Proc. Estonian Acad. Sci. Phys. Math., 1995, 44, 4, 420—436
https://doi.org/10.3176/phys.math.1995.4.02

THE METHOD OF FINITE DIFFERENCES IN SOLVING AN INVERSE PROBLEM FOR OSCILLATION OF NONLINEAR VISCOELASTIC ROD

Jaan JANNO and Arvi RAVASOO

Eesti Teaduste Akadeemia Küberneetika Instituut (Institute of Cybernetics, Estonian Academy of Sciences), Akadeemia tee 21, EE-0026 Tallinn, Eesti (Estonia)

Presented by J. Engelbrecht

Received 9 September, 1994, accepted 19 June 1995

Abstract. An equation of motion of the one-dimensional homogeneous nonlinear viscoelastic medium is deduced. An inverse problem to determine two kernels from this equation is formulated. The data of the problem are measured from two independent experiments of rod oscillation. The inverse problem is reduced to a system of hyperbolic and second-kind Volterra equations. The method of finite differences is applied to this system.

Key words: hyperbolic equation, inverse problem, difference scheme.

1. INTRODUCTION

The determination of relaxation kernels of viscoelastic materials is a problem of both theoretical and practical interest. Often the kernels are reconstructed from experiments of wave propagation. This leads to inverse problems for viscoelastic equations of motion [1–4].

Several methods [1,2,5,6] are proposed to solve inverse problems related to linear viscoelastic models. Among them the method of optimization [1,2] is applicable in nonlinear cases too. The optimal solution is searched from a given finite-dimensional space so that the best accordance with initial data of a problem is attained. In linear cases the adjoint state functions (see e.g. [1]) are used to reduce the amount of the computational work. Unfortunately, in nonlinear cases one cannot use the adjoint state functions and the optimization technique turns out to be very time-consuming.

In this paper we shall present an essentially quicker method of solving an inverse problem related to a nonlinear viscoelastic equation of motion. The method is based on an approximation with finite differences. Analogous methods have successfully been applied to inverse problems of electrodynamics and seismology [7–9].

In Sec. 2, on the basis of ideas presented in [10], we deduce the equation of motion of one-dimensional nonlinear homogeneous viscoelastic medium (2.7). The equation includes two kernels, R_1 and R_2. In Sec. 3 we formulate an inverse problem to determine R_1, R_2 from two independent experiments of the rod oscillation. We shall reduce this problem to a system of hyperbolic and second-kind Volterra equations (Sec. 4), and apply the method of finite differences to the obtained system (Sec. 5). As a result, the kernels are determined in discrete values of their arguments.

The presented method is applicable to the linear model as well. This case together with a numerical example is analysed in Sec. 6.

2. ONE-DIMENSIONAL EQUATION OF MOTION FOR NONLINEAR VISCOELASTIC MEDIUM

We consider the initially isotropic and homogeneous medium, and assume that applied deformations are small but finite. The Lagrangian strain tensor E_{KL} has, in the Lagrangian rectangular coordinate system, the form

$$2E_{KL}=U_{K,L}+U_{L,K}+U_{M,L}U_{M,K}, \tag{2.1}$$

where U_K denotes the displacement vector. All indices in Eq. (2.1) run over 1,2,3, so the usual summation convention is used. The indices after comma indicate differentiation with respect to x_I. All displacement vectors, strain and stress tensors are dependent on time t and spatial coordinate x_I. We use the Kirchhoff pseudostress tensor T_{IJ}, and propose that it is the continuous and continuously differentiable functional in point x_I and it can be expanded in a Fréchet series [11]. Constitutive equations of various accuracy follow [10]. The first term in the series permits one to construct the linear constitutive equation of the medium. The second term permits to take into account the quadratic nonlinearity of the stress response to strain. The third term extends this response to cubic nonlinearity, and so on. Further we keep two first terms to this series under consideration.

Following the ideas presented in [10], we derive the constitutive equation of the viscoelastic medium in terms of the instantaneous part ${}_ET_{IJ}$ of Kirchhoff pseudostress tensor T_{IJ} and its regular part ${}_DT_{IJ}$:

$$T_{IJ}(t)={}_ET_{IJ}(t)-{}_DT_{IJ}(t). \tag{2.2}$$

The instantaneous part ${}_ET_{IJ}$ of Eq. (2.2)

$$\begin{aligned}{}_ET_{IJ}(t)=&\lambda(0)\delta_{IJ}E_{KK}(t)+2\mu(0)E_{IJ}(t)+3\nu_1(0)\delta_{IJ}E_{KK}(t)E_{LL}(t)+\\&+\nu_2(0)[\delta_{IJ}E_{KL}(t)E_{KL}(t)+2E_{KK}(t)E_{IJ}(t)]+\\&+3\nu_3(0)E_{IK}(t)E_{KJ}(t)\end{aligned} \tag{2.3}$$

coincides exactly with the constitutive equation of the nonlinear theory of elasticity [12]. Here δ_{IJ} denotes the Kroneker delta, constants $\lambda(0)$, $\mu(0)$ may be regarded as Lamé constants and $\nu_1(0)$, $\nu_2(0)$, and $\nu_3(0)$ as the third-order elastic constants. The regular part of Eq. (2.2) may be presented in a similar form

$$\begin{aligned}{}_DT_{IJ}(t)=&\int_0^t[\lambda_1(t-\tau)\delta_{IJ}E_{KK}(\tau)+2\mu_1(t-\tau)E_{IJ}(\tau)]d\tau+\\&+F_1(\tau,\tau,\tau)+F_2(\eta,t,\eta)+F_3(\tau,\tau,t)+F_4(\tau,\eta,\tau,\eta),\end{aligned}$$

$$\begin{aligned}F_j(\alpha,\beta,\gamma)=&\int_0^t\{3\nu_1^j(\tau-\alpha)\delta_{IJ}E_{KK}(\beta)E_{LL}(\gamma)+\\&+\nu_2^j(t-\alpha)[\delta_{IJ}E_{KL}(\beta)E_{KL}(\gamma)+2E_{KK}(\beta)E_{IJ}(\gamma)]+\\&+3\nu_3^j(t-\alpha)E_{IK}(\beta)E_{KJ}(\gamma)\}d\alpha,\end{aligned} \tag{2.4}$$

$$F_4(\tau,\eta,\tau,\eta)=\int_0^t\int_0^t \{3\nu_1^4\,(t-\tau,\ t-\eta)\delta_{IJ}E_{KK}(\tau)E_{LL}(\eta)+$$
$$+\nu_2^4\,(t-\tau,\ t-\eta)\,[\delta_{IJ}E_{KL}(\tau)E_{KL}(\eta)+2E_{KK}(\tau)E_{IJ}(\eta)]+$$
$$+3\nu_3^4\,(t-\tau,\ t-\eta)E_{IK}(\tau)E_{KJ}(\eta)\}\,d\tau\,d\eta,$$
$$\nu_j^2\,(t)=\nu_j^3\,(t),\quad j=1,\ 2,\ 3.$$

Equations (2.2), (2.3), and (2.4) represent the modified constitutive equation of the quasi-linear theory of viscoelasticity. The peculiarity of this equation is that kernel functions $\lambda_1(t)$, $\mu_1(t)$, and $\nu_i^j\,(t)$ in the regular part of it are related to stress and strain in a similar way as constants $\lambda(0)$, $\mu(0)$, and $\nu_i^j(0)$ are in the instantaneous part. The difference is in their dependence on time only. This facilitates the understanding of the physical meaning of numerous kernel functions.

Henceforth we shall consider the one-dimensional case $x\in\mathbf{R}$, and use the indices x, t for partial derivatives.

