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SURFACES WITH A PARALLEL NORMAL CURVATURE TENSOR

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Abstract. The class of parallel submanifolds M^m in Euclidean spaces E^n, characterized by $\bar{\nabla}h=0$, is extended to the class of M^m in E^n with $\bar{\nabla}R^{\perp}=0$. A surface M^2 in E^n, satisfying $\bar{\nabla}R^{\perp}=0$, is proved to be locally either (i) a M^2 with flat $\nabla^{\perp}$ or (ii) a M^2 in a $E^4\subset E^n$ or in a $S^4(r)\subset E^5\subset E^n$, whose normal curvature ellipses have the same constant area. Here the additional condition for M^2 to be minimal yields: (i) M^2 lies minimally in a $E^3\subset E^n$, (ii) M^2 is a Veronese surface in $S^4(r)\subset E^5\subset E^n$ or its open part (minimal in $S^4(r)$).

Key words: normal curvature tensor, parallel surfaces, minimal surfaces.

1. INTRODUCTION

1.1. Parallel (or symmetric, extrinsically) submanifolds. Let M^m be a submanifold in a Euclidean space E^n and h its second fundamental form. A series of interesting investigations is made concerning M^m in E^n whose h is parallel with respect to the van der Waerden—Bortolotti connection $\bar{\nabla}=\nabla\oplus\nabla^{\perp}$, i.e. $\bar{\nabla}h=0$. For example, such a M^m is proved to have a totally geodesic Gauss image [1] and to be locally (and, if complete, globally) symmetric [2] with respect to its normal subspaces in E^n. The result [2] that every such complete M^m is a standardly imbedded symmetric R-space makes it possible to use the classification of these spaces [3]. The submanifolds M^m with $\bar{\nabla}h=0$ are called *parallel* [4] or, especially if they are complete, *symmetric* (extrinsically) [2, 5] also the *symmetric orbits* [6].

For surfaces the above-mentioned classification means that a parallel surface M^2 in E^n is an open part of one of such symmetric orbits as plane $E^2\subset E^n$, round cylinder $S^1(r)\times E^1\subset E^3\subset E^n$, Clifford surface $S^1(r')\times\times S^2(r'')\subset E^4\subset E^n$, sphere $S^2(r)\subset E^3\subset E^n$, Veronese surface $V^2(\tilde{r})\subset\subset S^4(r)\subset E^5\subset E^n$ $(\tilde{r}=r\sqrt{3})$. The first three have flat $\bar{\nabla}$ (i.e. ∇ and $\nabla^{\perp}$ are both flat), a $S^2(r)$ has flat $\nabla^{\perp}$ but nonflat ∇, a $V^2(\tilde{r})$ has nonflat $\nabla^{\perp}$ and nonflat ∇.

1.2. Semiparallel (or semisymmetric, extrinsically) submanifolds. A submanifold M^m in E^n, satisfying the integrability condition $R\circ h=0$ of the system $\bar{\nabla}h=0$, where $\bar{R}=R\oplus R^{\perp}$ is the curvature operator of $\bar{\nabla}$, is called *semiparallel* [7], also *semisymmetric* (extrinsically) [8, 9]. Geometrically such a M^m can be characterized as a 2nd-order envelope of parallel submanifolds [10].

A parallel M^m is obviously semiparallel. The general semiparallel submanifolds M^m in E^n are up to now completely classified and described only by $m=2$ (see [7]) and $m=3$ (see [11]), also by flat $\nabla^\perp$ [12, 13] (in particular, by $m=n-1$ and $m=n-2$).

For surfaces this gives that a semiparallel but nonparallel M^2 in E^n is either a M^2 with flat $\bar{\nabla}$ (i.e. $\bar{R}=0$) or a nontrivial 2nd-order envelope of Veronese surfaces; the existence of the latter by $n>5$ is proved in [14, 15].

An interesting subclass of semiparallel submanifolds M^m in E^n consists of 2-parallel submanifolds, characterized by $\bar{\nabla}(\bar{\nabla}h)=0$, i.e. the third fundamental form $\bar{\nabla}h$ is parallel. They are classified completely by $m=2$, by $m=3$, and by flat $\nabla^\perp$ (see [16, 17, 18]).

Note that semi-2-parallel submanifolds M^m in E^n with $\bar{R}\circ\bar{\nabla}h=0$ are not investigated properly yet, even by low dimensions.

1.3. Submanifolds with a parallel normal curvature tensor. These are the submanifolds M^m in E^n with $\bar{\nabla}R^\perp=0$, i.e. $R^\perp$ is parallel with respect to $\bar{\nabla}$. Their class is another extension of the class of parallel submanifolds, because $\bar{\nabla}h=0$ yields $\bar{\nabla}R^\perp=0$ due to the simple tensor algebraic relation between $R^\perp$ and h. All normally flat submanifolds, characterized by $R^\perp=0$ (i.e. $\nabla^\perp$ is flat), belong obviously to this class.

