Proc. Estonian Acad. Sci. Phys. Math., 1995, 44, 4, 459—465
https://doi.org/10.3176/phys.math.1995.4.05

ECCENTRIC INTERACTING ELECTRON

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Presented by A. Sapar

Received 30 November 1994, accepted 19 June 1995

Abstract. New equations of motion of radiating elementary particles are derived by taking account of the induced angular momentum (eccentricity) of the particles. The equations satisfy all ten conservation laws and generalize the Lorentz—Dirac equation.

Key words: radiation reaction, equations of motion in classical electrodynamics.

1. INTRODUCTION

Equations of motion of interacting point charges taking account of radiation reaction are a highly debated subject [1–5]. There exists an enormous literature on the problem (see e. g. references to be found in the surveys [4–7]). In spite of numerous ingenious efforts, no better equation than the "old" Lorentz—Dirac equation has been found. This third-order equation, however, leads to unphysical runaway solutions, therefore additional restrictions must be imposed upon solutions to exclude the unphysical ones. There occur also undesirable effects such as pre-acceleration and others (for details, see Jackson's textbook [8]). In this paper we outline a completely new approach to the problem.

Firstly, we make a fuller use of the invariants of the Poincaré group as characteristics of elementary particles. Usually elementary particles are characterized by their mass and spin. Shirokov [9] found two further invariants of the Poincaré group, usually not applied in the theory. One of these "neglected" invariants refers to the centre of inertia. In this paper we admit that the interacting electron can be eccentric, i. e. the location of the point charge and electron's centre of mass (as well as the corresponding velocities) need not coincide. We shall call the velocity of the centre of mass dynamic velocity and the velocity of the point charge kinematic velocity (cf. Dixon [10]).

Secondly, we specify further a generic property of a fourforce, its orthogonality to the four-velocity [11], and assume that the product of dynamic velocity and the total fourforce vanishes, the latter being defined as the sum of the Lorentz force and the radiation reaction force (the radiated energy and momentum with the opposite sign).

The above two assumptions together with ten conservation laws give us a new system of equations of motion, the Lorentz—Dirac equations being the first approximation to the full system.

Equations of motion of regular multipole particles following from the conservation laws have already been derived by Mathisson [7, 12], renormalization problems in the case of point charges have also been solved [6]. However, this does not mean that we have a unique translational law of motion: different solutions of the "dipole" (rotational) part of Mathisson's equations provide different translational laws of motion. This law cannot be neglected, because point charges radiate the angular momentum besides momentum, more exactly, they radiate the components of the four-dimensional angular momentum which are related to the motion of the centre of mass. If we do not admit the existence of the corresponding induced angular momentum of the point particle, it is not possible to require the persistence of all conservation laws.

In Sec. 2, we outline the derivation of the Lorentz—Dirac equation from the Mathisson equations [6, 7, 13], and on the basis of this derivation we continue to argue that the former is an equation of motion in a noninertial frame of reference. In Sec. 3, new equations are derived. They provide an explanation to radiated energy: the Lorentz force becomes dissipative due to the "roughness" of an eccentric electron. Solutions of the new equations are free from "runaways", and the rest masses of elementary particles turn out to be strictly constant.

The following notation is used throughout the paper: v^α — kinematic velocity, u^α — dynamic velocity, $a^\alpha = \frac{dv^\alpha}{d\tau}$, $F^\alpha_{(\mathrm{react})}$ — radiation reaction force in the Lorentz—Dirac equation, $F^\alpha_{(\mathrm{rad})}$ — the negative of the total energy and momentum radiated per unit of proper time, it is the force that should be properly called radiation reaction force. Greek indices take values 0, 1, 2, 3; the Einstein summation convention is used. Only orthogonal Cartesian coordinates are used, indices are raised and lowered with the Minkowski metric tensor $\eta^{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$; $v^\alpha = (v^0, \vec{v}/\sqrt{1 - v^2/c^2})$.