The one-dimensional constitutive equation for the nonlinear viscoelastic medium may be derived on the basis of Eq. (2.2) by using the equality $U_2=U_3=0$ in Eq. (2.1). We simplify this resulting constitutive equation for our purposes by setting functions $F_2(\eta,\ t,\ \eta)$, $F_3(\tau,\ \tau,\ t)$ and $F_2(\tau,\ \eta,\ \tau,\ \eta)$ equal to zero, and redenoting $U=U_1$. Introducing this constitutive equation into the equation of motion (ϱ_0 is density) [12]

$$[T_{11}(1+U_x)]_x-\varrho_0U_{tt}=0, \tag{2.5}$$

the one-dimensional nonlinear equation of motion for the nonlinear viscoelastic medium follows:

$$(\lambda(0)+2\mu(0))U_{xx}+3[\lambda(0)+2\mu(0)+2(\nu_1(0)+\nu_2(0)+$$
$$+\nu_3(0))]U_xU_{xx}-(\lambda_1(t)+2\mu_1(t))*U_{xx}-$$
$$-[\lambda_1(t)+2\mu_1(t)+6(\nu_1^1\,(t)+\nu_2^1\,(t)+\nu_3^1\,(t))]*(U_xU_{xx})-$$
$$-[(\lambda_1(t)+2\mu_1(t))*U_{xx}]U_x-[(\lambda_1(t)+2\mu_1(t))*U_x]U_{xx}=\varrho_0U_{tt}. \tag{2.6}$$

Here $*$ denotes the convolution integral:

$$[w_1(x,\cdot)*w_2(x,\cdot)](t)=\int_0^t w_1(x,t-s)w_2(x,s)\,ds.$$

Equation (2.6) may be presented in the form

$$U_{xx}+k_1U_xU_{xx}+R_1*U_{xx}+R_2*(U_xU_{xx})+(R_1*U_{xx})U_x+$$
$$+(R_1*U_x)U_{xx}=c^{-2}U_{tt},$$
$$k_0=(\lambda(0)+2\mu(0))^{-1},$$
$$k_1=3[1-2k_0(\nu_1(0)+\nu_2(0)+\nu_3(0))],$$
$$c^{-2}=k_0\varrho_0,$$
$$R_1(t)=-k_0[\lambda_1(t)+2\mu_1(t)],$$
$$R_2(t)=-k_0[\lambda_1(t)+2\mu_1(t)+6(\nu_1^1\,(t)+\nu_2^1\,(t)+\nu_3^1\,(t))]. \tag{2.7}$$

Equation (2.7) describes the motion of the medium with nonlinear elastic and nonlinear viscous properties. For the real media there may be cases when it is possible to describe the elastic or viscous behaviour of the media on the basis of the linear theory. This leads to simplifications in Eq. (2.7). There may be three principle types of simplified equations of motion:

1. Linear motion of the viscoelastic medium with linear elastic and linear viscous properties

$$U_{xx}+R_1*U_{xx}=c^{-2}U_{tt}; \qquad (2.8)$$

2. Nonlinear motion of the viscoelastic medium with nonlinear elastic and weak linear viscous properties

$$U_{xx}+k_1U_xU_{xx}+R_1*U_{xx}=c^{-2}U_{tt}; \qquad (2.9)$$

3. Nonlinear motion of the medium with nonlinear elastic and linear viscous properties

$$U_{xx}+k_1U_xU_{xx}+R_1*U_{xx}+R_1*(U_xU_{xx})+(R_1*U_{xx})U_x+ +(R_1*U_x)U_{xx}=c^{-2}U_{tt}. \qquad (2.10)$$

Equation (2.7) also describes the motion of linear or nonlinear purely elastic media ($R_1(t)=R_2(t)=0$) and the motion of purely viscous linear or nonlinear media ($\lambda(0)=\mu(0)=\nu_1(0)=\nu_2(0)=\nu_3(0)=0$).

3. SETUP OF THE INVERSE PROBLEM

Let us have a viscoelastic rod located between spatial points $x=0$, $x=X$. Consider two experiments that consist in releasing the pre-deformed rod and forcing its oscillations at the point $x=0$. Due to the equation of motion (2.7), the displacements of the rod U^i, $i=1$, 2, are governed by the mixed problems

$$(1+k_1U^i_x(x,t))U^i_{xx}(x,t)-c^{-2}U^i_{tt}(x,t)+\int_0^t R_1(t-s)U^i_{xx}(x,s)ds+$$
$$+\int_0^t R_2(t-s)U^i_x(x,s)U^i_{xx}(x,s)ds+\int_0^t R_1(t-s)[U^i_x(x,t)U^i_{xx}(x,s)+$$
$$+U^i_x(x,s)U^i_{xx}(x,t)]ds=0,\quad 0\leqslant x\leqslant X,\quad 0\leqslant t\leqslant T,\quad i=1,\ 2, \qquad (3.1)$$
$$U^i(x,0)=a_i(x),\quad U^i_t(x,0)=0,\quad 0\leqslant x\leqslant X,$$
$$U^i(0,t)=\varphi_i(t),\quad 0\leqslant t\leqslant T,\quad U^i(X,t)=0,\quad 0\leqslant t\leqslant T.$$

Here i is the experiment index. In addition to given initial and boundary conditions a_i, φ_i, we suppose that the histories of the deformation of the rod are measured at $x=0$, i.e. we know

$$U^i_x(0,t)=\psi_i(t),\quad 0\leqslant t\leqslant T,\quad t=1,\ 2. \qquad (3.2)$$

We formulate the following inverse problem: given a_i, φ_i, ψ_i ($i=1$, 2), k_1, c find $R_1(t)$, $R_2(t)$, $0\leqslant t\leqslant T$, such that the solutions U^i of the mixed problems (3.1) satisfy (3.2).

Conditions of solvability and uniqueness have been established for inverse problems that are formulated to more simple nonlinear [13] as well as linear [3, 14, 15] hyperbolic integrodifferential equations. Nevertheless, let us assume a priori that the solution of (3.1), (3.2) exists,

$$R_1,\ R_2\in C[0,\ T], \qquad (3.3)$$

and the corresponding solutions of (3.1) satisfy

$$U^i\in C^5([0,\ X]\times[0,\ T]), \qquad (3.4)$$

$$1+k_1U^i_{\ x}(x,t)+\int_0^t R_1(t-\tau)U^i_{\ x}(x,\tau)\,d\tau>0,$$
$$(x,t)\in[0,X]\times[0,T],\ i=1,\ 2. \tag{3.5}$$

Inequality (3.5) is necessary for the hyperbolity of Eq. (2.7). Physically it holds since model (2.7) is valid only in case of small deformations of the medium:

$$|U^i_{\ x}(x,t)|\ll 1,\ (x,t)\in[0,X]\times[0,T],\ i=1,\ 2 \tag{3.6}$$

(see beginning of Sec. 2 and [11]).

4. REDUCTION TO A SYSTEM OF HYPERBOLIC AND VOLTERRA EQUATIONS

4.1. Formulation of main results

Theorem 1. *Let* (3.3)—(3.5) *be satisfied and*

$$\det D(t;(U^1_{\ xx}(0,t),U^2_{\ xx}(0,t)))\neq 0,\quad 0\leqslant t\leqslant T, \tag{4.1.1}$$

where D is defined by (4.2.5). *Denote*

$$u_1=U^1_{\ x},\ u_2=U^2_{\ x},\ u_3=U^1_{\ xx},\ u_4=U^2_{\ xx},\ u_5=U^1_{\ xt},$$
$$u_6=U^2_{\ xt},\ u_7=U^1_{\ xxx},\ u_8=U^2_{\ xxx},\ u_9=U^1_{\ xxt},\ u_{10}=U^2_{\ xxt}, \tag{4.1.2}$$

Then Eqs. (4.3.1), (4.3.5), (4.4.5), (4.5.1), (4.5.4), (4.5.3) *are valid for* u_i, $1\leqslant i\leqslant 10$, R_1, R_2, *where the quantities* f_{ij}, f_i, $g^i_{\ i}$, $u_{0,i}$, $u_{1,i}$, K_i γ_i, $\hat{\gamma}_i$, d_{ij}, e_{ij}, $\overline{H}_i$, H_i, χ_i, Λ^i, Υ^i, ε^i, p_i, Θ, $\varkappa^j_{\ i}$ *are defined by formulae* (4.2.1)—(4.2.11).

The proof consisting in deriving formulae (4.3.1), (4.3.5), (4.4.5), (4.5.1), (4.5.4), (4.5.3) is located in Subsec. 4.3—4.5 of the paper.