Here also the prefix "semi-" can be added, replacing the system $\bar{\nabla}R^\perp=0$ by its integrability condition $\bar{R}\circ R^\perp=0$. The submanifolds, satisfying the last condition, are called the submanifolds with a semiparallel normal curvature tensor. All semiparallel submanifolds belong to this class, because $\bar{R}\circ h=0$ implies $\bar{R}\circ R^\perp=0$.

1.4. Surfaces with parallel $R^\perp$. The submanifolds M^m in E^n of the new classes, introduced above, have not been investigated in general yet. The aim of this paper is to start with surfaces, i.e. with the case $m=2$. Below all surfaces with $\bar{\nabla}R^\perp=0$ are classified and described, also all minimal surfaces among them are found out.

Theorem A. *A surface M^2 in E^n has the parallel normal curvature tensor $R^\perp$ (i.e. $\bar{\nabla}R^\perp=0$) if and only if every its open connected part is either*
(i) *a surface with flat $\nabla^\perp$ (i.e. $R^\perp=0$), or*
(ii) *a surface in a $E^4\subset E^n$ or in a $S^4(r)\subset E^5\subset E^n$ whose normal curvature ellipses have the same area $k=$*const.

Here the normal curvature ellipse of a surface M^2 in E^n or $S^n(r)$ at a point $x\in M^2$ is the locus of end points of the normal curvature vectors $h(X, X)$ applied from x for all $X\in T_xM^2$, $\|X\|=1$, i.e. is $\{z\,|\,\overrightarrow{xz}=h(X, X)\}$.

The parallel surfaces are here included in (i), except the Veronese surfaces $V^2(\tilde{r})$, which belong to subclass (ii), because the normal curvature ellipses of a $V^2(\tilde{r})$ are congruent circles of the area $\tilde{r}^{-2}$. Moreover, a $V^2(\tilde{r})$ is a minimal surface of $S^4(r)$, $\tilde{r}=r\sqrt{3}$, i.e. every such circle is centred at $x\in V^2(\tilde{r})$ in spherical geometry.

The problem is whether there are any other minimal surfaces among the surfaces of Theorem A. The following theorem gives an answer to it.

Theorem B. *The only minimal surface of Theorem A are the minimal* $M^2 \subset E^3 \subset E^n$ *(type* (i)) *and the Veronese surfaces* $V^2(\tilde{r}) \subset \subset S^4(r) \subset E^5 \subset E^n$, $\tilde{r}=\sqrt{3}$ *(type* (ii)).

Remark 1. By means of the formulae below (Section 2.2.) it is easy to establish that every surface M^2 in E^n has a semiparallel normal curvature tensor, i.e. $\bar{R} \circ R^{\perp}=0$ is satisfied identically for the surfaces.

Remark. 2. The fact that the only minimal surface in a 4-dimensional space form with normal curvature ellipses of constant area k is a $V^2(r\sqrt{3})$ in $S^4(r)$, $k=r^{-2}$, was established in [19] more than thirty years ago. Since [19] is not easily available now, this fact is proved again in the course of the proof of Theorem B.

Remark 3. For semiparallel surfaces in $S^n(r)$ with nonflat $\nabla^{\perp}$ the following assertion is proved in [20, 21]: such a surface is minimal if and only if it is a Veronese surface in a $S^4(r) \subset S^n(r)$. In the proof a result of [22] is used that a minimal surface with the Gaussian curvature $\frac{1}{3}r^{-2}$ in $S^n(r)$ is a Veronese surface in $S^4(r)$; for $n=4$ this result is deduced already in [19] by more general assumptions (only constancy of the Gaussian curvature is needed).

2. APPARATUS

2.1. Adapted orthonormal frame bundle. Let M^m be a submanifold in E^n. The bundle $O(E^n)$ of orthonormal frames $\{x;\ e_1, \ldots, e^n\}$ (where a point $x \in E^n$ and its radius vector with respect to an origin $o \in E^n$ are identified) with derivation formulae

$$dx=e_I\omega^I,\quad de_I=e_J\omega_I^J,\quad \omega_I^J+\omega_J^I=0$$

(independent of o) and structure equations

$$d\omega^I=\omega^J \wedge \omega_J^I,\quad d\omega_I^J=\omega_I^K \wedge \omega_K^J$$

