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ON RIGHT INVERTIBILITY OF RECURSIVE NONLINEAR SYSTEMS[a]

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Abstract. The right invertibility problem is studied for a class of recursive nonlinear systems (RNS), i.e. for systems, modelled by recursive nonlinear input–output equations involving only a finite number of input values and a finite number of output values. The concept of delay orders and the special case of right invertibility, the notion of $(d_1, \ldots, d_p)$-forward time shift (FTS) right invertibility, known for discrete-time nonlinear systems in state space form, are extended to RNS. Necessary and sufficient conditions for local $(d_1, \ldots, d_p)$-FTS right invertibility are proposed. Finally, it is shown how to generalize the special case of right invertibility – $(d_1, \ldots, d_p)$-FTS right invertibility – using the ideas from the state space theory.

Key words: recursive nonlinear systems, delay orders, right invertibility.

1. INTRODUCTION

The fundamental question of the existence of right inverses for nonlinear discrete-time dynamical systems is discussed in this paper. Such a right inverse is intuitively understood as a second nonlinear system such that when the original system is applied in series with this right inverse, then its outputs are equal to the inputs of the right inverse system. Because of the inherent delay typically found in dynamical systems, in a great number of cases such inversion is not possible, and the problem is of limited interest. Greater generality is obtained by considering a notion of forward time shift (FTS) right inverse, in which the input to the right inverse system is not just the reference signal $y_{ref}(t)$ but the reference signal at some future time

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instant $t + \alpha$, i.e. $y_{ref}(t + \alpha)$. The determination of the smallest possible value for α is a question of practical and theoretical importance. In case of multi-output systems, the value of the smallest possible α, in general, is not the same for all components of the reference signal. A very important concept in treating system inversion from this generalized point of view is that of delay orders.

Except the paper [1] on finding the inverse Volterra representation, previous work on this subject concentrates mostly on systems having a state space representation [2–13]. The purpose of the present paper is to study the FTS right invertibility problem for a class of systems described by recursive nonlinear input–output equations involving only a finite number of input values and a finite number of output values. Such systems are called recursive nonlinear systems (RNS) [14–16]. We shall extend to this class of nonlinear systems the concept of delay orders and the notion of $(d_1, \ldots, d_p)$ -FTS right invertibility. The latter is a special case of FTS right invertibility in case of which the smallest possible values for $\alpha_1, \ldots, \alpha_p$ may be realized.

Note that in [17, 18] for single-input single-output RNS the question of dead-beat control has been studied. The latter problem is closely related to the right invertibility problem. Unlike [17, 18], we consider the multi-input multi-output systems and concentrate on the local solutions around an equilibrium point of the system. However, the paper does not give any algorithm which explicitly constructs the right inverse system; it only presents necessary and sufficient conditions under which the right inverse system locally exists and describes how to obtain it. The solution given in the paper relies on the application of the implicit function theorem.

2. THE DESCRIPTION OF THE RNS

In this section, besides recalling the notion of the RNS [14, 17], we establish some notations and introduce some preliminary material.

We denote by $S(R^m)$ the set of all two-sided infinite sequences of the form

$$\{z(t)\} = (\ldots, z(-1), z(0), z(1), z(2), \ldots),$$

where $z(t) \in R^m$ for all integers t.

Dynamical system. A dynamical system is a map

$$\Sigma : S(R^m) \mapsto S(R^p) : \{u(t)\} \mapsto \{y(t)\}$$

which transforms the input sequence $\{u(t)\}$ into the output sequence $\{y(t)\}$.

Given two systems $\Sigma_1 : S(R^m) \mapsto S(R^p)$ and $\Sigma_2 : S(R^p) \mapsto S(R^q)$, we denote by $\Sigma_2 \circ \Sigma_1 : S(R^m) \mapsto R(^q)$ the system represented by the composite map.

A finite subsequence of the infinite sequence $\{z(t)\}$ between time instants t and $t-\tau$ stacked in the column vector is denoted by

$$Z(t,t-\tau) = (z^T(t), z^T(t-1), \ldots, z^T(t-\tau))^T, \quad \tau \geq 0.$$

If $\tau \leq 0$, it is understood that $Z(t,t-\tau)$ denotes an empty subsequence of $\{z(t)\}$.