Observe that the preliminary inverse problem (3.1)—(3.2) has been transformed to a system consisting of Volterra integral equations of the second kind (4.5.3) for R_1, R_2 and hyperbolic equations (4.3.1) for u_i, $1\leqslant i\leqslant 10$, together with the initial conditions (4.3.5) and boundary conditions (4.4.5), (4.5.1), (4.5.4). The left-hand-side boundary conditions of the first kind (4.5.1) and the third kind (4.5.4) are in an implicit form.

The assumption (4.1.1) in Theorem 1 is necessary in deriving the equations of the second kind (4.5.3) for R_1, R_2. However, condition (4.1.1) is hard to interpret from the physical point of view. The following theorem gives a more interpretable particular case.

Theorem 2. *Let* $U^i\in C^2([0,X]\times[0,T])$ *and* (3.5) *be valid. Assume that*

$$|a_2'(0)|>|a_1'(0)|\,\frac{1+\Gamma}{1-\Gamma},\ |a_i''(0)|\neq 0,\ i=1,\ 2, \tag{4.1.3}$$

where

$$|\psi_i(t)|=|U^i_{\ x}(0,t)|\leqslant\Gamma<1,\ 0\leqslant t\leqslant T,\quad i=1,\ 2 \tag{4.1.4}$$

(cf. (3.6)*). Then there exist* $\eta, \delta > 0$ *such that if*

$$|\varphi''(t)| \leqslant \delta \quad \text{for} \quad t \geqslant \eta, \tag{4.1.5}$$

then (4.1.1) *is satisfied.*

Consequently, choosing the functions a_1, a_2 in a suitable way (cond. (4.1.3)), and forcing the perturbations of the rod only in a short time interval (cond. (4.1.5)), inequality (4.1.1) holds.

The proof of Theorem 2 is located in Subsec. 4.6 of the paper.

4.2. Auxiliary quantities

For arguments $x \in [0, X]$, $t \in [0, T]$, $s \in [0, t]$, $\varrho_1, \varrho_2 \in \mathbf{R}$, $(z_1, \ldots, z_{10})$, $(y_1, \ldots, y_{10}) \in \mathbf{R}^{10}$, we define

$$f_{ij}(z_1, \ldots, z_{10}), \quad f_i(z_1, \ldots, z_{10}, y_1, \ldots, y_{10}),$$

$$g_i^1(x, z_1, \ldots, z_{10}), \quad g_i^2(x), \quad 1 \leqslant i, \ j \leqslant 10:$$

$$\begin{aligned}
&f_{77} = 4z_3, \quad f_{88} = 4z_4, \quad f_{97} = z_5, \quad f_{10,8} = z_6, \quad f_{99} = 3z_3, \quad f_{10,10} = 3z_4,\\
&f_1 = z_3 y_3, \quad f_2 = z_4 y_4, \quad f_3 = 3z_3 y_7, \quad f_4 = 3z_4 y_8,\\
&f_5 = 2z_3 y_9 + z_5 y_7, \quad f_6 = 2z_4 y_{10} + z_6 y_8,\\
&f_7 = 3z_3 y_7, \quad f_8 = 3z_8 y_8, \quad f_9 = 3z_7 y_9, \quad f_{10} = 3z_8 y_{10},\\
&g_5^1 = a_1'''(x)(1+z_1) + 2a_1''(x) z_3 + a_1'(x) z_7,\\
&g_6^1 = a_2'''(x)(1+z_2) + 2a_2''(x) z_4 + a_2'(x) z_8,\\
&g_9^1 = a_1^{iv}(x)(1+z_1) + 3a_1'''(x) z_3 + 3a_1''(x) z^7,\\
&g_{10}^1 = a_2^{iv}(x)(1+z_2) + 3a_2'''(x) z_4 + 3a_2''(x) z_8,\\
&g_5^2 = (a_1''(x))^2 + a_1'(x) a_1'''(x), \quad g_6^2 = (a_2''(x))^2 + a_2'(x) a_2'''(x),\\
&g_9^2 = 3a_1''(x) a_1'''(x) + a_1'(x) a_1^{iv}(x), \quad g_{10}^2 = 3a_2''(x) a_2'''(x) + a_2'(x) a_2^{iv}(x),
\end{aligned} \tag{4.2.1}$$

the functions f_{ij}, f_i, g_i^j, not included in (4.2.1), are equal to zero,

$$u_{0,i}, \ u_{1,i}, \ 1 \leqslant i \leqslant 10: \tag{4.2.2}$$

$$\begin{aligned}
&u_{0,1} = a_1'(x), \quad u_{0,2} = a_2'(x), \quad u_{0,3} = a_1''(x),\\
&u_{0,4} = a_2''(x), \quad u_{0,7} = a_1'''(x), \quad u_{0,8} = a_2'''(x),\\
&u_{0,5} = u_{0,6} = u_{0,9} = u_{0,10} = u_{1,1} = u_{1,2} = u_{1,3} = u_{1,4} = u_{1,7} = u_{1,8} = 0,\\
&u_{1,5} = c^2[(1 + k_1 a_1'(x)) a_1'''(x) + k_1 (a_1''(x))^2],\\
&u_{1,6} = c^2[(1 + k_1 a_2'(x)) a_2'''(x) + k_1 (a_2''(x))^2],\\
&u_{1,9} = c^2[(1 + k_1 a_1'(x)) a_1^{iv}(x) + 3k_1 a_1''(x) a_1'''(x)],\\
&u_{1,10} = c^2[(1 + k_1 a_2'(x)) a_2^{iv}(x) + 3k_1 a_2''(x) a_2'''(x)],
\end{aligned} \tag{4.2.3}$$

$$K_i(t;s;\varrho_1;\varrho_2;\ (z_1,\ldots,z_i);\ (y_1,\ldots,y_{i-4})),\ 1\leqslant i\leqslant 10,$$

$$\gamma_i(t;(y_1,\ldots,y_{i-4})),\ 1\leqslant i\leqslant 10,\ \hat{\gamma}_i(t),\ 1\leqslant i\leqslant 8:$$

$$K_1=K_2=K_5=K_6=0,$$

$$K_i=(\varrho_1(1+\psi_{p_t}(t))+\varrho_2\psi_{p_t}(s))z_i,\quad i=3,\ 4,$$

$$K_i=(\varrho_1(1+\psi_{p_t}(t))+\varrho_2\psi_{p_t}(s))z_i+2\varrho_1 z_{i-4}y_{i-4}+\varrho_2(z_{i-4})^2,$$

$$i=7,\ 8, \tag{4.2.4}$$

$$K_i=(\varrho_1(1+\psi_{p_t}(t))+\varrho_2\psi_{p_t}(s))z_i+\varrho_1(y_{i-6}z_{i-4}+z_{i-6}y_{i-4})+$$

$$+\varrho_2 z_{i-6}y_{i-6},\quad i=9,\ 10,$$

$$\gamma_1=\gamma_2=\gamma_5=\gamma_6=\hat{\gamma}_3=\hat{\gamma}_4=\hat{\gamma}_7=\hat{\gamma}_8=0,$$

$$\hat{\gamma}_1=\psi_1,\quad \hat{\gamma}_2=\psi_2,\quad \gamma_3=c^{-2}\varphi''_1,\quad \gamma_4=c^{2}\varphi''_2,\quad \hat{\gamma}_5=\psi'_1,\quad \hat{\gamma}_6=\psi'_2,$$

$$\gamma_i=c^{-2}\psi''_{p_t}(t)-k^1(y_{i-4})^2,\quad i=7,\ 8,$$

$$\gamma_i=c^{-2}\varphi'''_{p_t}(t)-k_1y_{i-6}y_{i-4},\quad i=9,\ 10,$$

$$(d_{ij})_{i,j=1,2}=D(t;\ (y_3,\ y_4))=$$

$$=\begin{pmatrix} a''_1(0)(1+\psi_1(t))+a'_1(0)y_3 & a'_1(0)a''_1(0)\\ a''_2(0)(1+\psi_2(t))+a'_2(0)y_4 & a'_2(0)a''_2(0)\end{pmatrix}, \tag{4.2.5}$$