2. THE LORENTZ—DIRAC EQUATION

More than a century ago H. Lorentz derived the radiation reaction force $\vec{F}_{(\mathrm{react})}$ and equations of motion of point charges in the exterior electromagnetic field by *approximately* balancing radiated and bound energy, and considering *quasiperiodic* motions [5]. The resulting Lorentz—Abraham equation is

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{F}_{(\mathrm{react})}, \tag{1}$$

where $\vec{F}$ is the external force, and the radiation reaction force is

$$\vec{F}_{(\mathrm{react})} = \frac{2}{3}\frac{e^2}{c^3}\frac{d^2}{dt^2}\vec{v}. \tag{2}$$

Here $\vec{v}$ is the velocity of a point charge e with mass m and c is the velocity of light. In case the external force $\vec{F}$ vanishes, Eq. (1) has a physically unacceptable runaway solution

$$\vec{v} = \mathrm{const.} \cdot \exp(3mc^3 t/2e^2).$$

The relativistic generalization of Eq. (1), the Lorentz—Dirac equation, can be found if we replace $\vec{v}$ with components of the four-velocity v^α, derivatives d/dt with the proper time derivatives $d/d\tau$, make use of the Lorentz force F^α and radiation reaction force $F^\alpha_{(\mathrm{react})}$:

$$m\frac{dv^\alpha}{d\tau}=F^\alpha+F^\alpha_{(\mathrm{react})}, \tag{3}$$

$$F^\alpha=\frac{e}{c}F^{\alpha\beta}v_\beta, \tag{4}$$

$$F^\alpha_{(\mathrm{react})}=\frac{2e^2}{3c^3}\left(\frac{d^2v^\alpha}{d\tau^2}-\frac{v^\alpha v_\beta}{c^2}\frac{d^2v^\beta}{d\tau^2}\right). \tag{5}$$

Here $F^{\alpha\beta}$ is the antisymmetric electromangetic field tensor. The second term in the brackets on the r. h. s. (right-hand side) of expression (5) has been found from the orthogonality condition [11]

$$F^\alpha_{(\mathrm{react})}v_\alpha=0. \tag{6}$$

Next we reproduce Mathisson's derivation of Eqs. (3) to (5). From exact energy, momentum, and angular momentum balance the following equations can be derived [6, 7, 13]:

$$\frac{dP^\alpha}{d\tau}=F^\alpha-Wv^\alpha/c^2, \tag{7}$$

$$P^\alpha=mv^\alpha+n^\alpha,\ \ n^\alpha v_\alpha=0, \tag{8}$$

$$\frac{dS^{\alpha\beta}}{d\tau}=2P^{[\alpha}v^{\beta]}+L^{\alpha\beta}_{(\mathrm{rad})}. \tag{9}$$

Here square brackets denote antisymmetrization, $-Wv^\alpha/c^2$ is an alternative radiation reaction force, equal to radiated energy momentum with the opposite sign, $L^{\alpha\beta}_{(\mathrm{rad})}$ is the radiation reaction torque, and $S^{\alpha\beta}$ is the angular momentum of a point charge.

$$S^{\alpha\beta}=B^{\alpha\beta}+D^{\alpha\beta}, \tag{10}$$

where $B^{\alpha\beta}=-B^{\beta\alpha}$ is the intrinsic angular momentum (spin), and

$$D^{\alpha\beta}=2v^{[\beta}D^{\alpha]} \tag{11}$$

is the induced angular momentum; $B^{\alpha\beta}v_\beta=0$, $D^\alpha v_\alpha=0$. In the case of the Liénard—Wiechert potentials one has

$$W=-\frac{2}{3}\frac{e^2}{c^3}a_\alpha a^\alpha, \tag{12}$$

$$L^{\alpha\beta}_{(\mathrm{rad})}=\frac{4}{3}\frac{e^2}{c^3}a^{[\alpha}v^{\beta]}. \tag{13}$$

If we want to take into account the magnetic moment of the electron, the external torque $L^{\alpha\beta}$ (to be found in [6, 14]) should be included into Eq. (9) and terms describing magnetic dipole radiation must be added to expressions (12) and (13).