If for every input sequence $\{u(t)\}$ the corresponding output sequence $\{y(t)\}$ of the system Σ satisfies the equation

$$y(t) = F(Y(t-1,t-\mu), U(t-1,t-\nu)), \tag{1}$$

where $F : R^{\mu p+\nu m} \mapsto R^p$ is a C^ω map and $1 \leq \mu < \infty$, $1 \leq \nu < \infty$, then the system Σ is said to have a causal finite dimensional realization.

Definition 2.1. Recursive nonlinear system. *A recursive nonlinear system (RNS) is a system which has a causal finite dimensional realization of the form* (1).

Definition 2.2. Equilibrium point. *The pair of constant values* (u^0, y^0) *is called the equilibrium point of the RNS* (1) *if* (u^0, y^0) *satisfies the equality* $y^0 = F(Y^0, U^0)$, *where* $Y^0 = (y^{0,T}, \ldots, y^{0,T})^T$, $U^0 = (u^{0,T}, \ldots, u^{0,T})^T$.

From now on, we consider the RNS (1) at non-negative time steps in a finite time interval $0 \leq t \leq t_F$ under the initial conditions

$$x(0) = \begin{bmatrix} Y(-1,-\mu) \\ U(-1,-\nu) \end{bmatrix} = \begin{bmatrix} (y^T(-1), \ldots, y^T(-\mu))^T \\ (u^T(-1), \ldots, u^T(-\nu))^T \end{bmatrix}.$$

Then the system (1) has as inputs the sequence $\mathbf{u} = \{u(t); 0 \leq t \leq t_F\}$.

Throughout the paper we shall adopt a local viewpoint. More precisely, we work around an equilibrium point (u^0, y^0) of the system (1). Let us denote by $\mathcal{U}^0$ (resp. $\bar{U}$) the set of control sequences $\mathbf{u} = \{u(t); 0 \leq t \leq t_F\}$ (resp. $U(t-1,t-\nu)$) such that the controls $u(t)$ for every t are sufficiently close to u^0, i.e. that $\| u(t) - u^0 \| \leq \delta$ for some $\delta > 0$. Analogously, let us denote by $\mathcal{Y}^0$ (resp. $\bar{Y}$) the set of output sequences $\{y(t); 0 \leq t \leq t_F\}$ (resp. $Y(t-1,t-\mu)$) such that the outputs $y(t)$ for every t are sufficiently close to y^0, i.e. that $\| y(t) - y^0 \| < \epsilon$ for some $\epsilon > 0$. Denote by x^0 a $(\mu p + \nu m)$-dimensional vector $(y^{0,T}, \ldots, y^{0,T}, u^{0,T}, \ldots, u^{0,T})^T$. Finally, let us denote by X^0 the neighbourhood of x^0 such that for every $x \in X^0$, $\| x - x^0 \| < \gamma$ for some $\gamma > 0$.

For difference equation (1) under initial conditions $x(0)$, as long as F is a well-defined function of $R^{\mu p+\nu m}$, there is no problem regarding the existence and uniqueness of its solution $y(t; 0 \leq t \leq t_F)$, for an arbitrary control sequence $\mathbf{u} \in \mathcal{U}^0$, and an arbitrary initial condition $x(0) \in X^0$. Such a solution will be denoted as $y(t, x(0), \mathbf{u})$ which is a shorthand writing for $y(t, x(0), u(0), \ldots, u(t-1))$.

3. THE DELAY ORDERS WITH RESPECT TO THE CONTROL

For discrete-time nonlinear systems, described by state equations, the delay orders $d_i, i = 1, \ldots, p$ (in literature also referred to as the characteristic numbers, the relative orders or the dead times) with respect to the control have been defined, one for each output component. These structural parameters of the system tell us how many inherent delays there are between the ith component y_i of the output and the control, or equivalently, for how many first time instants y_i is completely defined by the initial conditions and which is the first time instant for which the possibility arises to change y_i arbitrarily.

In this section we shall extend the concept of delay orders to the class of RNS. Define the ith component of F in (1) by F_i.

At first sight it may seem that one can define d_i as the smallest positive integer k such that

$$\frac{\partial}{\partial u(t-k)} F_i(Y(t-1, t-\mu), U(t-1, t-\nu))$$

is not identically zero. Using the above definition, a RNS (1) with delay orders $d_1, \ldots, d_p$ admits a representation of the form

$$y_i(t+d_i) = F_i(Y(t+d_i-1, t+d_i-\mu), u(t), U(t-1, t+d_i-\nu)),$$
$$i = 1, \ldots, p. \tag{2}$$

However, in general, the above definition is not in complete correspondence with the state space formulation of the concept, since it does not show for how many first time instants y_i is completely defined by the initial conditions (observe that (2) contains $y(t+d_i-1), \ldots, y(t)$ which are not the part of initial conditions and may depend on the control). The following example serves as an illustration.

