# GEODESIC MULTIPLICATION AND GEOMETRICAL BRST-LIKE OPERATORS 

Piret KUUSK ${ }^{\text {a }}$, Jüri ÖRD ${ }^{\text {b }}$, and Eugen PAALc

a Eesti Teaduste Akadeemia Füüsika Instituut (Institute of Physics, Estonian Academy of Sciences), Riia 142, EE-2400 Tartu, Eesti (Estonia)

- Institute of Theoretical Physics, Chalmers University of Technology and University of Göteborg, S-412 96 Göteborg, Sweden; Tartu Ulikool (University of Tartu), Tähe 4, EE-2400 Tartu, Eesti (Estonia)
c Tallinna Tehnikaülikooli matemaatika instituut (Department of Mathematics, Tallinn Technical University), Ehitajate tee 5, EE-0026 Tallinn, Eesti (Estonia)

Presented by V. Hizhnyakov
Received 26 September 1994, accepted 19 June 1995


#### Abstract

A generalization of Poincaré translations into a nonassociative algebraic system called the local geodesic loop is proposed. Two BRST-like nilpotent operators based on the local geodesic left and right translation matrices are constructed. As an explicit example, the local geodesic translation matrices are calculated in the spacetime of a weak plane gravitational wave.


Key words: nonassociative algebras, geodesic multiplication, BRST operator, quantum gravity.

## 1. INTRODUCTION

The BRST quantization [ ${ }^{1}$, which is up to now the most advanced method of quantization, is a generalization of Dirac's scheme of quantization for physical systems with constraints and gauge freedom. Both methods coincide if the algebra of constraints closes and its structure constants do not involve fields. In the case of open gauge algebras and field-dependent structure functions only the BRST quantization is applicable. Usually an explicit construction of the BRST generator is based on the algebra of constraints. However, one of the main points of the BRST quantization lies in the statement that different physical states correspond to the different cohomology classes of the nilpotent BRST operator, and physical observables must commute with it.

There exist physical systems for which a consistent quantum theory cannot be constructed by means of any of the available quantization schemes, e. g. the gravitational field. Nevertheless, we can try to construct a reasonable nilpotent operator that can be used for defining cohomology classes and investigate the resulting quantum theory. $\AA$ geometrical BRST operator $Q$ has been proposed by Bars and Yankielowicz [ ${ }^{2}$ ]. It involves an infinite-dimensional algebra of the modified

Poincaré group where the torsion tensor and the curvature tensor act as structure functions. The nilpotency of $Q$ turns out to be a consequence of the differential geometrical Bianchi identities. An analogous geometrical BRST-like operator for a $N$-dimensional differentiable manifold with a zero curvature but a nonzero torsion tensor has recently been given also by Okubo $\left[{ }^{3}\right]$. In the present paper we investigate a possibility of using our earlier work [ ${ }^{4}$ ] on the role of geodesic multiplication in the theory of gravity for constructing BRST-like operators according to Okubo's scheme, and for discussing their possible geometrical and physical meaning.

Geodesic multiplication of points of a differentiable manifold with an affine connection is a generalization of constant translations of a flat torsionless manifold which allows to transform straight lines (geodesics) into parallel straight lines. In the case of a flat spacetime, constant translations form a subgroup of the Poincare group, the Abelian group of Poincaré translations. It can be gauged, e. g. constant translations $x \rightarrow x+a, a=$ const. can be replaced by point-dependent translations, $x \rightarrow x+a(x)$. The corresponding gauge group is the group of general coordinate transformations (diffeomorphisms). and the gauge field is the local frame of reference $e_{A}{ }^{\mu}(x)\left[{ }^{5}\right]$. But the mathematical structure of the following gauge theory differs in some essential points from that of a standard gauge theory. We propose to generalize the group of Poincaré translations into a geometrically defined algebraic system called the geodesic loop which has the geodesic multiplication as a binary operation. Due to the nonassociativity of the geodesic multiplication, it does not form a group and the methods of the conventional gauge theory cannot be applied. However, using Okubo's construction it is possible to write down two BRST-like operators containing the algebra of vector fields which generate left and right infinitesimal geodesic translations of space-time points. Cohomologies of these BRST-like operators turn out to be analogous to the de Rham cohomology of the space-time.

The paper is organized as follows. In Sec. 2. Okubo's construction of a BRST-like operator is briefly reviewed. In Sec. 3, the notions of the geodesic multiplication and the geodesic loop are introduced and their main algebraic properties are described. In Sec. 4, the left and the right geodesic translation matrices are used for constructing two geometrical BRST-like operators. Our main idea is to derive these operators not directly from the geometry of the space-time, but from the geometry of the geodesic loop. In Sec. 5, explicit expressions for the left and the right geodesic translation matrices are calculated in the case of a weak plane gravitational wave and the corresponding parallelizing torsions of the geodesic loop are determined. Section 6 is devoted to the problem of physical state vectors and their cohomologies.