We note that momentum P^α and angular momentum $S^{\alpha\beta}$ in Eqs. (7) to (9) are phenomenological inputs, the outcomes of a (nonunique) renormalization procedure [6]: from infinite bound electromagnetic energy-momentum tensor infinite matter energy-momentum tensor is subtracted (containing the Poincaré stresses [8]). In complete analogy with point dipoles in classical electrodynamics, charged point particles with mass dipole moment D^α can exist. More rigorously, they are described by induced angular momentum (11).

Assuming that $dS^{\alpha\beta}/d\tau=0$ and inserting expressions (8) and (13) into Eq. (9), one obtains [6, 7, 13]

$$n^\alpha=-\frac{2}{3}\frac{e^2}{c^3}a^\alpha, \tag{14}$$

and the translational law of motion (7) can be written in the following form

$$\frac{d}{d\tau}\left(mv^\alpha-\frac{2}{3}\frac{e^2}{c^3}a^\alpha\right)=F^\alpha+\frac{2}{3}\frac{e^2}{c^5}a^\beta a_\beta v^\alpha. \tag{15}$$

Here the explicit form (12) of W has been used. Shifting $\frac{2}{3}\frac{e^2}{c^3}\frac{da^\alpha}{d\tau}$ from the l.h.s. of Eq. (15) to the r.h.s. and taking account of identities $v^\alpha a_\alpha=0$, $\frac{d}{d\tau}(v_\alpha a^\alpha)=a_\alpha a^\alpha+v_\alpha\frac{da^\alpha}{d\tau}=0$, we obtain from Eq. (15) the Lorentz—Dirac equation (3).

From expressions (4) and (5) it follows that the Lorentz force (4) and the radiation reaction force (5) satisfy separately orthogonality conditions, $F^\alpha v_\alpha=0$, $F^\alpha_{(\text{react})}v_\alpha=0$. This may be a drawback of the theory, indicating that the external force has a nondissipative character. Herrera [1] claims that the only source of radiation energy can be the work done by the external force, indicating thereby its dissipative nature. Dissipative terms have been added to the Lorentz force by Mo and Papas [15].

It is difficult to interpret the Lorentz—Dirac equation properly. The trouble comes from the first term in the brackets on the r. h. s. of expression (5), the so-called Schott term. By integrating Eq. (3) one has $(p^\alpha=mv^\alpha)$

$$p^\alpha(\tau_2)-p^\alpha(\tau_1)=\frac{e}{c}\int_{\tau_1}^{\tau_2}F^\alpha d\tau-$$

$$-\frac{1}{c^2}\int_{\tau_1}^{\tau_2}Wv^\alpha d\tau+\frac{2}{3}\frac{e^2}{c^3}\left[a^\alpha(\tau_2)-a^\alpha(\tau_1)\right]. \tag{16}$$

Some authors argue (see e. g. [4]) that the nonvanishing last term on the r. h. s. of integral (16) of the Lorentz—Dirac equation indicates that the law of conservation of energy and momentum is violated. There are others who claim that the missing energy comes from the Schott term. This point of view is criticized by Herrera [1] who writes that "term" is not a material object and therefore it is not intrinsically endowed with energy. We propose the following interpretation. The momentum p^α on the l. h. s. of expression (16) is evaluated in a noninertial frame of reference, while the true momentum P^α is constructed from the very beginning as a quantity in an inertial frame of reference [6, 13]. By accelerating the inertial frame of reference by a quantity $\frac{1}{m}\frac{dn^\alpha}{d\tau}=-\frac{2e^2}{3c^3m}\frac{da^\alpha}{d\tau}$ we enforce the vanishing of the part n^α of the momentum P^α in Eq. (7) and introduce new terms into the radiation reaction force. True, shifting terms from one side of the equation to the other side does not change integrals of the equation, but it changes the orthogonality conditions of the four-force and the four-velocity. Hence, we must find new solutions to Eqs. (7) to (9), satisfying the orthogonality conditions in inertial frames of reference.