$$(e_{ij})_{i,j=1,2}(t;\ (y_3,\ \ldots,\ y_8))=$$

$$=\begin{pmatrix} a'''_1(0)(1+\psi_1(t))+2a''_1(0)y_3+a'_1(0)y_7 & a'_1(0)a'''_1(0)+(a''_1(0))^2\\ a'''_2(0)(1+\psi_2(t))+2a''_2(0)y_4+a'_2(0)y_8 & a'_2(0)a'''_2(0)+(a''_2(0))^2\end{pmatrix}, \tag{4.2.6}$$

$$\overline{H}_i(t;s;\varrho_1;\varrho_2)=\varrho_1(1+\psi_{p_t}(t))+\varrho_2\psi_{p_t}(s),\quad i=9,\ 10,$$

$$H_i(t;s;\varrho_1;\varrho_2;(z_1,\ \ldots,\ z_i);\ (y_1,\ \ldots,\ y_8))=\varrho_1(y_{i-4}z_{i-2}+z_{i-4}y_{i-2})+$$

$$+\varrho_2 z_{i-4}z_{i-2}+\frac{e_{p_t,1}(d_{12}K_{10}-d_{22}K_9)+e_{p_t,2}(d_{21}K_9-D_{11}K_{10})}{\det D},\quad i=9,\ 10, \tag{4.2.7}$$

$$\chi_i(t;\ (y_1,\ \ldots,\ y_8))=c^{-2}\psi'''_{p_t}(t)-k_1y_{i-4}y_{i-2}+$$

$$+\frac{e_{p_t,1}(d_{12}\gamma_{10}-d_{22}\gamma_9)+e_{p_t,2}(d_{21}\gamma_9-d_{11}\gamma_{10})}{\det D},\quad i=9,\ 10,$$

$$\Lambda^i(t;s;\varrho_1;\varrho_2;(z_1,\ldots,z_{10});(y_1,\ldots,y_6))=\frac{d_{2,3-i}K_9-d_{1,3-i}K_{10}}{\det D},$$

$$i=1,\ 2, \tag{4.2.8}$$

$$\Upsilon^i(t;\ (y_1,\ldots,\ y_6))=\frac{d_{2,3-i}\gamma_9-d_{1,3-i}\gamma_{10}}{\det D},\quad i=1,\ 2,$$

$$\varepsilon^i(t;\ (y_3,\ \ldots,\ y_8))=\frac{d_{3-i,2}e_{i,1}-d_{3-i,1}e_{i,2}}{\det D},\quad i=1,\ 2.$$

Here

$$p_i=\frac{3}{2}+(-1)^i\frac{1}{2}. \tag{4.2.9}$$

Moreover, define

$$\Theta(\tau)=1, \quad \tau>0, \quad \Theta(\tau)=0, \quad \tau\leqslant 0, \tag{4.2.10}$$

$$\varkappa_1^2=\varkappa_2^2=\varkappa_5^2=\varkappa_6^2=\varkappa_7^2=\varkappa_8^2=\varkappa_3^1=\varkappa_4^1=\varkappa_9^1=\varkappa_{10}^1=0,$$

$$\varkappa_1^1=\varkappa_2^1=\varkappa_5^1=\varkappa_6^1=\varkappa_7^1=\varkappa_8^1=\varkappa_3^2=\varkappa_4^2=\varkappa_9^2=\varkappa_{10}^2=1. \tag{4.2.11}$$

4.3. Differentiated equations and initial conditions

Taking the derivatives ∂_x, ∂_x^2, $\partial_x\partial_t$, ∂_x^3, $\partial_x^2\partial_t$ from Eqs. (3.1) in view of notations (4.1.2), (4.2.1), (4.2.2), (4.2.9), (4.2.10), we come to the following system of equations:

$$[1+k_1u_{p_i}(x,t)+\int_0^t R_1(t-\tau)u_{p_i}(x,\tau)\,d\tau]\,(u_i)_{xx}(x,t)-c^{-2}(u_i)_{tt}(x,t)+$$

$$+\int_0^t [R_1(t-s)\,(1+u_{p_i}(x,t)+R_2(t-s)\,u_{p_i}(x,s)]\,(u_i)_{xx}(x,s)\,ds+$$

$$+L\{R_1;R_2;\ \sum_{j=1}^{10} f_{ij}(u(x,\tau_1))\cdot(u(x,\tau_2))_x+f_i(u(x,\tau_1),u(x,\tau_2))\}(t)+$$

$$+(g_i^1(x;u(x,t))+\Theta(i-8)\,a'_{i-8}(x)\,(u_{i-2}(x,t))_x)\cdot R_1(t)+$$

$$+g_i^2(x)R_2(t), \quad 0\leqslant x\leqslant X, \quad 0\leqslant t\leqslant T, \quad 1\leqslant i\leqslant 10. \tag{4.3.1}$$

Here

$$L\{R_1;R_2;w(\tau_1,\tau_2)\}(t)=k_1w(t,t)+\int_0^t R_2(t-s)\,w(s,s)\,ds+$$

$$+\int_0^t R_1(t-s)\,[w(t,s)+w(s,t)]\,ds. \tag{4.3.2}$$

Differentiating the initial conditions (3.1), and taking into account (4.1.2), (4.2.3), we immediately obtain

$$u_i(x,0)=u_{0,i}(x), \quad 1\leqslant i\leqslant 10, \quad (u_i)_t(x,0)=u_{1,i}(x), \quad i=1,\,2,\,3,\,4,\,7,\,8. \tag{4.3.3}$$

Let us set $t=0$ for $i=1,2,3,4$ in (4.3.1). According to (4.2.1), (4.2.2), we have

$$(u_i)_{xx}(1+k_1u_{p_i})-c^{-2}(u_i)_{tt}+f_i(u;u)\ |_{t=0}=0,$$

$$0\leqslant x\leqslant X, \quad i=1,\,2,\,3,\,4. \tag{4.3.4}$$

The complex $(u_i)_{tt}(x,0)$, $1\leqslant i\leqslant 4$ in (4.3.4) is equivalent to $(u_i)_t(x,0)$, $i=5$, 6, 9, 10 and the functions $u_i(x,0)$, $(u_i)_{xx}(x,0)$ are replaced with the derivatives of $u_{0,i}$ (cf. (4.3.3)). According to (4.2.3), Eqs. (4.3.4) become $(u_i)_t(x,0)=u_{1,i}(x)$, $i=5$, 6, 9, 10, which, together with (4.3.3), yield

$$u_i(x,0)=u_{0,i}(x), \quad (u_i)_t(x,0)=u_{1,i}(x), \quad 0\leqslant x\leqslant X, \quad 1\leqslant i\leqslant 10. \tag{4.3.5}$$

4.4. Right-hand-side boundary conditions

Let us set $x=X$ in Eqs. (3.1). We obtain

$$U^i_{xx}(X,t)+\int_0^t \frac{R_1(t-s)(1+k_1U^i_x(X,t))+R_2(t-s)U^i_x(X,s)}{1+k_1U^i_x(X,t)+\int_0^t R_1(t-\tau)U^i_x(X,\tau)\,d\tau}\times$$

$$\times U^i_{xx}(X,s)\,ds=c^{-2}U^i_{tt}(X,t),\quad 0\leqslant t\leqslant T,\quad i=1,\ 2. \tag{4.4.1}$$

Since $U^i(X,t)\equiv 0$ and (3.3)—(3.5) hold, the relations (4.4.1) are homogeneous Volterra equations of the second kind with bounded kernels. Therefore,

$$U^i_{xx}(X,t)=0,\quad 0\leqslant t\leqslant T,\quad i=1,\ 2. \tag{4.4.2}$$

Differentiating Eqs. (3.1) twice by x, we get Volterra equations for $U^i_{xxxx}(X,t)$ as well. These equations are also homogeneous because $U^i_{xxtt}(X,t)\equiv 0$ (cf. (4.4.2)). Thus,

$$U^i_{xxxx}(X,t)=0,\quad 0\leqslant t\leqslant T,\quad i=1,\ 2. \tag{4.4.3}$$

The boundary conditions $U^i(X,t)=0$, $0\leqslant t\leqslant T$, $i=1,\ 2$ together with (4.4.2), (4.4.3) imply

$$U^i_t(X,t)=U^i_{xx}(X,t)=U^i_{xxt}(X,t)=U^i_{xxxx}(X,t)=0,$$
$$0\leqslant t\leqslant T,\quad i=1,\ 2. \tag{4.4.4}$$