Example 3.1. Consider the system

$$\begin{aligned} y_1(t) &= u_1(t-1) + u_2(t-2)y_2(t-1), \\ y_2(t) &= u_2(t-2) - u_2(t-3)y_1(t-2) + y_3(t-1), \\ y_3(t) &= -u_2(t-1) - u_3(t-2)y_2(t-2). \end{aligned}$$

Compute

$$y_2(t+2) = u_2(t) - u_2(t-1)y_1(t) + y_3(t+1). \tag{3}$$

By the above definition, $d_2 = 2$. However, actually there is no possibility to change $y_2(t+2)$ arbitrarily since

$$y_3(t+1) = -u_2(t) - u_3(t-1)y_2(t-1), \tag{4}$$

and if we replace $y_3(t+1)$ in (3) by the right-hand side (RHS) of (4), we see that in (3) $y_2(t+2)$ depends completely on the initial conditions:

$$y_2(t+2) = -u_2(t-1)y_1(t) - u_3(t-1)y_2(t-1).$$

Next we shall give a proper definition of the delay orders for RNS.

The FTS operator δ is defined as $\delta y(t) = y(t+1)$.

Apply the one-step forward shift operator to Eq. (1) and replace in the latter $y(t)$ via the initial conditions, i.e. via the RHS of (1), in order to obtain

$$\begin{aligned} y(t+1) &= F(y(t), Y(t-1, t-\mu+1), u(t), U(t-1, t-\nu+1)) \\ &= F(F(Y(t-1, t-\mu), U(t-1, t-\nu)), Y(t-1, t-\mu+1), \\ &\quad u(t), U(t-1, t-\nu+1)) \\ &= F^1(Y(t-1, t-\mu), U(t-1, t-\nu), u(t)). \end{aligned}$$

Denote the ith component of F^1 by F_i^1 and compute for $i = 1, \dots, p$ the derivative

$$\frac{\partial}{\partial u(t)} F_i^1(Y(t-1, t-\mu), U(t-1, t-\nu), u(t)).$$

From the analyticity of the system (1) it follows that either the vector $\partial F_i^1(\cdot)/\partial u(t)$ is nonzero for all $(Y(t-1, t-\mu), U(t-1, t-\nu))$ belonging to an open and dense subset O_i of $\bar{Y} \times \bar{U}$ or this vector vanishes for all $(Y(t-1, t-\mu), U(t-1, t-\nu)) \in \bar{Y} \times \bar{U}$. In the first case we define $d_i = 1$, whereas in the latter case we can see that the function F_i^1 does not depend on $u(t)$, i.e. it depends completely on the initial conditions, and so we may write

$$y_i(t+1) = F_i^1(Y(t-1, t-\mu), U(t-1, t-\nu)). \tag{5}$$

Apply again a forward shift operator to Eq. (5) and replace in the latter $y(t)$ via the RHS of (1):

$$\begin{aligned} y(t+2) &= F_i^1(y(t), Y(t-1, t-\mu+1), u(t), U(t-1, t-\nu+1)) \\ &= F_i^1(F(Y(t-1, t-\mu), U(t-1, t-\nu)), Y(t-1, t-\mu+1), \\ &\quad u(t), U(t-1, t-\nu+1)) \\ &= F_i^2(Y(t-1, t-\mu), U(t-1, t-\nu), u(t)). \end{aligned}$$

Compute in an analogous fashion the derivative

$$\frac{\partial}{\partial u(t)} F_i^2(Y(t-1, t-\mu), U(t-1, t-\nu), u(t)).$$

If this vector is nonzero on an open and dense subset O_i of $\bar{Y} \times \bar{U}$, we set $d_i = 2$; otherwise we continue with

$$y_i(t+2) = F_i^2(Y(t-1, t-\mu), U(t-1, t-\nu)). \tag{6}$$

In this way the number d_i, if it exists, determines the inherent delay between the inputs and the ith output.

