3^j, \quad \gamma_0 = \kappa_0 - \tilde{\kappa}_0, \\ &\gamma_1 = \kappa_1 - \tilde{\kappa}_1\,. \end{aligned} \tag{6.26}$$

Now due to (6.25), (6.26) we see that Theorem 2 and Lemma 5 imply the following theorem:

Theorem 3. *Let* $u \in C^3(D)$, $\phi, R \in C^1[0,T]$, $\tilde{\kappa}_0 \neq 0$ *and let the inequality* (5.7) *be valid for* τ *and* h. *Then the difference of solutions of*

(6.2)–(6.7) *and* (4.2)–(4.7) *can be estimated as follows*:

$$\max_{0\le j\le M} \| u^j - v^j \|_\infty + \max_{1\le j\le M} \| \hat\partial_t(u^j - v^j) \|_2$$

$$+ \max_{1\le j\le M} \| \hat\partial_x(u^j - v^j)]|_2 + \max_{0\le j\le M} | \phi^j - \psi^j |$$

$$+ \max_{0\le j\le M} | R^j - Q^j | + \max_{0\le j\le M} | r^j - q^j |$$

$$\le \hat c_0(X, T, B_u, B_\phi, B_R, B_Q, a, \sigma, \tilde\kappa_0, \kappa_1)\Big\{ \| \alpha - \tilde\alpha \|_\infty$$

$$+ \| \hat\partial_x(\alpha - \tilde\alpha)]|_2 + |[\beta - \tilde\beta]|_2 + \tau \sum_{j=1}^{M-1} \| f^j - \tilde f^j \|_2$$

$$+ | \rho - \tilde\rho | + \max_{0\le j\le M} | g^j - \tilde g^j | + | \kappa_0 - \tilde\kappa_0 |$$

$$+ | \kappa_1 - \tilde\kappa_1 | + h + \tau \Big\}, \qquad (6.27)$$

where

$$B_Q = \max_{0\le j\le M} | Q^j | + \tau \sum_{j=0}^{M-1} | \partial_t Q^j |$$

$$= \max_{0\le j\le M} | Q^j | + \tau \sum_{j=0}^{M-1} | q^j |, \qquad (6.28)$$

the quantities B_u, B_ϕ, B_R *are defined by* (6.18) *and* $\hat c_0$ *is a certain constant.*

Theorem 3 says that the convergence of the solution (v, ψ, Q, q) of the difference scheme (4.2)–(4.7) to the exact solution of the problem (2.2)–(2.7) in nodes (x_i, t_j) takes place under the following conditions:

(1) $(\tilde\alpha, \tilde\beta, \tilde f, \tilde\rho, \tilde g, \tilde\kappa_0, \tilde\kappa_1)$ converges to $(\alpha, \beta, f, \rho, g, \kappa_0, \kappa_1)$ in norms indicated on the right-hand side of (6.27),

(2) h and τ tend to zero in the coordinated manner (5.7),

(3) the quantity B_Q that depends on the approximate solution remains bounded in the process of approximation.

The condition (3) is necessary because B_Q is included in the coefficient $\hat c_0$ in (6.27). To get rid of the restriction (3), we must estimate B_Q in terms $\tilde\alpha, \tilde\beta, \tilde f, \tilde\rho, \tilde g, \tilde\kappa_0\, \tilde\kappa_1$. This is a quite complicated task since the problem (4.2)–(4.7) is nonlinear. However, the method of weighted norms of Bielecki type (see [4]) may help here, because the nonlinearities in (4.2)–(4.7) have the form of discrete convolutions.

Finally we describe the entire procedure of solving the inverse problem (1.1)–(1.4). Suppose that instead of the exact data A, B, F, G we know certain approximations $\tilde A$, $\tilde B$, $\tilde F$, $\tilde G$. Let the error of the data be δ, i.e.