## 2. OKUBO'S CONSTRUCTION FOR A BRST-LIKE OPERATOR

Okubo $\left.{ }^{3}{ }^{3}\right]$ has proposed the following formal construction of an anticommuting nilpotent operator for a $N$-dimensional differentiable manifold with local coordinates $x^{\mu}$. Let us introduce a $N$-bein field $e_{A}^{\mu}(x)$ that is invertible, i.e. there exists also the inverse matrix $e_{\mu}^{A}(x)$,

$$
\begin{equation*}
e_{A}^{\mu} e_{\mu}^{B}=\delta_{A}^{B}, \quad e_{A}^{\mu} e_{v}^{A}=\delta_{v}^{\mu} . \tag{1}
\end{equation*}
$$

In the framework of the Cartan formalism, $e_{\mu}^{A}(x)$ determines the basis 1 -forms $\omega^{A}$,

$$
\begin{equation*}
\omega^{A}=e_{\mu}^{A} d x^{\mu} \tag{2}
\end{equation*}
$$

Let us suppose that the connection 1 -forms $\omega_{B}^{A} \equiv \Gamma_{B D}^{A} \omega^{D}$ vanish,

$$
\begin{equation*}
\omega_{B}^{A}=0, \quad \Gamma_{B D}^{A}=0 . \tag{3}
\end{equation*}
$$

From the Cartan structure equations

$$
\begin{align*}
& d \omega^{A}+\omega_{B}^{A} \wedge \omega^{B}=\Omega^{A},  \tag{4}\\
& d \omega_{B}^{A}+\omega_{D}^{A} \wedge \omega_{B}^{D}=\Omega_{B}^{A} \tag{5}
\end{align*}
$$

it follows that the curvature 2 -form $\Omega_{B}^{A}$ also vanishes, $\Omega_{B}^{A}=0$, and the components of the torsion 2 -form $\Omega^{A}=\frac{1}{2} S_{\mu \nu}^{A} d x^{\mu} \wedge d x^{\nu}$ are determined by the inverse $N$-bein field $c_{\mu}^{A}$,

$$
\begin{equation*}
\partial_{\mu} e_{v}^{A}-\partial_{v} e_{\mu}^{A}=S_{\mu v}^{A} \tag{6}
\end{equation*}
$$

Although the connection 1 -forms $\omega_{B}^{A}$ and the Riemann curvature tensor vanish, the connection coefficients $\Gamma_{\mu \nu}^{\lambda}$ in local holonomic curvilinear coordinates $x^{\mu}$ may acquire nonvanishing values due to the coordinate transformation from anholonomic flat coordinates $y_{A}$ to $x_{\mu}$ :

$$
\begin{gather*}
\frac{\partial y^{A}(x)}{\partial x^{\mu}}=e_{\mu}^{A}(x),  \tag{7}\\
\Gamma_{\mu \nu}^{\lambda} \equiv e_{D}^{\lambda}\left(e_{\mu}^{A} e_{v}^{B} \Gamma_{A B}^{D}+\partial_{v} e_{\mu}^{D}\right)=e_{D}^{\lambda} \partial_{v} e_{\mu}^{D} . \tag{8}
\end{gather*}
$$

For constructing a BRST-like operator $Q$, Okubo introduced coordinateindependent anticommuting ghost-like operators $c^{A}, b_{A}$ satisfying

$$
\begin{gather*}
b_{A} b_{B}+b_{B} b_{A}=0, \quad c^{A} C^{B}+c^{B} C^{A}=0,  \tag{9}\\
b_{A} c^{B}+c^{B} b_{A}=\delta_{A}^{B} . \tag{10}
\end{gather*}
$$

They are covariantly constant if considered in holonomic coordinates $x^{\mu}$ :

$$
\begin{gather*}
b_{\mu}(x)=e_{\mu}^{A}(x) b_{A}, \quad b_{v ; \mu} \equiv \partial_{\mu} b_{v}-\Gamma_{v \mu}^{\lambda} b_{\lambda}=0,  \tag{11}\\
c^{v}(x)=e_{A}^{v}(x) c^{A}, \quad c_{; \mu}^{v} \equiv \partial_{\mu} c^{v}+\Gamma_{\lambda, \mu}^{v} c^{\lambda}=0 . \tag{12}
\end{gather*}
$$

The definition of $Q$ as given by Okubo $\left[{ }^{3}\right]$ reads
Direct computations using the Bianchi identity

$$
\begin{gather*}
Q=c^{\mu}(x) \partial_{\mu}+\frac{1}{2} c^{\mu}(x) c^{\nu}(x) S_{\mu \nu}^{\lambda}(x) b_{\lambda}(x) .  \tag{13}\\
d \Omega^{A}+\omega_{B}^{A} \wedge \Omega^{B}=\Omega_{D}^{A} \wedge \omega^{D} \tag{14}
\end{gather*}
$$

confirm that $Q$ is nilpotent,

$$
\begin{equation*}
2 Q^{2} \equiv\{Q, Q\}=0 . \tag{15}
\end{equation*}
$$

In a sense, operator $Q$ is a generalization of the exterior differential operator. It has been known already for a long time that the BRST generator in the classical constrained dynamics can be considered as an exterior differential operator along orbits of the gauge group in the phase space of a mechanical system $\left[{ }^{6}\right]$ or in the configuration space of a gauge field $\left[{ }^{7}\right]$. Okubo's operator $Q$ has an essential difference from the latter ones: $Q$ is a generalization of the exterior differential in the differentiable manifold (space-time) with holonomic coordinates $x^{\mu}$, but the conventional BRST generator is a generalization of the exterior differential in the space of the gauge group.

## 3. GEODESIC MULTIPLICATION

Let us consider a 4-dimensional differentiable manifold $M$ with an affine connection (the space-time). Its geodesic lines (autoparallels) $x^{\mu}(t)$ must satisfy the following differential equations:

$$
\begin{equation*}
\frac{d^{2} x^{\mu}}{d t^{2}}+\Gamma_{v \rho}^{\mu} \frac{d x^{v}}{d t} \frac{d x^{\rho}}{d t}=0 \tag{16a}
\end{equation*}
$$

where $\Gamma_{v \rho}^{\mu}(x)$ denote the affine connection coefficients. Let $M_{e} \subset M$ be such a neighbourhood of $e \in M$ where geodesic lines emerging from $e$ do not intersect. In general, $M_{e}$ is a finite region of $M$ which does not contain singular points. Solutions of Eq. (16a) at initial values $x^{\mu}(0)=e^{\mu},\left.\frac{d x^{\mu}}{d t}\right|_{t=0}=X^{\mu}$ determine the exponential mapping $T_{e} M \rightarrow$ $\rightarrow M_{e}: X \rightarrow x:=\exp _{e} X:=x(1 ; X)$. A parallel transport mapping $\tau_{y}^{e}: T_{e} M \rightarrow T_{y} M$ along a geodesic line $y(s)$ emerging from $e$ is given as a solution of the Cauchy problem

$$
\begin{equation*}
\frac{d X^{\prime \mu}}{d s}+\Gamma_{v \rho}^{\mu} \frac{d y^{v}}{d s} X^{\prime \rho}=0, \quad X^{\prime \mu}(0)=X^{\mu} \tag{16b}
\end{equation*}
$$