3. NEW EQUATIONS OF MOTION

Let us start by imposing the orthogonality condition upon the dynamic velocity and the total force $F^\alpha_{(tot)} \equiv F^\alpha - \frac{1}{c^2} W v^\alpha$. By definition the dynamic velocity u^α is [10]

$$u^\alpha \equiv cP^\alpha/P, \qquad P^2 \equiv P^\alpha P_\alpha.$$

Inserting the value of P^α from expression (8) we have

$$u^\alpha = (mv^\alpha + n^\alpha)\, c/\sqrt{m^2c^2 + n_\alpha n^\alpha}, \tag{17}$$

and the orthogonality condition reads

$$F^\alpha_{(tot)} u_\alpha \equiv F^\alpha u_\alpha - \frac{1}{c^2} W v^\alpha u_\alpha = 0. \tag{18}$$

The dissipative character of the Lorentz force F^α in the last equation is evident. From definitions (4) and (8) it follows $F^\alpha v_\alpha = 0$, $n^\alpha v_\alpha = 0$, and from Eq. (18) one easily finds

$$W = \frac{1}{m} F^\alpha n_\alpha. \tag{19}$$

Equation (9) determines the induced angular momentum (11). This equation can be satisfied by any value of a^α, n^α and $B_{\alpha\beta}B^{\alpha\beta} = \text{const}$.

Let us find the restrictions put by condition (19) upon the translational law of motion (7), which can be written in the following form

$$\frac{dm}{d\tau} v^\alpha + ma^\alpha + \frac{dn^\alpha}{d\tau} = F^\alpha - \frac{1}{c^2} W v^\alpha. \tag{20}$$

Multiplication of the last equation by v^α, a_α, and n^α and summation over the index α gives $(v^\alpha v_\alpha = c^2)$

$$c^2 \frac{dm}{d\tau} + v^\alpha \frac{dn_\alpha}{d\tau} = -W, \tag{21}$$

$$ma_\alpha a^\alpha + a_\alpha \frac{dn^\alpha}{d\tau} = F^\alpha a_\alpha, \tag{22}$$

$$ma_\alpha n^\alpha + \frac{1}{2} \frac{d}{d\tau} (n_\alpha n^\alpha) = F^\alpha n_\alpha, \quad \text{or} \tag{23}$$

$$\frac{1}{2} \frac{d}{d\tau} (n_\alpha n^\alpha) = -ma^\alpha n_\alpha + mW. \tag{24}$$

By definition we have $v^\alpha n_\alpha \equiv 0$, hence $\frac{d(v^\alpha n_\alpha)}{d\tau} = 0$ or $v_\alpha \frac{dn^\alpha}{d\tau} = -a^\alpha n_\alpha$ and an equivalent form of Eq. (21) is

$$c^2 m \frac{dm}{d\tau} = m a_\alpha n^\alpha - mW. \tag{25}$$

By adding Eqs. (24) and (25) we obtain

$$\frac{dm^2}{d\tau} = -\frac{1}{c^2} \frac{d}{d\tau} (n_\alpha n^\alpha). \tag{26}$$

The square of mass M of an elementary particle is defined as the following invariant of the Poincaré group $M^2 = \frac{1}{c^2} P_\alpha P^\alpha$. Taking account of definition (8), we have

$$M^2 = m^2 + n_\alpha n^\alpha / c^2. \tag{27}$$

From the last expression and Eq. (26) it follows that the mass of any elementary particle satisfying condition (18) is a constant.