A RNS (1) with delay orders $d_i, i = 1, \ldots, p$ admits a representation of the form

$$\begin{array}{rcl} y_1(t+d_1) & = & F_1^{d_1}(Y(t-1,t-\mu),U(t-1,t-\nu),u(t)), \\ & \vdots & \\ y_p(t+d_p) & = & F_p^{d_p}(Y(t-1,t-\mu),U(t-1,t-\nu),u(t)) \end{array} \tag{7}$$

or in the vector form

$$\begin{bmatrix} y_1(t+d_1) \\ \vdots \\ y_p(t+d_p) \end{bmatrix} = A(x(t),u(t)), \tag{8}$$

where

$$x(t) = \begin{bmatrix} Y(t-1,t-\mu) \\ U(t-1,t-\nu) \end{bmatrix}.$$

By definition of [17, 18] a prediction model with the prediction horizon d for a RNS is a recursive map which allows the computation of the output at future time instants $t+d$ from inputs up to time t and outputs up to time $t-1$. According to the above definition, the representation (7) is actually a prediction model for (1) with the prediction horizon d_i for the ith output.

Using the proper definition of the delay orders, we may compute for Example 3.1 $d_1 = d_3 = 1, d_2 = 3$ and the representation (7) takes the following form

$$\begin{array}{rcl} y_1(t+1) & = & u_1(t) + u_2(t-1)[u_2(t-2) - u_2(t-3)y_1(t-2) \\ & & +y_3(t-1)], \\ y_2(t+3) & = & -u_2(t)u_1(t) - [u_2(t)u_2(t-1) + u_3(t)][u_2(t-2) \\ & & -u_2(t-3)y_1(t-2) + y_3(t-1)], \\ y_3(t+1) & = & -u_2(t) - u_3(t-1)y_2(t-1). \end{array}$$

4. THE CONCEPT OF FTS RIGHT INVERTIBILITY

It is natural to say that the system Σ is right invertible if the map Σ is surjective, or equivalently, if there exists another system $\Sigma_R^{-1} : S(R^p) \mapsto S(R^m)$, called the right inverse, such that the input–output map of the composition of Σ_R^{-1} and Σ is the identity map $\mathcal{I}_p$:

$$\Sigma \circ \Sigma_R^{-1} = \mathcal{I}_p : S(R^p) \mapsto S(R^p).$$

If Σ is invertible in the above sense, then it is possible to reproduce an arbitrary p-dimensional sequence $\{y_{ref}(t); 0 \leq t \leq t_F\}$ as an output of Σ by manipulating the input sequence.

This definition is certainly too restrictive for most systems and obviously useless for strictly causal RNS of the form (1), where the map F does not depend on $u(t)$. Such systems cannot be right invertible in the above sense, since the output y at $t = 0$ is not affected by the input and is completely defined by the initial conditions $x(0)$:

$$y(0) = F(Y(-1, -\mu), U(-1, -\nu)) = F(x(0)).$$

In general, the output may be defined completely by $x(0)$ also at a few next time instants $t = 1, 2, \ldots, d-1$. Therefore, for those systems it is useless to require that all sequences would be reproducible. The best we can achieve is that all sequences could be reproducible beginning from the time instant $t = d$. For example, in the case of the system (1) having delay orders $d_1, \ldots, d_p$, we have for $i = 1, \ldots, p$

$$\begin{aligned} y_i(0) &= F_i(x(0)), \\ y_i(1) &= F_i^1(x(0)), \\ &\vdots \\ y_i(d_i - 1) &= F_i^{d_i-1}(x(0)) \end{aligned}$$

and the output y_i is only affected by the input $u(0)$ d_i steps later:

$$y_i(d_i) = F_i^{d-i}(x(0), u(0)).$$

We shall modify the definition of right invertibility according to the above observations and introduce the notion of $(d_1, \ldots, d_p)$-FTS right invertibility around an equilibrium point (u^0, y^0) of the RNS (1).

Definition 4.1. $(d_1, \ldots, d_p)$**-FTS right invertibility**. *The RNS* (1) *is called locally* $(d_1, \ldots, d_p)$*-FTS right invertible in a neighbourhood of its equilibrium point* (u^0, y^0) *if there exist sets* $\mathcal{U}^0$, $\mathcal{Y}^0$, *and* X^0 *such that given* $x(0) \in X^0$, *we are able to find for any sequence* $\{y_{ref}(t); 0 \le t \le t_F\} \in \mathcal{Y}^0$ *a control sequence* $\{u_{ref}(t); 0 \le t \le t_F\} \in \mathcal{U}^0$ *(not necessarily unique) yielding*

$$y_i(t, x(0), u_{ref}(0), \ldots, u_{ref}(t)) = y_{ref,i}(t), \;\; d_i \le t \le t_F, \;\; i = 1, \ldots, p.$$

Denote by $\mathcal{Y}_i^0$ the set of sequences $\{y_i(t); 0 \le t \le t_F\} \in \mathcal{Y}_i^0$.