Using the exponential mapping and the parallel transport mapping, the local geodesic multiplication of points $x, y \in M_{e}$ can be introduced [ ${ }^{8,9}$ ]:

$$
\begin{equation*}
x \cdot y \equiv L_{x} y \equiv R_{y} x=\left(\exp _{y}{ }^{\circ} \tau_{y}^{e} \circ{ }^{\circ} \exp _{e}^{-1}\right) x \tag{17}
\end{equation*}
$$

The local geodesic multiplication can be constructed in such a neighbourhood $M_{e}$ where all required exponential mappings and parallel transport operations are well-defined local diffeomorphisms.

In general, the local geodesic multiplication need not be commutative and associative. In the Riemann normal coordinates with the origin in $e$, equations for geodesic lines (16a) and parallel transport (16b) can be solved, using expansions in local coordinates. Direct calculations [ ${ }^{10}$ ] demonstrate that the commutator and the associator of the local geodesic multiplication are intimately related to the torsion $S_{v \rho}^{\mu}(x)$ and the curvature tensor $R_{v \rho \sigma}^{\mu}(x)$ of the space-time $M$ :

$$
\begin{gather*}
\left((y \cdot x)_{L}^{-1} \cdot(x \cdot y)\right)^{\mu}=C_{v \rho}^{\mu} x^{v} y^{\rho}+\ldots,  \tag{18a}\\
C_{v \rho}^{\mu}=2 S_{v \rho}^{\mu}(e)  \tag{18b}\\
\left((x \cdot(y \cdot z))_{L}^{-1}((x \cdot y) \cdot z)\right)^{\mu}=A_{v \rho \sigma}^{\mu} x^{v} y^{\rho} z^{\sigma}+\ldots,  \tag{19a}\\
A_{v \rho \sigma}^{\mu}=R_{v \rho \sigma}^{\mu}(e)-\nabla_{\sigma} S_{v \rho}^{\mu}(e) \tag{19b}
\end{gather*}
$$

Here $x_{L}^{-1}$ denotes the left inverse element of $x, x_{L}^{-1} \cdot x=e, \nabla_{v}$ is the covariant differentiation operator and dots mean higher-order terms.

The local geodesic multiplication converts the neighbourhood $M_{e}$ into the space of an algebraic system called the local geodesic loop [ ${ }^{11,12}$ ]. Point $e \in M$ is the unit element of the loop.

Local geodesic multiplication (17) determines the following infinitesimal left (L) and right (R) translation matrices:

$$
\begin{align*}
(x \cdot y)^{\mu} & =y^{\mu}+L_{v}^{\mu}(y) x^{v}+\ldots,\left.\quad L_{v}^{\mu}(y) \equiv \frac{\partial(x \cdot y)^{\mu}}{\partial x^{v}}\right|_{x=e}  \tag{20a}\\
& =x^{\mu}+R_{v}^{\mu}(x) y^{v}+\ldots,\left.\quad R_{v}^{\mu}(x) \equiv \frac{\partial(x \cdot y)^{\mu}}{\partial y^{v}}\right|_{y=e} \tag{20b}
\end{align*}
$$

At general coordinate transformations $x^{\prime}=x^{\prime}(x)$ they transform as bitensors, i.e. they are contravariant vectors in $x$ and covariant vectors in $e\left[{ }^{13}\right]$ :

$$
\begin{align*}
& L_{v^{\prime}}^{\mu^{\prime}}\left(x^{\prime}\right)=A_{\mu}^{\mu^{\prime}}(x) L_{v}^{\mu}(x)\left(A^{-1}\right)_{v^{\prime}}^{v}(e),  \tag{21a}\\
& R_{v^{\prime}}^{\mu^{\prime}}\left(x^{\prime}\right)=A_{\mu}^{\mu^{\prime}}(x) R_{v}^{\mu}(x)\left(A^{-1}\right)_{v^{\prime}}^{v}(e), \tag{21b}
\end{align*}
$$

where $A_{\mu}^{\mu^{\prime}}=\partial x^{\mu^{\prime}} / \partial x^{\mu}$. Although the upper and the lower indices of $L_{v}^{\mu}, R_{v}^{\mu}$ do not transform independently, these matrices can be considered as defining two preferred local vierbein fields in the neighbourhood $M_{e} \subset M$.

Let the differentiable manifold $M$ with an affine connection be torsionless, $S_{v \rho}^{\mu}(x)=0$, and endowed with a metric $g_{\mu v}(x)$ that is compatible with the connection,

$$
\begin{gather*}
\nabla_{\rho} g_{\mu \nu}=0  \tag{22a}\\
\Gamma_{\mu \nu}^{\lambda}=\frac{1}{2} g^{\lambda \sigma}\left(g_{\mu \sigma, v}+g_{v \sigma, \mu}-g_{\mu v, \sigma}\right) . \tag{22b}
\end{gather*}
$$

Then the main part of commutator (18) vanishes and the main part of associator (19) equals to the curvature tensor,

$$
A_{v \rho \sigma}^{\mu}=R_{v \rho \sigma}^{\mu}(e) .
$$

If the metric tensor $g_{\mu v}(x)$ is a solution of the Einstein equations, then the associator $A_{v \rho \sigma}^{\mu}$ equals to the value of the curvature tensor $R_{v \rho \sigma}^{\mu}(e)$ in the point $e$ of a physical (dynamical) space-time.