Boundary conditions (4.4.4) rewritten in notation (4.1.2), (4.2.11) are

$$\varkappa^1_i(u_i)_x(X,t)+\varkappa^2_i u_i(X,t)=0,\quad 0\leqslant t\leqslant T,\quad 1\leqslant i\leqslant 10. \tag{4.4.5}$$

4.5. Left-hand-side boundary conditions and equations for R_1, R_2

Let us derive the following formulae on the boundary $x=0$:

$$u_i(0,t)+$$
$$+\int_0^t \frac{K_i(t;s;R_1(t-s);R_2(t-s);(u_1,\ldots,u_i)}{1+k_1\psi_{p_i}(t)+\int_0^t R_1(t-\tau)\psi_{p_i}(\tau)\,d\tau}\times$$
$$\times\frac{(0,s);(u_1,\ldots,u_{i-4})(0,t))}{1+k_1\psi_{p_i}(t)+\int_0^t R_1(t-\tau)\psi_{p_i}(\tau)\,d\tau}\,ds=$$
$$=\frac{\gamma_i(t;(u_1,\ldots,u_{i-4})(0,t))}{1+k_1\psi_{p_i}(t)+\int_0^t R_1(t-\tau)\psi_{p_i}(\tau)\,d\tau}+\hat{\gamma}_i(t),\quad 0\leqslant t\leqslant T,\quad 1\leqslant i\leqslant 8, \tag{4.5.1}$$

$$d_{p_i,1}(t;(u_3,u_4)(0,t))R_1(t)+d_{p_i,2}R_2(t)=$$
$$=-u_i(0,t)\left(1+k_1\psi_{p_i}(t)+\int_0^t R_1(t-\tau)\psi_{p_i}(\tau)\,d\tau\right)-$$
$$-\int_0^t K_i(t;s;R_1(s);R_2(s);(u_1,\ldots,u_i)(0,t-s);(u_1,\ldots,u_{i-4})(0,t))\,ds+$$
$$+\gamma_i(t;(u_1,\ldots,u_{i-4})(0,t)),\quad 0\leqslant t\leqslant T,\quad i=9,\ 10. \tag{4.5.2}$$

In cases $i=1, 2, 5, 6$ formula (4.5.1) follows simply from (3.2) and (4.2.4). Formulae (4.5.1) for $i=3, 4, 7, 8$ and (4.5.2) are inferred from the Eqs. (3.1) and their first-order derivatives at $x=0$. We must only replace the functions U^i_{xx}, U^i_{xxx}, U^i_{xxt} with u_3, u_4, u_7, u_8, u_9, u_{10} and the functions $U^i_x(0,t)$, $U^i_{tt}(0,t)$, $U^i_{xtt}(0,t)$, $U^i_{ttt}(0,t)$ with suitable derivatives of φ_i, ψ_i. After that we can use notation (4.2.4).

Due to (4.1.1), we can transform the system of Volterra equations of the third kind (4.5.2) into the system of equations of the second kind:

$$R_i(t)=$$
$$=(-1)^i\Big[\int_0^t \Lambda^i(t;s;R_1(s);R_2(s);(u_1,\ldots,u_{10})(0,t-s);(u_1,\ldots$$
$$\ldots,u_6)(0,t))ds-\sum_{j=9}^{10}(-1)^j\left\{\frac{d_{3-p_j,\,3-i}}{\det D}\right\}(t;(u_3,u_4)(0,t))\,(1+k_1\psi_{p_i}(t)+$$
$$+\int_t^0 R_1(t-\tau)\psi_{p_i}(\tau)d\tau)u_j(0,t)-\Upsilon^{p_i}(t;(u_1,\ldots,u_6)(0,t))\Big],$$
$$0\leqslant t\leqslant T,\quad i=1,2. \tag{4.5.3}$$

Now let us compute the derivative $\partial_x\partial_t$ of (3.1), set $x=0$ and eliminate R_1, R_2 standing outside integrals by means of (4.5.3). Thereupon let us replace the derivatives of U^i with the functions $(u_i)_x$, $i=9, 10$, u_i, $1\leqslant i\leqslant 10$ or, if possible, with derivatives of φ_i, ψ_i. Then, in view of (4.2.6)—(4.2.8), we obtain

$$(u_i)_x(0,t)\cdot(1+k_1\psi_{p_i}(t)+\int_0^t R_1(t-\tau)\psi_{p_i}(\tau)d\tau)+2(k_1u_{i-6}(0,t)+$$
$$+\int_0^t R_1(t-\tau)u_{i-6}(0,\tau)d\tau)\cdot u_i(0,t)+\sum_{j=9}^{10}(-1)^j(1+k_1\psi_{p_j}(t)+$$
$$+\int_0^t R_1(t-\tau)\psi_{p_j}(\tau)d\tau)\varepsilon^{p_j}(t;(u_3,\ldots,u_8)(0,t))\cdot u_j(0,t)+$$
$$+\int_0^t \overline{H}_i(t;s;R_1(t-s)\,R_2(t-s))(u_i)_x(0,s)ds+$$
$$+\int_0^t H_i(t;s;R_1(t-s);R_2(t-s);(u_1,\ldots,u_{10})(0,s);(u_1,\ldots,u_8)(0,t))ds=$$
$$=\chi_i(t;(u_1,\ldots,u_8)(0,t)),\quad 0\leqslant t\leqslant T,\quad i=9,10. \tag{4.5.4}$$

4.6. Proof of Theorem 2

Since U^i_x, U^i_{xx} are continuous, $\psi_i(0)=a'_i(0)$, $U^i_{xx}(0,0)=a''_i(0)$, $i=1, 2$, we have

$$a''_i(0)(1+2a'_i(0))-|a''_i(0)+a'_i(0)|c_1(\eta)\leqslant a''_i(0)(1+\psi_i(t))+$$
$$+a'_i(0)U^i_{xx}(0,t)\leqslant a''_i(0)(1+2a'_i(0))+|a''_i(0)+a'_i(0)|c_1(\eta),$$
$$0\leqslant t\leqslant\eta,\quad i=1,2, \tag{4.6.1}$$

where

$$c_1(\eta)\to 0\quad\text{if}\quad\eta\to 0. \tag{4.6.2}$$

On the other hand, Eqs. (3.1) at $x=0$ yield

$$U^i_{xx}(0,t)+\int_0^t G^i(t,s)\,U^i_{xx}(0,s)\,ds=$$

$$=\frac{c^{-2}\varphi''_i(t)}{1+k_1\psi_{p_i}(t)+\int_0^t R_1(t-\tau)\,\psi_{p_i}(\tau)\,d\tau},\quad 0\leqslant t\leqslant T,\quad i=1,\,2. \tag{4.6.3}$$

Here G^i are some bounded functions. Applying the Gronwall's lemma to (4.6.3), estimating and making use of (3.5), (4.1.5), we obtain

$$|U^i_{xx}(0,t)|\leqslant c_3|\varphi''_i(t)|+c_4\int_0^t|\varphi''_i(\tau)|\,d\tau\leqslant c_2(\eta+\delta),\quad \eta<t\leqslant T, \tag{4.6.4}$$

$c_2=c_3=c_4=\text{const}$. Now it follows from (4.6.4), (4.1.4) that

$$(1-\Gamma)\,|a''_i(0)|-c_2(\eta+\delta)\,|a'_i(0)|\leqslant|a''_i(0)\,(1+\psi_i(t))+$$

$$+a'_i(0)\,U^i_{xx}(0,t)|\leqslant(1+\Gamma)\,|a''_i(0)|+c_2(\eta+\delta)\,|a'_i(0)|,$$

$$\eta<t\leqslant T,\quad i=1,\,2. \tag{4.6.5}$$

Choosing η, δ small enough, estimates (4.1.3), (4.6.1), (4.6.2), (4.6.5) imply (4.1.1).