Then the above definition says that for the ith output component it is possible to reproduce locally all sequences $\mathrm{y}_{ref,i}$ from $\mathcal{Y}_i^0$, beginning from the time instant d_i. But $(d_1, \ldots, d_p)$-FTS right invertibility does not allow us to reproduce the first d_i terms in the arbitrary sequence $\{y_{ref,i}(t);$ $0 \le t \le t_F\} \in \mathcal{Y}_i^0$.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR FTS RIGHT INVERTIBILITY

Consider the RNS (1) with delay orders $d_i < \infty, i = 1, \ldots, p$, i.e. the system described by Eqs. (7).

We introduce the so-called decoupling matrix $K(x,u)$ for the system (1) in the following way

$$K(x,u) = \frac{\partial}{\partial u} \begin{bmatrix} F_1^{d_1}(x,u) \\ \vdots \\ F_p^{d_p}(x,u) \end{bmatrix}$$

From the definition of the d_is the rows of the matrix $K(x,u)$ are nonzero vector functions around (u^0, y^0). It is obvious that the rank of $K(x,u)$ is, in general, input and output dependent. However, we shall assume that $K(x,u)$ has a constant rank around (u^0, y^0). This assumption is formalized in the notion of regularity of an equilibrium point.

Definition 5.1. Regularity of an equilibrium point. *We call the equilibrium point* (u^0, y^0) *of the system* (1) *regular with respect to* $(d_1, \ldots, d_p)$*-FTS right invertibility if the rank of the decoupling matrix* $K(x,u)$ *of the system* (1) *is constant around* (u^0, y^0).

Theorem 5.2. *Assume that for the system* (1) $d_i < \infty$, $i = 1, \ldots, p$. *Then the RNS* (1) *is locally* $(d_1, \ldots, d_p)$*-FTS right invertible around a regular equilibrium point* (u^0, y^0) *if and only if rank* $K(x^0, u^0) = p$.

Proof. Sufficiency. Consider the system of equations (7). By the definition of the equilibrium point we have $y^0 = A(x^0, u^0)$. Observe that the Jacobian matrix of the RHS of (7) with respect to the control u equals to the decoupling matrix $K(x,u)$. By the assumption of the theorem the rank of the decoupling matrix $K(x,u)$ is equal to p at the equilibrium point (x^0, u^0). So, we may apply the Implicit Function Theorem in order to solve the system of equations (7) with respect to the control u. After a possible reordering of the control components we may assume that the Jacobian matrix of the RHS of (7) with respect to $u^1 = (u_1, \ldots, u_p)^T$ around the point (x^0, u^0) has the full row rank p. Therefore, Eqs. (7) can be solved for $u^1(t)$ uniquely around (x^0, u^0). Define $u^2 = (u_{p+1}, \ldots, u_m)^T$.

The Implicit Function Theorem says that in some (possibly small) neighbourhood (x^0, u^0, y^0) there exists a smooth function φ of variables $x(t)$, $y_1(t+d_1), \ldots, y_p(t+d_p)$, and $u^2(t)$, i.e.

$$u^1(t) = \varphi(x(t), y_1(t+d_1), \ldots, y_p(t+d_p), u^2(t)), \tag{9}$$

which is such that

$$\varphi(x^0, y^0, u^{20}) = u^{10}$$

and

$$\begin{aligned} &[y_1(t + d_1), \ldots, y_p(t+d_p)]^T \\ &\quad \equiv A(x(t), \varphi(x(t), y_1(t+d_1), \ldots, y_p(t+d_p), u^2(t)), u^2(t)). \end{aligned}$$