Let $B^{\mu\nu} = 0$. From Eqs. (8) to (11) and (13) we find

$$\left(\delta^\alpha_\beta - \frac{v^\alpha v_\beta}{c^2} \right) \frac{dD^\beta}{d\tau} = -n^\alpha - \frac{2}{3} \frac{e^2}{c^3} a^\alpha \tag{28}$$

and by taking account of Eqs. (12), (25), and (26) we have

$$m a_\alpha \frac{dD^\alpha}{d\tau} = \frac{1}{2} \frac{d}{d\tau} (n_\alpha n^\alpha). \tag{29}$$

We see that equations describing the backreaction of radiation are nonlinear, and we propose to integrate them by the successive approximation method. The order-of-magnitude analysis of terms describing the radiation reaction as well as the description of the cases when they are relevant may be found in Jackson's textbook [8]. In the case of the electron we have $2e^2/3mc^3 = 6.26\ 10^{-24}$ sec. In most cases of physical interest (when extremely rapid changes of the external force are excluded), the small parameter of the dimension of time $2e^2/3mc^3$ can be used as an expansion parameter. In the zeroth approximation we have the well-known equation

$$m a^\alpha = \frac{e}{c} F^{\alpha\beta} v_\beta.$$

From Eqs. (28) and (29) it follows that n^α is the first-order quantity, and D^α is the second-order quantity. In the first approximation

$$n^\alpha = -\frac{2}{3} \frac{e^2}{mc^3} \cdot m a^\alpha. \tag{30}$$

This solution coincides with solution (14), which gave us the Lorentz—Dirac equation.

In the full theory self-accelerating solutions are excluded already by condition (19): when the external force F^α vanishes, the radiated energy W vanishes as well, which in turn means that no acceleration of charge is admitted. (References to papers discussing this aspect of the problem may be found in [4].)

In this paper not all aspects of the radiation reaction theory were treated. It is planned to continue the work in this field.

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EKSTSENTRILINE KIIRGAV ELEKTRON

Väino UNT

Kiirgavate fundamentaalosakeste liikumisvõrranditel, nende hulgas ka üldkasutataval Lorentzi—Diraci võrrandil, on tõsiseid puudusi. Käesoleva töö eesmärk on tuletada uued liikumisvõrrandid, tuues kiirguspidurduse teooriasse «märkamata jäänud» Poincaré rühma invariandi, elektroni ekstsentrilisuse, mis kirjeldab punktlaengu asukoha ja tema massikeskme asukoha erinevust. Lisaks sellele on postuleeritud summaarse nelijõu ja dünaamilise nelikiiruse ortogonaalsus. Nii on saadud uued liikumisvõrrandid (19), (20) ja (28), mis sisaldavad Lorentzi—Diraci võrrandit kui esimest lähendit täpsele võrrandisüsteemile. Uutel võrranditel puuduvad «ärajooksvaid» osakesi kirjeldavad lahendid, nad annavad elementaarosakeste massi konstantsed väärtused ja seletavad kiirgust ekstsentrilisele elektronile mõjuva Lorentzi jõu dissipatiivse iseloomuga.

ЭКСЦЕНТРИЧЕСКИЙ ИЗЛУЧАЮЩИЙ ЭЛЕКТРОН

Вяйно УНТ

Все предложенные к настоящему времени уравнения движения излучающих фундаментальных частиц, включая общепризнанное уравнение Лоренца—Дирака, имеют серьезные недостатки. В данной работе выводятся новые уравнения движения путем введения в теорию торможения излучением «незамеченного» инварианта группы Пуанкаре — эксцентриситета электрона. То есть допускается, что местонахождение точечного заряда и его центр инерции могут не совпадать. В дополнение к этому постулируются ортогональность полной четырехсилы и динамической четырехскорости. Новые уравнения (19), (20) и (28) содержат в виде первого приближения уравнение Лоренца—Дирака, у них отсутствуют «убегающие» решения и они дают постоянные значения для масс элементарных частиц. Механизм излучения объясняется диссипативностью силы Лоренца в случае эксцентрического электрона.