5. DIFFERENCE SCHEME

Let us discretize problem (4.3.1), (4.3.5), (4.4.5), (4.5.1), (4.5.4), (4.5.3), making use of finite differences. We shall use the uniform mesh

$$\{(mh,n\tau):0\leqslant m\leqslant M,\ \ 0\leqslant n\leqslant N\},\quad \tau=TN^{-1},\quad h=XM^{-1} \tag{5.1}$$

and the notation

$$v^i_{mn}\approx u_i(mh,n\tau),\quad r^i_n\approx R_i(n\tau),\quad 0\leqslant m\leqslant M,\quad 0\leqslant n\leqslant N. \tag{5.2}$$

The derivatives will be approximated by the following formulae:

$$(u_i)_{xx}(mh,n\tau)\approx\frac{v^i_{m+1,n}-2v^i_{mn}+v^i_{m-1,n}}{h^2};$$

$$(u_i)_{tt}(mh,n\tau)\approx\frac{v^i_{m,n+1}-2v^i_{mn}+v^i_{m,n-1}}{\tau^2}; \tag{5.3}$$

$$(u_i)_x(mh,n\tau)\approx\frac{v^i_{m+1,n}-v^i_{m-1,n}}{2h};\quad (u_i)_x(0,n\tau)\approx\frac{v^i_{1,n}-v^i_{0,n}}{h}\ \text{ in (4.5.4)};$$

$$(u_i)_x(Mh,n\tau)\approx\frac{v^i_{M,n}-v^i_{M-1,n}}{h}\ \text{ in (4.4.5)};\quad (u_i)_t(mh,0)\approx\frac{v^i_{m,1}-v^i_{m,0}}{\tau}.$$

For integrals we shall use the rectangular rule:

$$\int_0^{n\tau}f(s)\,ds\approx\tau\sum_{l=1}^{n}f(l\tau). \tag{5.4}$$

Moreover, let us introduce the following functionals being discrete analogues of (4.3.2):

$$L_n: \mathbf{R}^{n^2+2n} \to \mathbf{R}: L_n\{\varrho^1_{l_1};\ \varrho^2_{l_1};\ w_{l_1l_2}: 1\leqslant l_1,\ l_2\leqslant n\} =$$

$$= k_1 w_{nn} + \tau\sum_{l=1}^{n}\varrho^2_{n-l+1}\, w_{ll} + \tau\sum_{l=1}^{n}\varrho^1_{n-l+1}\,(w_{ln}+w_{nl}),\quad n\geqslant 1. \tag{5.5}$$

Substituting approximations (5.2)—(5.4) into the initial conditions (4.3.5) and Eqs. (4.5.3), we immediately derive formulae for v^i_{m0}, v^i_{m1}, r^1_0, r^i_1:

$$v^i_{m0} = u_{0,i}(mh),\quad v^i_{m1} = v^i_{m0} + \tau u_{1,i}(mh),\quad 0\leqslant m\leqslant M,\quad 1\leqslant i\leqslant 10, \tag{5.6}$$

$$r^1_0 = (-1)^i\Bigg[\sum_{j=9}^{10}(-1)^{j+1}\left\{\frac{d_{3-p_j,3-i}}{\det D}\right\}(0;\ (v^3_{00},\ v^4_{00}))\times$$

$$\times(1+k_1\psi_{p_j}(0))\,v^j_{00} - \Upsilon^{p_i}(0;\ (v^1_{00},\ldots,v^6_{00}))\Bigg],\quad i=1,\ 2, \tag{5.7}$$

$$r^i_1 - (-1)^i\tau\Lambda^i(\tau;\tau;\ r^1_1;\ r^2_1;\ (v^1_{00},\ldots,v^{10}_{00});\ (v^1_{01},\ldots,v^6_{01})) =$$

$$= (-1)^i\Bigg[\sum_{j=9}^{10}(-1)^{j+1}\left\{\frac{d_{3-p_j,3-i}}{\det D}\right\}(0;\ (v^3_{01},\ v^4_{01}))\times$$

$$\times(1+(k_1+\tau r^1_0)\,\psi_{p_j}(\tau))\,v^j_{01} - \Upsilon^{p_i}(\tau;\ (v^1_{01},\ldots,v^6_{01}))\Bigg],\quad i=1,\ 2. \tag{5.8}$$

Let us define

$$S^i_{mn} = 1 + k_1 v^i_{mn} + \tau\sum_{l=1}^{n} r^i_{n-l}\, v^i_{ml},\quad 0\leqslant m\leqslant M,\quad 1\leqslant i\leqslant 10 \tag{5.9}$$

and

$$\hat{S}^i_{0,n+1} = 1 + k_1\psi_i((n+1)\tau) + \tau\sum_{l=1}^{n+1} r^i_{n+1-l}\,\psi_i(l\tau),\quad i=1,\ 2. \tag{5.10}$$

Let us fix $t=n\tau$, $x=mh$ in Eqs. (4.3.1) and $t=n\tau$ in conditions (4.4.5), (4.5.1), (4.5.4), (4.5.3). By means of approximations (5.2)—(5.4) in view of (5.5), (5.9), (5.10), we obtain the following discrete analogues:

$$v^i_{m,n+1} = c^2\Bigg[2v^i_{mn} - v^i_{m,n-1} + S^{p_i}_{mn}\frac{\tau^2}{h^2}(v^i_{m+1,n} - 2v^i_{mn} + v^i_{m-1,n}) +$$

$$+ \frac{\tau^3}{h^2}\sum_{l=1}^{n}(r^1_{n-l}(1+v^{p_i}_{mn}) + r^2_{n-l}v^{p_i}_{ml})(v^i_{m+1,l} - 2v^i_{ml} + v^i_{m-1,l}) + \tau^2\times$$

$$\times L_n\Bigg\{r^1_{l_1};\ r^2_{l_1};\ \sum_{j=1}^{10} f_{ij}(v_{m,l_1})\,\frac{v^j_{m+1,l_2} - v^j_{m-1,l_2}}{2h} +$$

$$+ f_i(v_{m,l_1};\ v_{m,l_2}): 1\leqslant l_1,\ l_2\leqslant n\Bigg\} + \tau^2\Big(g^1_i(mh;\ v_{mn}) +$$

$$+ \Theta(i-8)\,a'_{i-8}(mh)\,\frac{v^{i-2}_{m+1,n} - v^{i-2}_{m-1,n}}{2h}\Big)r^1_n + \tau^2 g^2_i(mh)\,r^2_n\Bigg],$$

$$1\leqslant m\leqslant M,\quad 1\leqslant i\leqslant 10, \tag{5.11}$$

$$v^i_{M,n+1}=\frac{\varkappa^1_i}{\varkappa^1_i+h\varkappa^2_i}v^i_{M-1,n+1},\quad 1\leqslant i\leqslant 10, \tag{5.12}$$

$$\hat S^{p_i}_{0,n+1}v^i_{0,n+1}+$$
$$+\tau K_i((n+1)\tau;\,(n+1)\tau;\,r^1_0;\,r^2_0;\,(v^1_{0,n+1},\dots,v^1_{0,n+1});\,(v^1_{0,n+1},\dots$$
$$\dots,v^{i-4}_{0,n+1}))=-\tau\sum_{l=1}^{n}K_i((n+1)\tau;\,l\tau;\,r^1_{n+1-l};\,r^2_{n+1-l};\,(v^1_{0l},\dots$$
$$\dots,v^i_{0l});\,(v^1_{0,n+1},\dots,v^{i-4}_{0,n+1}))+\gamma_i((n+1)\tau;\,(v^1_{0,n+1},\dots,v^{i-4}_{0,n+1})),$$
$$1\leqslant i\leqslant 8, \tag{5.13}$$