Necessity. Suppose that the system (1) is locally $(d_1, \ldots, d_p)$-FTS right invertible around its regular equilibrium point (u^0, y^0). This implies, in particular, that at the time instant $t = d_i$ at the ith output y_i of the system (1) we can reproduce by the suitable choice of $u(0) = u_{ref}$ arbitrary $y_{ref,i}$ sufficiently close to y_i^0, i.e. the following holds

$$F_i^{d_i}(x(0), u_{ref}) = y_{ref,i}\,, \quad i = 1, \ldots, p\,.$$

Assume that rank $K(x^0, u^0) = k < p$. As by regularity of (u^0, y^0), k is constant in some neighbourhood of (u^0, y^0), the rank of $K(x, u)$ in this neighbourhood is less than p. This implies that the functions $F_i^{d_i}(x(0), u_{ref})$, $i = 1, \ldots, p$ of u_{ref} are functionally dependent, i.e. there exists the map $R(\cdot)$ such that

$$R(F_1^{d_1}, \ldots, F_p^{d_p}, x(0)) = R(y_{ref,1}, \ldots, y_{ref,p}, x(0)) = 0\,.$$

The last equality means that y_{ref} is not arbitrary but satisfies the equation $R(y_{ref,1}, \ldots, y_{ref,p}, x(0)) = 0$ which gives a contradiction. This completes the proof.

Remark 5.3. *Clearly, rank $K(x^0, u^0) = p$ requires $m \geq p$. So, $p \leq m$ is always a necessary condition for a system to have a $(d_1, \ldots, d_p)$-FTS right inverse, i.e. the system must have at least as many inputs as outputs.*

Remark 5.4. *We should like to stress that the assumption of the regularity of the equilibrium point (x^0, u^0) in Theorem 5.2 is extremely vital. If the point (u^0, y^0) is not regular, i.e. around the point (u^0, y^0) the rank of the decoupling matrix $K(x, u)$ is not necessarily constant, then the condition $K(x^0, u^0) = p$ is not necessary for $(d_1, \ldots, d_p)$-FTS right invertibility.*

The illustration of this phenomenon is given in the following simple example:

$$y(t) = u(t-1)^3 \tag{10}$$

We have

$$K(x, u) = \frac{\partial}{\partial u} A(x, u) = 3u^2\,.$$

At the equilibrium point $u^0 = 0$, $y^0 = 0$ the rank of $K(x, u)$ is equal to 0 which is less than $p = 1$. Still, the arbitrary sequences are reproducible for $1 \leq t \leq t_F$ by the choice of the control

$$u(t) = \sqrt[3]{y(t+1)}\,.$$

The reason is that the point $u^0 = 0$, $y^0 = 0$ is not a regular equilibrium point. The rank of the matrix $K(x, u)$ is equal to 1 at all points $u \neq 0$.

From Eqs. (9) it is clear that the set of recursive nonlinear systems of the form (1) is not closed under system inversion: the inverse system in general depends on the future values of the outputs of the original system.

There exists no right inverse for a strictly causal recursive system Σ such that the input to the original system can be computed as the output of the inverse system without using the future values of the reference signal, or equivalently, without using forward shift operators on the reference signal. Actually, for $(d_1, \ldots, d_p)$-FTS right invertible systems, the output of the right inverse system (i.e. the control of the original system) at the time instant t will depend on the ith component of the reference output at the time instant $t + d_i$. As $\mathbf{y}_{ref}$ is generated by the designer, in the actual control law design this implies that a change of the reference signal must be preplanned some time steps ahead, which is often a realistic assumption. If this is possible, the right inverse system can be realized.

If the reference signal can be generated from a model M, the need for future values of model inputs is avoided under the conditions that the delay orders of the model are equal to or greater than the corresponding delay orders of the ($(d_1, \ldots, d_p)$-FTS right invertible) original system.

6. EXAMPLES

Example 6.1. *Consider the RNS*

$$\begin{aligned} y_1(t) &= u_1(t-1) + u_2(t-2)y_2(t-1), \\ y_2(t) &= u_2(t-2)[y_1(t-2)+1] \end{aligned}$$

for $t \geq 0$ under the initial conditions $[U^T(-1,-2), Y^T(-1,-2)]^T$.