## 4. THE CONSTRUCTION OF LOCAL BRST-LIKE OPERATORS

Let us consider the space of the geodesic loop $M_{e}$ with the left and the right infinitesimal geodesic translation operators $L_{\mu}(x), R_{\mu}(x)$ as defined by Eq. (20),

$$
\begin{equation*}
L_{\alpha}(x)=L_{\alpha}^{v}(x) \frac{\partial}{\partial x^{v}}, \quad R_{\alpha}(x)=R_{\alpha}^{v}(x) \frac{\partial}{\partial x^{v}} \tag{23}
\end{equation*}
$$

They generate two vector field algebras,

$$
\begin{gathered}
{\left[L_{\alpha}(x), L_{\beta}(x)\right]=A_{\alpha \beta}^{\nu}(x) L_{\gamma}(x), \quad\left[R_{\alpha}(x), R_{\beta}(x)\right]=B_{\alpha \beta}^{\gamma}(x) R_{\gamma}(x),} \\
A_{\alpha \beta}^{\gamma}=\left(L_{\alpha}^{\mu}\left(\partial_{\mu} L_{\beta}^{v}\right)-L_{\beta}^{\mu}\left(\partial_{\mu} L_{\alpha}^{v}\right)\right) L_{\nu}^{\gamma} \\
B_{\alpha \beta}^{\gamma}=\left(R_{\alpha}^{\mu}\left(\partial_{\mu} R_{\beta}^{v}\right)-R_{\beta}^{\mu}\left(\partial_{\mu} R_{\alpha}^{v}\right)\right) R_{v}^{\nu} .
\end{gathered}
$$

In the case of left (right) invariant vector fields on a group manifold, we have $A_{\alpha \beta}^{\gamma}=-B_{\alpha \beta}^{\gamma}=$ const $=-C_{\alpha \beta}^{\gamma}$, and $\left[L_{\alpha}, R_{\beta}\right]=0$. In the case of a loop manifold these relations do not hold.

Both vector fields, $L_{\alpha}(x)$ and $R_{\alpha}(x)$, can be considered as a preferred local frame of reference that can be used for introducing a parallelizing torsion in the space of the geodesic loop. Let us suppose that the connection 1 -forms and the curvature 2 -forms vanish so that the second Cartan structure equation (5) is identically satisfied. Then the corresponding connection coefficients in local holonomic coordinates $L_{\mu \nu}^{\lambda}, R_{\mu \nu}^{\lambda}$ are given by vierbein fields according to Eq. (8):

$$
\begin{equation*}
L_{\mu \nu}^{\lambda}=L_{\sigma}^{\lambda} \partial_{\nu}\left(L^{-1}\right)_{\mu}^{\sigma}, \quad R_{\mu \nu}^{\lambda}=R_{\sigma}^{\lambda} \partial_{v}\left(R^{-1}\right)_{\mu}^{\sigma} . \tag{24}
\end{equation*}
$$

This means that we can introduce three different connection coefficients to the same neighbourhood $M_{e}$, the affine connection coefficients $\Gamma_{\mu \nu}^{\lambda}$ of the space-time and the connection coefficients $L_{\mu \nu}^{\lambda}, R_{\mu \nu}^{\lambda}$ induced by the local geodesic multiplication. Note that $L_{\mu \nu}^{\lambda}$ and $R_{\mu \nu}^{\lambda}$ depend in a sophisticated way on $\Gamma_{\mu \nu}^{\lambda}$, since the local geodesic multiplication (17) is determined by geodesics (16a) and parallel transport operator (16b).

From Eqs. (24) two torsion tensors of the space of the local geodesic loop can be calculated as well:

Note that the corresponding Cartan first structure equation (4) for them,

$$
d L^{\alpha}=\theta^{\alpha}, \quad d R^{\alpha}=\theta^{\alpha}
$$

$$
\mathrm{L} \quad \mathrm{R}
$$

can be considered as having been obtained from the structure equation of the Abelian group of Poincare translations,

$$
\left[P_{\alpha}, P_{\beta}\right]=0, \quad d P^{\alpha}=0
$$

as soft-group manifolds $\left[{ }^{14}\right],\left[{ }^{15}\right]$. However, in the general case of a soft group, the left (right) invariant vector fields $P_{\alpha}$ are substituted by arbitrary vector fields. In our case, the vector fields $L_{\alpha}, R_{\beta}$ are not arbitrary but are generated by the geodesic multiplication.

Now, following Okubo's construction, two local anticommuting nilpotent BRST-like operators can be defined using definition (13):

$$
\begin{align*}
& Q_{\mathrm{L}}^{Q}=c_{\mathrm{L}}^{\mu}(x) \partial_{\mu}+\frac{1}{2} c_{\mathrm{L}}^{\mu}(x) c_{\mathrm{L}}^{\nu}(x) \underset{\mathrm{L}}{\theta_{\mu \nu}^{\lambda}}(x) b_{\lambda}(x),  \tag{26}\\
& Q=c_{\mathrm{R}}^{\mu}(x) \partial_{\mu}+\frac{1}{2} c_{\mathrm{R}}^{\mu}(x) c_{\mathrm{R}}^{\nu}(x) \underset{\mathrm{R}}{\theta_{\mu \nu}^{\lambda}}(x) b_{\lambda}(x) . \tag{27}
\end{align*}
$$

They can be considered as generalizations of exterior derivative, not in the space-time but in the space of the geodesic loop. Okubo's construction implies
and indicates that

$$
\begin{equation*}
\{Q, Q\} \neq 0 \tag{28b}
\end{equation*}
$$

L R

The possible physical meaning of BRST operators $\underset{\mathrm{L}}{Q}$ and $\underset{\mathrm{R}}{Q}$ must follow from the properties of their action on suitably defined space of state vectors.