$$\| \tilde A - A \|_A \le \delta, \quad \| \tilde B - B \|_B \le \delta, \quad \| \tilde F - F \|_F \le \delta, \quad \| \tilde G - G \|_G \le \delta \qquad (6.29)$$

in some norms $\|\cdot\|_A, \|\cdot\|_B, \|\cdot\|_F, \|\cdot\|_G$. The first stage of solving the inverse problem is linear but ill-posed. We have to compute the approximations $\tilde{\alpha}, \tilde{\beta}, \tilde{f}, \tilde{\rho}, \tilde{g}, \tilde{\kappa}_0\ \tilde{\kappa}_1$ for the quantities $\alpha, \beta, f, \rho, g, \kappa_0, \kappa_1$ on the basis of the formulae (2.8). To this end we apply some regularized methods for evaluating the derivatives of the functions $\tilde{A}, \tilde{B}, \tilde{F}, \tilde{G}$. Suppose that the errors of the obtained approximations satisfy the following estimates:

$$\| \tilde{\alpha} - \alpha \|_\infty \le m_1(\delta), \quad |[\tilde{\beta} - \beta]|_2 \le m_1(\delta),$$

$$\tau \sum_{j=1}^{M-1} \| \tilde{f}^j - f^j \|_2 \le m_1(\delta), \quad \max_{0\le j\le M} | \tilde{g}^j - g^j | \le m_1(\delta), \tag{6.30}$$

$$| \tilde{\rho} - \rho | \le m_1(\delta), \quad | \tilde{\kappa}_0 - \kappa_0 | \le m_1(\delta), \quad | \tilde{\kappa}_1 - \kappa_1 | \le m_1(\delta),$$

$$\| (\tilde{\alpha} - \alpha)' \|_{L^2(0,X)} \le m_2(\delta), \quad \| (\tilde{\alpha} - \alpha)'' \|_{C[0,X]} \le m_3, \tag{6.31}$$

where m_1, m_2 are some continuous functions, $m_1(0) = m_2(0) = 0$, and m_3 is independent of δ. It follows from Lemma 2 and (6.31) that

$$\| \hat{\partial}_x(\tilde{\alpha} - \alpha)]|_2 \le \text{const} \left(\sqrt{m_2(\delta)} + h \right) \tag{6.32}$$

if $m_2(\delta)$ is small.

The second stage of solving (1.1)–(1.4) is nonlinear but well-posed. From the scheme (4.2)–(4.7) we evaluate the mesh functions $v_i^j \approx u(x_i, t_j)$, $\psi^j \approx \phi(t_j)$, $Q^j \approx R(t_j)$, $q^j \approx r^j = R'(t_j)$, $0 \le i \le N$, $0 \le j \le M$. According to Theorem 3 and (6.30), (6.32), we have

$$\begin{aligned}
\max_{0\le j\le M} \| u^j - v^j \|_\infty &+ \max_{1\le j\le M} \| \hat{\partial}_t(u^j - v^j) \|_2 \\
&+ \max_{1\le j\le M} \| \hat{\partial}_x(u^j - v^j)]|_2 + \max_{0\le j\le M} | \phi^j - \psi^j | \\
&+ \max_{0\le j\le M} | R^j - Q^j | + \max_{0\le j\le M} | r^j - q^j | \\
&\le \text{const} \left(m_1(\delta) + \sqrt{m_2(\delta)} + h + \tau \right).
\end{aligned} \tag{6.33}$$

We point out that the estimate (6.33) holds provided the approximation of the function α is good enough, i.e. (6.31) is satisfied. Otherwise the stage of solving the system (4.2)–(4.7) is also ill-posed. For instance, if (6.30) holds but (6.31) not, then

$$\| \hat{\partial}_x(\tilde{\alpha} - \alpha)]|_2 \le 4\sqrt{X} \| \tilde{\alpha} - \alpha \|_\infty \frac{1}{h} \le 4\sqrt{X} \frac{m_1(\delta)}{h}$$

and we must set the step h depending on δ.

The second stage provides the values for the relaxation kernel R and its derivative. To determine the second component U of the solution (R, U), we must additionally solve the problem (2.10).

Remark. We can apply the method of finite differences to multidimensional analogues of the problem (1.1)–(1.4) as well. Only in this case the error analysis needs higher energy estimates, i.e. estimates with differences of higher order than in the expression of J^k (see (5.15)). This is due to the breakdown of Lemma 1 for the first differences in the multidimensional case.

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ÜHEMÕÕTMELISE VISKOELASTSE KESKKONNA LIIKUMISVÕRRANDIGA SEOTUD PÖÖRDÜLESANDE LAHENDAMISEST DIFERENTSMEETODIL

Jaan JANNO

Ühemõõtmelise lineaarse viskoelastse keskkonna liikumisvõrrandiga seotud pöördülesanne on taandatud hüperboolset ja Volterra teist liiki võrrandeid sisaldavale süsteemile. Saadud süsteemi on diskretiseeritud diferentsmeetodiga. On tõestatud meetodi koonduvus.