The delay orders of this system are $d_1 = 1, d_2 = 2$ and the system can be represented in the form

$$\begin{aligned} y_1(t+1) &= u_1(t) + u_2(t-1)u_2(t-2)[y_1(t-2)+1], \\ y_2(t+2) &= u_2(t)[u_1(t-1) + u_2(t-2)y_2(t-1)+1]. \end{aligned}$$

This system is $(1,2)$-FTS right invertible around an equilibrium point with $y_1^0 \neq -1$. The equations of the right inverse are:

$$\begin{aligned} u_1(t) &= y_1(t+1) - u_2(t-1)u_2(t-2)[1 + y_1(t-2)], \\ u_2(t) &= y_2(t+2)/[1 + u_1(t-1) + u_2(t-2)y_2(t-1)]. \end{aligned}$$

Example 6.2. *Consider the RNS*

$$\begin{aligned} y_1(t) &= u_2(t-3) - u_1(t-2), \\ y_2(t) &= u_1(t-2)/[y_1^2(t-2)+1] \end{aligned}$$

for $t \geq 0$ under the initial conditions $[Y^T(-1,-2), U^T(-1,-3)]^T$.

The delay orders of this system are $d_1 = d_2 = 2$ and so the system can be represented in the form

$$\begin{aligned} y_1(t+2) &= u_2(t-1) - u_1(t), \\ y_2(t+2) &= u_1(t)/[u_2^2(t-3) - 2u_2(t-3)u_1(t-2) + u_1^2(t-2)+1]. \end{aligned}$$

It is clear that this system is not locally $(2,2)$-FTS right invertible, since the rank of the matrix K is equal to one for all possible equilibrium points. However, the arbitrary reference signals can be generated at the first output starting from the time instant 3 and at the second output starting from the time instant 2 by the choice of the following control

$$\begin{aligned}
u_1(t) &= y_{ref,2}(t+2)[1+(u_2(t-3)-u_1(t-2))^2], \\
u_2(t) &= y_{ref,1}(t+3)+y_{ref,2}(t+3)[1+(u_2(t-2)-u_1(t-1))^2].
\end{aligned}$$

7. DISCUSSION AND CONCLUSIONS

For system Σ defined by (1) no possibilities exist to reproduce at the output an arbitrary reference signal starting from the time instant $t=0$. We are able to reproduce the reference signals at the output with some time-shifts and the smallest possible value of the time-shift is d_i (the delay order) for the ith output component. These smallest values can be realized if the system of equations

$$\begin{bmatrix} y_1(t+d_1) \\ \vdots \\ y_p(t+d_p) \end{bmatrix} = A(x(t),u(t)) \tag{11}$$

can be solved for $u(t)$ for arbitrary $[y_i(t+d_1),\ldots,y_p(t+d_p)]^T$.

Note that we cannot solve the system of equations (11) for $u(t)$ in case of the arbitrary left-hand side if some components of the vector function $A(x,u)$, as functions of the control, depend functionally on the others, or equivalently, if

$$\operatorname{rank} \frac{\partial}{\partial u} A(x,u) < p\,.$$

The idea to generalize the notion of right invertibility is to represent the functionally dependent components via the independent ones and apply to the dependent equations the one-step FTS operator and repeat the whole procedure (say α times) until we obtain a system of equations which can be solved for the control $u(t)$ in terms of $x(t)$ and $y(t+1), y(t+2), \ldots, y(t+\alpha)$ in case of an arbitrary reference signal, or it will become clear that the latter is impossible. If it is possible to obtain a system of equations which can be solved for the control, then we are able to reproduce at the ith output y_i an arbitrary reference signal starting from a certain time instant $\alpha_i \geq d_i$ with $\alpha_i > d_i$ for some $j \in \{1,\ldots,p\}$. Generalizations along these lines will be the topic of another paper.

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MITTELINEAARSETE REKURSIIVSETE SÜSTEEMIDE PAREMALT PÖÖRATAVUS

Ülle KOTTA

On uuritud mittelineaarsete rekursiivsete süsteemide klassi kuuluvate süsteemide paremalt pööratavust, s.t. sisendeid ja väljundeid siduvate kõrgemat järku diferentsiaalvõrranditega kirjeldatavate süsteemide paremalt pööratavust. Olekumudeli puhul tuntud mõisted, nagu

hilistumisjärgud ja paremalt pööratavuse erijuht, nn. $(d_1, \ldots, d_p)$-nihkega paremalt pööratavus, on üldistatud rekursiivsetele mittelineaarsetele süsteemidele. On tuletatud tarvilikud ja piisavad tingimused rekursiivsete mittelineaarsete süsteemide $(d_1, \ldots, d_p)$-nihkega paremalt pööratavuseks. Lõpuks on näidatud, et samuti nagu olekumudelitel baseeruvate süsteemide korral, on ka rekursiivsete mittelineaarsete süsteemide puhul võimalik vaadelda üldisemat pööratavuse mõistet.