## 5. AN EXAMPLE: THE SPACE-TIME OF A WEAK PLANE GRAVITATIONAL WAVE

For explicitly determining the expression of the local geodesic product of two arbitrary points $x, y \in M_{e}$ of a manifold $M$ with a given affine connection $\Gamma_{\mu \nu}^{\lambda}(x)$, we need to integrate Eqs. (16a, b) for geodesic lines and parallel transport in arbitrary directions. It turns out to be analytically a rather complicated task even in the seemingly simple case of a 2 -sphere. To give an example that can be analytically worked out, let us obtain explicit expressions for the left and the right translation operators and the corresponding parallelizing torsions in the case of the physical space-time of the weak plane gravitational wave.

The metric tensor can be given as perturbations around the Minkowski metric:

$$
\begin{gathered}
g_{\mu \nu}=\eta_{\nu \mu}+h_{\mu v} \\
\eta_{\mu v}=\operatorname{diag}(-1,+1,+1,+1) .
\end{gathered}
$$

In the case of a polarized weak plane gravitational wave moving in the direction of $x$ the only nonzero components of $h_{\mu v}$ in the TT-gauge $\left[{ }^{16}\right]$ are

$$
h_{y y}=-h_{z z}=A \cos \omega(t-x) .
$$

Here $A=$ const., $A \ll 1$, is the wave amplitude, and all subsequent equations hold in the linear approximation in $A$.

In these coordinates the equation of a geodesic line with a tangent vector $X^{\mu}$ at a point $e$ can be easily integrated, yielding

$$
g^{\mu}(t)=e^{\mu}+X^{\mu} t+A U^{\mu}[\sin \omega B(\cos \omega C t-1)+\cos \omega B(\sin \omega C t-\omega C t)],
$$

where we have denoted

$$
\begin{gathered}
B=e^{0}-e^{1}, \quad C=X^{0}-X^{1}, \\
U^{0}=U^{1}=-\frac{1}{2 \omega C^{2}}\left[\left(X^{2}\right)^{2}-\left(X^{3}\right)^{2}\right], \\
U^{2}=-\frac{X^{2}}{\omega C}, \quad U^{3}=\frac{X^{3}}{\omega C} .
\end{gathered}
$$

Let us choose the point $e$ to be a unit element of the geodesic loop and let $g, h$ be two points from its neighbourhood. We denote by $X^{\mu}$ and $Y^{\mu}$ the tangent vectors of geodesic lines, joining the point $e$ with the points $g$ and $h$, respectively.

To calculate the product of the points $g$, $h$, we must integrate the corresponding equations of geodesics (16a) and of parallel transport (16b) of the tangent vector $X^{\mu}$. Direct but lengthy calculations give the following result:

$$
\begin{aligned}
k^{\mu} \equiv(h \cdot g)^{\mu}= & \\
=g^{\mu}+h^{\mu}-e^{\mu} & +A U^{\mu}\left[\left(\sin \omega B^{\prime}-\sin \omega B\right)(\cos \omega C-1)+\right. \\
& \left.+\left(\cos \omega B^{\prime}-\cos \omega B\right)(\sin \omega C-\omega C)\right]+ \\
& +\frac{A V^{\mu}}{2 D}[\cos \omega B-\cos \omega(B+D)]
\end{aligned}
$$

where

$$
\begin{gathered}
B^{\prime}=h^{0}-h^{1}, \quad D=Y^{0}-Y^{1} \\
V^{0}=V^{1}=X^{2} Y^{2}-X^{3} Y^{3}
\end{gathered}
$$

$$
V^{2}=C Y^{2}+D X^{2}, \quad V^{3}=-C Y^{3}-D X^{3}
$$

The previous equations contain no singularities if $C \equiv X^{0}-X^{1}=0$ or $D \equiv Y^{0}-Y^{1}=0$. The equations corresponding to these special cases can be obtained by just taking the limit $C \rightarrow 0$ or $D \rightarrow 0$.

From the expression of the geodesic multiplication we can calculate the matrices of the left and right translations, respectively (we have taken $e=0$ for simplicity):

$$
\begin{gathered}
L=I+A \xi\left[\begin{array}{cccc}
0 & 0 & \frac{y}{t-x} & -\frac{z}{t-x} \\
0 & 0 & \frac{y}{t-x} & -\frac{z}{t-x} \\
\frac{y}{t-x} & -\frac{y}{t-x} & 1 & 0 \\
-\frac{z}{t-x} & \frac{z}{t-x} & 0 & -1
\end{array}\right], \\
R=I+A \xi\left(\begin{array}{cccc}
\frac{y^{2}-z^{2}}{(t-x)^{2}} & -\frac{y^{2}-z^{2}}{(t-x)^{2}} & 0 & 0 \\
\frac{y^{2}-z^{2}}{(t-x)^{2}} & -\frac{y^{2}-z^{2}}{(t-x)^{2}} & 0 & 0 \\
\frac{2 y}{t-x} & -\frac{2 y}{t-x} & 0 & 0 \\
-\frac{2 z}{t-x} & \frac{2 z}{t-x} & 0 & 0
\end{array}\right)
\end{gathered}
$$

where

$$
\xi=\sin ^{2} \frac{\omega(t-x)}{2}
$$

and $I$ is the unit matrix. Taking into account that we are working in the linear approximation in $A$, their inverses differ only by the sign in front of the second terms.