$$\hat S^{p_i}_{0,n+1}v^i_{0,n+1}-$$
$$-[2h(k_1v^{i-6}_{0,n+1}+\tau\sum_{l=1}^{n+1}r^1_{n+1-l}v^{i-6}_{0l})+\tau\overline{H}_i((n+1)\tau;\,(n+1)\tau;\,r^1_0;\,r^2_0)]\times$$
$$\times v^i_{0,n+1}+h\sum_{j=9}^{10}(-1)^j\hat S^{p_j}_{0,n+1}\varepsilon^{p_j}((n+1)\tau;\,(v^3_{0,n+1},\dots,v^8_{0,n+1}))v^j_{0,n+1}-$$
$$-h\tau H_i((n+1)\tau;\,(n+1)\tau;\,r^1_0;\,r^2_0;\,(v^1_{0,n+1},\dots,v^{10}_{0,n+1});$$
$$(v^1_{0,n+1},\dots,v^8_{0,n+1}))=$$
$$=h\tau\sum_{l=1}^{n}H_i((n+1)\tau;\,l\tau;\,r^1_{n+1-l};\,r^2_{n+1-l};\,(v^1_{0l},\dots,v^{10}_{0l});$$
$$(v^1_{0,n+1},\dots,v^8_{0,n+1}))+$$
$$+\tau\sum_{l=1}^{n}\overline{H}_i((n+1)\tau;\,l\tau;\,r^1_{n+1-l};\,r^2_{n+1-l})(v^i_{1l}-v^i_{0l})+$$
$$+\tau\overline{H}_i((n+1)\tau;\,(n+1)\tau;\,r^1_0;\,r^2_0)v^i_{1,n+1}-$$
$$-h\chi_i((n+1)\tau;\,(v^1_{0,n+1},\dots,v^8_{0,n+1})),\quad i=9,\ 10, \tag{5.14}$$

$$r^i_{n+1}-(-1)^i\tau\Lambda^i((n+1)\tau;\,(n+1)\tau;\,r^1_{n+1};\,r^2_{n+1};\,(v^1_{00},\dots,v^{10}_{00});$$
$$(v^1_{0,n+1},\dots,v^6_{0,n+1}))=$$
$$=(-1)^i\Big[\tau\sum_{l=1}^{n}\Lambda^i((n+1)\tau;\,l\tau;\,r^1_l;\,r^2_l;\,(v^1_{0,n+1-l},\dots,v^{10}_{0,n+1-l});$$
$$(v^1_{0,n+1},\dots,v^6_{0,n+1}))-$$
$$-\sum_{j=9}^{10}(-1)^j\Big\{\frac{d_{3-p_j,\,3-i}}{\det D}\Big\}((n+1)\tau;\,(v^3_{0,n+1},v^4_{0,n+1}))S^{p_j}_{0,n+1}v^j_{0,n+1}-$$
$$-\Upsilon^{p_i}((n+1)\tau;\,(v^1_{0,n+1},\dots,v^6_{0,n+1}))\Big],\quad i=1,\ 2. \tag{5.15}$$

The approximate solution (5.2) can be found from expressions (5.6)—(5.15). The explicit formulae (5.6), (5.7) and the system of the second order (5.8) yield the first two levels of the solution:

$$v^i_{ml},\quad 0\leqslant m\leqslant M,\quad 1\leqslant i\leqslant 10,\quad r^i_l,\quad i=1,\ 2\ (l=1,\ 2).$$

Having found n levels:

$$v^i_{ml},\quad 0\leqslant m\leqslant M,\quad 1\leqslant i\leqslant 10,\quad r^i_l,\quad i=1,\ 2\ (l=1,\ \dots,\ n),$$

we compute the sums S^i_{mn}, $\hat{S}^i_{0,n+1}$ by (5.9), (5.10). Thereupon we find $V^i_{m,n+1}$, $1\leqslant i\leqslant 10$ for $m=1, \ldots, M$ making use of the explicit formulae (5.11), (5.12). The boundary values $v^i_{0,n+1}$, $1\leqslant i\leqslant 8$ are computed solving Eqs. (5.13) in the order of increasing i. Finally, solving the systems of the second order (5.14), (5.15), we find $v^9_{0,n+1}$, $v^{10}_{0,n+1}$, r^1_{n+1} r^2_{n+1}, which completes the level $n+1$:

$$v^i_{m,n+1},\quad 0\leqslant m\leqslant M,\quad 1\leqslant i\leqslant 10,\quad r^i_{n+1},\quad i=1,2.$$

Equations (5.13) and the systems of the second order (5.8), (5.14), (5.15) are linear. Moreover, systems (5.8), (5.14), (5.15) have dominating principle diagonals if τ and h are sufficiently small (cf. (3.5), (5.10)).

Note that even if the solution of the differentiated inverse problem depends continuously on functions (4.2.1)—(4.2.8), and if the presented numerical algorithm converges, the full procedure of solving the inverse problem contains an ill-posed stage of computing the derivatives of a_i, φ_i, ψ_i in (4.2.1)—(4.2.8).

6. THE CASE OF LINEAR EQUATION

We can deduce the difference algorithms of solving inverse problems for the equations of motion (2.8)—(2.10) as well. Since only one kernel is to be determined in (2.8)—(2.10), it suffices to carry out one experiment described in Sec. 3.

Let us consider the following problem: Determine $R_1(t)$, $0\leqslant t\leqslant T$ such that the solution U of

$$U_{xx}(x,t)+\int_0^t R_1(t-s)\,U_{xx}(x,s)\,ds=c^{-2}U_{tt}(x,t)+F(x,t),$$

$$0\leqslant x\leqslant X,\quad 0\leqslant t\leqslant T, \tag{6.1}$$

$$U(x,0)=a(x),\quad U_t(x,0)=0,\quad 0\leqslant x\leqslant X,$$

$$U(0,t)=\varphi(t),\quad U(X,t)=0,\quad 0\leqslant t\leqslant T$$

satisfies the condition

$$U_x(0,t)=\psi(t),\quad 0\leqslant t\leqslant T. \tag{6.2}$$

In comparison with the linear equation of motion (2.8), Eq. (6.1) contains the absolute term F. This is necessary for simplifying a construction of a numerical example below.

We assume that $F(X,t)=0$, the solution of (6.1), (6.2) exists, $R_1\in C$, $U\in C^5$ and

$$a''(0)\neq 0. \tag{6.3}$$

Condition (6.3) is an analogue of (4.1.1) in Theorem 1.

Denote $u=U_{xxt}$. We have

$$u_{xx}(x,t)+\int_0^t R_1(t-s)\,u_{xx}(x,s)\,ds-c^{-2}u_{tt}(x,t)=F_{xxt}(x,t)-a^{iv}(x)R(t),$$

$$0\leqslant x\leqslant X,\quad 0\leqslant t\leqslant T, \tag{6.4}$$

$$u(x,0)=0,\quad u_t(x,0)=c^2(a^{iv}(x)-F_{xx}(x,0)), \tag{6.5}$$

$$\left[u_x(0,t)-\frac{a'''(0)}{a''(0)}u(0,t)\right]+\int_0^t R_1(t-s)\left[u_x(0,s)-\frac{a'''(0)}{a''(0)}u(0,s)\right]ds=$$

$$=c^{-2}\psi'''(t)-F_{xt}(0,t)-\frac{a'''(0)}{a''(0)}[c^{-2}\varphi'''(t)-F_t(0,t)],$$

$$0\leqslant t\leqslant T, \tag{6.6}$$

$$u(X,t)=0,\quad 0\leqslant t\leqslant T, \tag{6.7}$$

$$R_1(t)+\frac{1}{a''(0)}\int_0^t u(0,t-s)R_1(s)\,ds=$$

$$=\frac{1}{a''(0)}[F_t(0,t)+c^{-2}\varphi'''(t)-u(0,t)],\quad 0\leqslant t\leqslant T. \tag{6.8}$$

Indeed, Eq. (6.4) and the former initial condition in (6.5) are obtained by differentiating Eq. (6.1) and the condition $U_t(x,0)=0$. Taking the second derivative by x from (6.1), setting $t=0$ and replacing $U_{xxxx}(x,0)=a^{iv}(x)$, we obtain the latter condition in (6.5). Since $U(X,t)=F(X,t)=0$, Eq. (6.1) at $x=X$ is a homogeneous Volterra equation of the second kind for $U_{xx}(X,t)$. Hence, $U_{xx}(X,t)=0$. But this implies $U_{xxt}(X,t)=0$, and (6.7) follows. The relation (6.8) can be derived, differentiating Eq. (6.1) by t, setting $x=0$ and replacing $U_{ttt}(0,t)$ with $\varphi'''(t)$. Finally, taking the derivative $\partial_x\partial_t$ from (6.1), setting $x=0$, replacing $U_{xtttt}(0,t)$ with $\psi'''(t)$, and eliminating R_1 standing out of integrals by means of (6.8), we come to (6.6).