Now let us calculate torsion tensors (25), corresponding to connections (24) in the space of the geodesic loop, obtained from the left and the right translations, respectively. To keep the expressions compact, we introduce some additional notations:

$$
\begin{gathered}
P=\frac{\omega \sin \omega(t-x)}{2(t-x)}, \\
Q=\frac{\sin ^{2} \frac{\omega(t-x)}{2}}{(t-x)^{2}}, \\
S=P-Q \\
T=(2 Q-P) \frac{y^{2}-z^{2}}{t-x} .
\end{gathered}
$$

Now we can write down the components of the torsion tensor obtained from the left translations

$$
\begin{aligned}
& { }_{\mathrm{L}}=A\left(\begin{array}{cccc}
0 & -y S & P(t-x)+y S & -Q(t-x) \\
y S & 0 & -y S & Q(t-x) \\
-P(t-x)-y S & y S & 0 & 0 \\
Q(t-x) & -Q(t-x) & 0 & 0
\end{array}\right), \\
& \theta^{\theta^{4}}=A\left(\begin{array}{cccc}
0 & Q(t-x)+z S & -z S & -P(t-x) \\
-Q(t-x)-z S & 0 & z S & 0 \\
z S & -z S & 0 & P(t-x) \\
P(t-x) & 0 & -P(t-x) & 0
\end{array}\right),
\end{aligned}
$$

and the right translations

$$
\begin{gathered}
\dot{R}_{\mathrm{R}}^{1}=\theta_{\mathrm{R}}^{2}=A\left(\begin{array}{cccc}
0 & 2 Q z+T & -T & -2 Q y \\
-2 Q z-T & 0 & T & 2 Q y \\
T & -T & 0 & 0 \\
2 Q y & -2 Q y & 0 & 0
\end{array}\right), \\
{\underset{R}{3}}_{\theta^{3}}=A\left(\begin{array}{cccc}
0 & -2 y S & 2 y S & -2 Q(t-x) \\
2 y S & 0 & -2 y S & 2 Q(t-x) \\
-2 y S & 2 y S & 0 & 0 \\
2 Q(t-x) & -2 Q(t-x) & 0 & 0
\end{array}\right), \\
\theta_{\mathrm{R}}^{4}=A\left(\begin{array}{cccc}
-2 Q(t-x)-2 z S & 2 Q(t-x)+2 z S & -2 z S & 0 \\
2 z S & 0 & 2 z S & 0 \\
0 & -2 z S & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{gathered}
$$

The BRST-like operators can be obtained in a straightforward way, substituting these values into Eqs. (26), (27). To ensure the correctness of the result, one can check that the Bianchi identities are indeed satisfied. Therefore it follows directly that $Q_{\mathrm{L}}^{2}=0, \underset{\mathrm{R}}{Q^{2}}=0$.

## 6. QUANTUM THEORY

In the conventional BRST quantization, the starting point is usually the Lagrangian of the physical system that determines the equations of motion and constraints. Upon quantization the Fourier coefficients in the solutions of the equations of motion are regarded as creation and annihilation operators. As distinct from this case, in our theory the solutions of the Einstein equations are not regarded as operators. Components of the classical curvature tensor act as structure functions of the geodesic loop. Its left and right translations determine two preferred local frames of reference that allow us to construct two BRST-like operators (26), (27). In a general case they do not commute, so the complete quantum theory must contain both of them. They possess a common system of ghost operators $c, b$ but do not contain any creation or annihilation operators of gravitons (quanta of the gravitational field). Consequently, Fock states of the corresponding quantum system contain only
ghost quanta. The background space-time remains to be a continuous differentiable manifold, possibly with a nontrivial topology.

Okubo considered in more detail the case where $b_{\mu}(x)$ are interpreted as annihilation operators. Then the vacuum state is defined by

$$
b_{\mu}(x) \mid 0>=0
$$

and the Fock space consists of vectors

$$
\left|\omega_{n}>=f_{\mu_{1} \ldots \mu_{n}}(x) c^{\mu_{1}}(x) \ldots c^{\mu_{n}}(x)\right| 0>
$$

He demonstrated that ghosts $c^{\mu}$ can be identified with differential forms $d x^{\mu}$ and the resulting cohomology of the BRST-like operator $Q$ turns out to be isomorphic to that of the standard de Rham cohomology of the underlying manifold $M$. In his theory, the local frame of reference $e_{A}{ }^{\mu}(x)$ is arbitrary. Frames of reference $L_{\alpha}^{\mu}(x)$ and $R_{\alpha}^{\mu}(x)$ used here have a definite geometrical meaning as the infinitesimal operators of the left and the right geodesic translations.

Let us define the vacuum state as

$$
b_{A} \mid 0>0 .
$$

It is equivalent to Okubo's definition, since the coordinate-dependent operators $b_{\mu}(x)$ are obtained from $b_{A}$ by means of multiplying it with a regular matrix, $b_{\mu}(x)=b_{A} e_{\mu}^{A}(x)$. In our case there are two preferred frames of reference, $L_{\alpha}^{\mu}(x)$ and $R_{\alpha}^{\mu}(x)$, and, respectively, two sets of annihilation operators ${\underset{\mathrm{L}}{ }}_{b_{\mu}}(x)=b_{\alpha}\left(L^{-1}\right)_{\mu}^{\alpha}(x)$ and $\underset{\mathrm{R}}{b_{\mu}}(x)=b_{\alpha}\left(R^{-1}\right)_{\mu}^{\alpha}(x)$
From the unique vacuum state

$$
b_{\alpha} \mid 0>=0
$$

we obtain two conditions,

$$
\begin{aligned}
& b_{\mathrm{L}}(x) \mid 0>=0, \\
& \underset{\mathrm{R}}{b_{\mu}(x) \mid 0>}=0 .
\end{aligned}
$$

Owing to the regularity of the matrices $L_{\alpha}^{\mu}(x)$ and $R_{\alpha}^{\mu}(x)$, postulating any of these conditions forces the validity of the other two. The Fock space can now be defined by

$$
\begin{aligned}
& \left|\omega_{n}>=f_{\alpha_{1} \ldots \alpha_{n}} c^{\alpha_{1}} \ldots c^{\alpha_{n}}\right| 0>= \\
& =f_{\mu_{1} \ldots \mu_{n}}(x) c_{\mathrm{L}}^{\mu_{1}}(x) \ldots c_{\mathrm{L}}^{\mu_{n}}(x) \mid 0>= \\
& =f_{\mathrm{R}} f_{\mu_{1} \ldots \mu_{n}}(x) c_{\mathrm{R}}^{\mu_{1}}(x) \ldots c_{\mathrm{R}}^{\mu_{n}}(x) \mid 0>
\end{aligned}
$$

Therefore, any state can be represented in terms of either only operators $c_{\mathrm{L}}^{\mu}(x)$ or only operators $c_{\mathrm{R}}^{\mu}(x)$. The coefficients $f_{\mu_{1} \ldots \mu_{n}}(x)$ are related through the transformation matrix from one preferred frame to anather

$$
\left(\Delta_{L R}\right)_{\mu}^{v}=\left(L^{-1}\right)_{\mu}^{\alpha} R_{\alpha}^{v} .
$$

Note that indices $\mu, v$ refer to the same coordinate system.
In the standard BRST quantization, the physical states are obtained as cohomologies of the BRST operator. By analogy, the cohomologies of our BRST-like operators are derived from the expressions for the action of nilpotent operators $\underset{\mathrm{L}}{Q} \underset{\mathrm{R}}{Q}($ see $(26),(27))$ in the space of state vectors $\left|\omega_{n}\right\rangle$ :

$$
\begin{aligned}
& \underset{\mathrm{L}}{Q} \mid \omega_{n}>=\underset{\mathrm{L}}{\partial_{\lambda} f_{\mu_{1}} \mu_{\mathrm{L}} \mu_{\mathrm{L}}^{\lambda} c_{\mathrm{L}}^{\mu_{1}} \ldots c_{\mathrm{L}}^{\mu_{n}} \mid 0>, ~}
\end{aligned}
$$

that have the same structure as differential $n$-forms. Notice that it is important to have here the same kind of creation operators in each expression. Then the cohomologies for ${\underset{L}{2}}_{Q}$ and $\underset{R}{Q}$ are analogous to the standard de Rham cohomologies. However, the physical states defined by these cohomologies are different because of the different creation operators associated with the same equivalence classes of coefficient functions. The reason for this is that ghosts reflect the symmetry under transformations and these can be either left or right shifts in our case.

There is also another possibility for defining ghost operators. We can identify the coordinate-dependent ghost operators.

$$
c^{\mu}(x)=\underset{\mathrm{L}}{c^{\mu}}(x)=\underset{\mathrm{R}}{c^{\mu}}(x), \quad b_{\mu}(x)=\underset{\mathrm{L}}{b_{\mu}}(x)=\underset{\mathrm{R}}{b_{\mu}}(x) .
$$

Then we have, in fact, started with two sets of coordinate-independent ghost operators, such that

$$
\begin{array}{ll}
L_{\alpha}^{\mu}(x) c_{\mathrm{L}}^{\alpha}=c^{\mu}(x), & \left(L^{-1}\right)_{\mu}^{\alpha}(x) b_{\alpha}=b_{\mu}(x), \\
R_{\alpha}^{\mu}(x) c_{\mathrm{R}}^{\alpha}=c^{\mu}(x), & \left(R^{-1}\right)_{\mu}^{\alpha}(x) b_{\alpha}=b_{\mu}(x) .
\end{array}
$$

The transition matrix between the left and right coordinate-independent ghost operators is now

$$
\left(\nabla_{L R}\right)_{\beta}^{\alpha}=\left(L^{-1}\right)_{\sigma}^{\alpha} R_{\beta}^{\alpha} .
$$

The Fock space is defined by

$$
\left|\omega_{n}>=f_{\mu_{1} \ldots \mu_{n}}(x) c^{\mu_{1}}(x) \ldots c^{\mu_{n}}(x)\right| 0>
$$

and the action of $\underset{L_{\mathrm{L}}}{Q}, \underset{R}{Q}$ is

$$
\underset{\mathrm{L}}{Q}\left|\omega_{n}>=\underset{R}{Q}\right| \omega_{n}>=\partial_{\lambda} f_{\mu_{1} \ldots \mu_{n}} c^{\lambda} C^{\mu_{1}} \ldots c^{\mu_{n}} \mid 0>.
$$

Its cohomology is analogous to the standard de Rham cohomology of differential forms.

## 7. DISCUSSION

There are several facts that indicate a possible role of geodesic
 space-time with orthonormal coordinates $x_{\mu}$, the right and left geodesic translations coincide, $R_{a}^{\text {rlat }}=L_{a}^{\text {flat }}$. The geodesic multiplication $x \rightarrow x \cdot a=$ $=a \cdot x$ describes a rigid shift of the space-time, $x^{\mu} \rightarrow x^{\mu}+a^{\mu}, a^{\mu}=$ const. The latter transformation is a Poincare translation. In the case of a curved space-time, matrices of infinitesimal left and right translations are different, as has explicitly been demonstrated in Sec. 5. They also have different meanings. An infinitesimal right translation given by Eq. (20b) describes an infinitesimal shift of a point $x$ (or a geodesic line $x(t)$ ) in the direction of $y$ (Fig. 1). An infinitesimal left translation given by Eq. (20a) describes an infinitesimal shift of a point $y$ in the direction that is parallel to the tangent vector $\frac{d x(e)}{d t}$ (Fig. 2).