We have reduced the inverse problem (6.1), (6.2) to a system consisting of the mixed problem (6.4)—(6.7) for u and the Volterra equation (6.8) for R_1.

Using the uniform mesh (5.1), the notation

$$v_{mn}\approx u(mh,n\tau),\quad 0\leqslant m\leqslant M,\quad 0\leqslant n\leqslant N,$$

$$r_n\approx R_1(n\tau),\quad 0\leqslant n\leqslant N, \tag{6.9}$$

and the approximations (5.3), (5.4), we can deduce the following explicit difference scheme for (6.4)—(6.8):

$$v_{m0}=0,\quad v_{m1}=v_{m0}+\tau c^2(a^{iv}(mh)-F_{xx}(mh,0)),$$
$$0\leqslant m\leqslant M, \tag{6.10}$$

$$r_0=\frac{1}{a''(0)}[F_t(0,0)+c^{-2}\varphi'''(0)-v_{00}], \tag{6.11}$$

$$r_1=\left(1+\frac{\tau}{a''(0)}v_{00}\right)^{-1}\frac{1}{a''(0)}[F_t(0,\tau)+c^{-2}\varphi'''(\tau)-v_{01}], \tag{6.12}$$

$$v_{m,j+1}=2v_{mj}-v_{m,j-1}+\frac{c^2\tau^2}{h^2}(v_{m+1,j}-2v_{mj}+v_{m-1,j})+$$

$$+\frac{\tau^3}{h^2}\sum_{l=1}^{j}r_{j-l}(v_{m+1,l}-2v_{ml}+v_{m-1,l})-\tau^2[F_{xxt}(mh,j\tau)-r_ja^{iv}(mh)],$$

$$1\leqslant m\leqslant M-1,\quad j\geqslant 1, \tag{6.13}$$

$$v_{N,j+1}=0,\quad j\geqslant 1, \tag{6.14}$$

$$v_{0,j+1}=\left[\left(1+\frac{a'''(0)}{a''(0)}h\right)(1+\tau r_0)\right]^{-1}\cdot\left[\tau\sum_{l=1}^{j} r_{j+1-l}\left(v_{1l}-v_{0l}-\right.\right.$$

$$\left.-\frac{a'''(0)}{a''(0)}hv_{0l}\right)+(1+\tau r_0)v_{1,j+1}-hc^{-2}\psi'''((j+1)\tau)+hF_{xt}(0,(j+1)\tau)+$$

$$\left.+\frac{a'''(0)}{a''(0)}h[c^{-2}\varphi'''((j+1)\tau)-F_t(0,(j+1)\tau)]\right],\quad j\geqslant 1, \qquad (6.15)$$

$$r_{j+1}=\left(1+\frac{\tau}{a''(0)}v_{00}\right)^{-1}\cdot\left[-\frac{\tau}{a''(0)}\sum_{l=1}^{j}v_{0,j+1-l}r_l+\right.$$

$$\left.+\frac{1}{a''(0)}(F_t(0,(j+1)\tau)+c^{-2}\varphi'''((j+1)\tau)-v_{0,j+1})\right],\quad j\geqslant 1. \qquad (6.16)$$

Example. Let us take $c=1$, $T=X=1$,

$$R(t)=e^{-t},\quad U(x,t)=\frac{1}{36}(x-1)^4(t-1)^3, \qquad (6.17)$$

and compute the functions a, φ, ψ, F from the relations (6.1), (6.2):

$$a=\frac{1}{36}(x-1)^4,\quad \varphi=\frac{1}{36}(t-1)^4,\quad \psi=-\frac{1}{9}(t-1)^4,$$

$$F=\frac{1}{3}(x-1)^2(t-1)^3-\frac{1}{6}(x-1)^4(t-1)-\frac{1}{3}(x-1)^2\times$$

$$\times[(t-1)^3-3(t-1)^2+6(t-1)-6+6e^{-t}]. \qquad (6.18)$$

We have a problem (6.1), (6.2), (6.18) with the known exact solution (6.17). After solving this problem by means of the method (6.10)—(6.16), we are able to estimate the error of the method.

The results of computation are presented in Table where RER stands for the relative error of R.

M	N	t	RER
50	25	0.5	−0.09
50	25	1.0	0.24
100	50	0.5	−0.04
100	50	1.0	0.13
200	100	0.5	−0.02
200	100	1.0	0.07

The results show that the method has presumably the first order of accuracy.

ACKNOWLEDGEMENT

The present study was completed with the help of the grant of the Estonian Science Foundation.

REFERENCES

1. Tobias, T. Proc. Estonian Acad. Sci. Phys. Math., 1984, **33**, 182—187.
2. Woldersdorf, L. v. Math. Nachr., 1993, **161**, 203—217.
3. Grasselli, M. J. Math. Anal. Appl., 1992, **171**, 27—60.
4. Grasselli, M. Jap. J. Indust. Appl. Math., 1994, **11**, 131—153.
5. Janno, J. Numer. Func. Anal. Optim., 1992, **13**, 277—286.
6. Kabanikhin, S. I., Karchevsky, A. L., Lorenzi, A. J. Inv. Ill-posed Problems, 1993, **1**, 115—140.
7. Романов В. Г., Кабанихин С. И., Пухначева Т. П. Обратные задачи электродинамики. ВЦ СО АН СССР, Новосибирск, 1984.
8. Алексеев А. С., Добринский В. И. Математ. проблемы геофизики, 6, ч. 2, ВЦ СО АН СССР, Новосибирск, 1975, 7—53.
9. Ayobi, S. Invariant imbedding and inverse problems (Albuquerque, NM, 1990). SIAM, Philadelphia, PA, 1992, 187—208.
10. Ильюшин А. А., Победря Б. Е. Основы математической теории термовязкоупругости. Наука, Москва, 1970.
11. Ravasoo, A. Proc. Estonian Acad. Sci. Phys. Math., 1991, **40**, 122—129.
12. Eringer, A. C. Mechanics of Continua. John Wiley & Sons Inc, New York, 1967.
13. Lorenzi, A. Nonlinear Anal. TMA, 1994, **22**, 21—44.
14. Grasselli, M., Kabanikhin, S. I., Lorenzi, A. Sibirsk. Math. Zh., 1992, **33**, 58—68.
15. Bukhgeim, A. L. J. Inv. Ill-posed Problems, 1993, **1**, 193—205.

DIFERENTSMEETOD MITTELINEAARSE VISKOELASTSE VARDA VÕNKUMISEGA SEOTUD PÖÖRDÜLESANDE LAHENDAMISEL

Jaan JANNO, Arvi RAVASOO

On tuletatud ühemõõtmelise homogeense mittelineaarse viskoelastse keskkonna liikumisvõrrand ja püstitatud pöördülesanne kahe tuumafunktsiooni määramiseks sellest võrrandist. Ülesande algandmed on saadud kahest võnkuva vardaga tehtud eraldi eksperimendist. Pöördülesanne on taandatud hüperboolsete ja Volterra teist liiki võrrandite süsteemile. Saadud süsteemi lahendamiseks on rakendatud diferentsmeetodit.

МЕТОД КОНЕЧНЫХ РАЗНОСТЕЙ, ИСПОЛЬЗУЕМЫЙ ДЛЯ РЕШЕНИЯ ОБРАТНОЙ ЗАДАЧИ, СВЯЗАННОЙ С КОЛЕБАНИЯМИ НЕЛИНЕЙНОГО ВЯЗКОУПРУГОГО СТЕРЖНЯ

Яан ЯННО, Арви РАВАСОО

Выводится уравнение движения одномерной нелинейной однородной вязкоупругой среды, на основе которой формулируется обратная задача определения двух ядер. Начальные данные задачи получены из двух независимых экспериментов, проведенных с колеблющимся стержнем. Обратная задача сводится к системе, состоящей из гиперболических уравнений и уравнений Вольтерра второго рода. Для решения полученной системы применяется метод конечных разностей.