Fig. 1. Right translation as an infinitesimal shift of a geodesic line $x(t)$ in the direction of $y$.


Fig. 2. Left translation as an infinitesimal shift of a point $y$ in the direction that is parallel to the tangent vector $\frac{d x(e)}{d t}$.

The geodesic loop has a close connection with the group of general coordinate transformations (diffeomorphisms) which is sometimes considered as the gauge group for the theory of gravity [ ${ }^{13}$ ]. Left and right translations of a loop generate a group that is called the Albert group of this loop $\left[{ }^{11,12}\right]$. Nonassociativity of the loop can be measured by the deviation from the unity of the following elements of its Albert group,

$$
\begin{equation*}
L(g, h) \equiv L_{g h}^{-1} L_{g} L_{h}, \quad R(g, h) \equiv R_{g h} R_{g}^{-1} R_{h}^{-1}, \quad M(g, h) \equiv R_{g} L_{h}^{-1} R_{g}^{-1} L_{h} \tag{29}
\end{equation*}
$$

In particular, the left and right translations (17) $L_{x}$ and $R_{x}$ of the local geodesic loop $M_{e}$ generate a subgroup in the group of space-time diffeomorphisms, the Albert group of $M_{e}$. Indeed, the left and right geodesic translations (17) determine diffeomorphisms given by

$$
\begin{equation*}
x \cdot y \equiv L_{x} y \equiv R_{y} x, \quad L_{x}, R_{y} \in \operatorname{Diff} M_{e} . \tag{30}
\end{equation*}
$$

The pair $(L, R)$ of the maps $x \rightarrow L_{x}, x \rightarrow R_{x}$ can be considered as a regular (bi)representation of the geodesic loop $M_{e}$. However, the analytical description of the Albert group of $M_{e}$ is extremely complicated due to the definition (17) of geodesic multiplication that involves integrations of geodesic and parallel transport equations.

In a full theory, the space-time may contain also quantized matter fields. According to the idea that the geodesic loop is the most natural generalization of the Poincare translations, matter fields may be described by suitable representations of the geodesic loop. In this way one can achieve the replacing of the infinite dimensional group of diffeomorphisms Diff $M$ by finite dimensional geodesic loop and its Albert group.

## ACKNOWLEDGEMENTS

The research described in this publication was in part made possible by grant No. LCROOO from the International Science Foundation. It was also supported by the Estonian Science Foundation grant No. 359. One of the authors (P. K. ) acknowledges the hospitality of the Institute of Theoretical Physics of Chalmers University of Technology, Göteborg, where a part of this work was carried out.

## REFERENCES

1. Henneaux, M. Phys. Repts., 1985, 126, 1-66.
2. Bars, I., Yankielowicz, S. Phys. Rev., 1987, D35, 3878-3889; Phys. Lett., 1987, B196, 329-335.
3. Okubo, S. Gen. Rel. Gravit., 1991, 23, 599-605.
4. Kuusk, P., Örd, J., Paal, E. J. Math. Phys., 1994, 35, 321-334.
5. Cho, Y. M. Phys. Rev., 1976, D14, 2521-2525.
6. Henneaux, M., Teitelboim, C. Commun. Math. Phys., 1988, 115, 213-230.
7. Bonora, L., Cotta-Ramusino, P. Commun. Math. Phys., 1983, 87, 589-603.
8. Kikkawa, M. J. Hiroshima Univ. Ser. A-1 Math., 1964, 28, 199-204.
9. Сабинин Л. В. Докл. АН СССР, 1977, 233, 800-803.
10. Акивис М. А. Сиб. матем. ж., 1978, 19, 243-253.
11. Albert, A. Trans. Am. Math. Soc., 1943, 55, 507-519.
12. Bruck, R. H. A Survey of Binary Systems. Springer, Berlin-Heidelberg-New York, 1971.
13. DeWitt, B. S. Dynamical Theory of Groups and Fields. Gordon and Breach, New York, 1965.
14. Ne'eman, Y., Takasugi, E., Thierry-Mieg, J. Phys. Rev., 1980, D22, 2371-2379.
15. Castellani, L. Int. J. Mod. Phys., 1992, A7, 1583-1625.
16. Misner, C. W., Thorne, K. S., Wheeler, J. A. Gravitation. Freeman, San Francisco, 1973.
17. Kuusk, P., Paal, E. Trans. Tallinn Techn. Univ., 1992, 733, 33-42.
18. Kuusk, P., Paal, E. Acta Appl. Math. (accepted for publication).

## GEODEETILINE KORRUTAMINE JA GEOMEETRILISED BRST-SARNASED OPERAATORID

Piret KUUSK, Jüri ÖRD, Eugen PAAL
Poincare translatsioonid on üldistatud mitteassotsiatiivseks algebraliseks süsteemiks, mida nimetatakse geodeetiliseks luubiks. Lokaalsete geodeetiliste vasak- ja paremnihete maatriksite abil on konstrueeritud kaks BRST-sarnast operaatorit. Näitena on arvutatud lokaalsete geodeetiliste nihete maatriksite ilmutatud kuju nõrka gravitatsioonilist tasalainet kirjeldavates aegruumides.

## ГЕОДЕЗИЧЕСКОЕ ПРОИЗВЕДЕНИЕ И ГЕОМЕТРИЧЕСКИЕ БРСТ-ПОДОБНЫЕ ОПЕРАТОРЫ

Пирет КУУСК, Юрий ЭРД, Эуген ПААЛ
Предложено обобщение трансляций Пуанкаре, приводящее к неассоциативной алгебраической системе, называемой геодезической лупой. При помощи матриц правых и левых геодезических трансляций сконструированы два БРСТ-подобных оператора. В качестве примера вычислены матрицы локальных левых и правых геодезических трансляций в пространстве-времени слабой плоской гравитационной волны,