(obtained from the previous ones by exterior differentiation, where I, J, K etc. run $\{1, \ldots, n\}$) can be reduced to the adapted bundle $O(M^m, E^n)$ taking $x \in M^m$, $e_i \in T_xM^m$; i, j etc. run $\{1, \ldots, m\}$. Then $e_\alpha \in T_x^{\perp}M^m$; α, β etc. run $\{m+1, \ldots, n\}$ and $\omega^\alpha=0$ hold. Hence $\omega^i \wedge \omega_i^\alpha=0$ and thus $\omega_i^\alpha=h_{ij}^\alpha\omega^j$, $h_{ij}^\alpha=h_{ji}^\alpha$, where h: $(X, Y) \to \to e_\alpha h_{ij}^\alpha X^iY^j$ for $X=e_iX^i$, $Y=e_jY^j$ is the second fundamental form. The next differential prolongation gives $\bar{\nabla}h_{ij}^\alpha=h_{ijk}^\alpha\omega^k$, $h_{ijk}^\alpha=h_{ikj}^\alpha$, where $\bar{\nabla}h_{ij}^\alpha:=dh_{ij}^\alpha-h_{kj}^\alpha\omega_i^k-h_{ik}^\alpha\omega_j^k+h_{ij}^\beta\omega_\beta^\alpha$ are the components of $\bar{\nabla}h$, and further

$$\bar{\nabla}h_{ijk}^\alpha \wedge \omega^k=-h_{kj}^\alpha\Omega_i^k-h_{ik}^\alpha\Omega_j^k+h_{ik}^\beta\Omega_\beta^\alpha,$$

where

$$\Omega_i^j:=d\omega_i^j-\omega_i^k \wedge \omega_k^j=-\frac{1}{2}R^j_{i,kl}\,\omega^k \wedge \omega^l,$$

$$\Omega_\alpha^\beta:=d\omega_\alpha^\beta-\omega_\alpha^\gamma \wedge \omega_\gamma^\beta=-\frac{1}{2}R^\beta_{\alpha,kl}\,\omega^k \wedge \omega^l$$

are the curvature 2-forms of ∇ and $\nabla^{\perp}$, respectively. Here $\Omega^j_i = \omega^{\alpha}_i \wedge \omega^j_{\alpha}$, $\Omega^{\beta}_{\alpha} = \omega^i_{\alpha} \wedge \omega^{\beta}_i$, thus

$$R^j_{i,kl} = \sum_{\alpha} (h^{\alpha}_{ik} h^{\alpha}_{jl} - h^{\alpha}_{il} h^{\alpha}_{jk}),$$

$$R^{\beta}_{\alpha,kl} = \sum_{i} (h^{\alpha}_{ik} h^{\beta}_{il} - h^{\alpha}_{il} h^{\beta}_{ik});$$

these are the components of the curvature tensors R and $R^{\perp}$ of ∇ and $\nabla^{\perp}$, respectively. The identity between two expressions of Ω^{β}_{α} gives by exterior differentiation $\bar{\nabla} R^{\beta}_{\alpha,kl} \wedge \omega^k \wedge \omega^l = 0$, where

$$\bar{\nabla} R^{\beta}_{\alpha,kl} = dR^{\beta}_{\alpha,kl} + R^{\gamma}_{\alpha,kl} \omega^{\beta}_{\gamma} - R^{\beta}_{\gamma,kl} \omega^{\gamma}_{\alpha} - R^{\beta}_{\alpha,pl} \omega^p_k - R^{\beta}_{\alpha,kp} \omega^p_l \tag{1}$$

are the components of $\bar{\nabla} R^{\perp}$. Since $\bar{\nabla}$ works as a differential operator, it is obvious that $\bar{\nabla} h = 0$ yields $\bar{\nabla} R^{\perp} = 0$ (see Section 1.3.). By exterior differentiation these systems yield their integrability conditions, respectively, $\bar{R} \circ h = 0$ and $\bar{R} \circ R^{\perp} = 0$, where the left sides are componentwise correspondingly

$$h^{\alpha}_{kj} \Omega^k_i + h^{\alpha}_{ik} \Omega^k_j - h^{\beta}_{ij} \Omega^{\alpha}_{\beta} = 0$$

and

$$R^{\beta}_{\alpha,kj} \Omega^k_i + R^{\beta}_{\alpha,ik} \Omega^k_j + R^{\beta}_{\gamma,ij} \Omega^{\gamma}_{\alpha} - R^{\gamma}_{\alpha,ij} \Omega^{\beta}_{\gamma} = 0. \tag{2}$$

Due to the expression of $R^{\beta}_{\alpha,kl}$ the first yields the second, i.e. every semiparallel submanifold M^m in E^n satisfies $\bar{R} \circ R^{\perp} = 0$.

2.2. The case of a surface; the canonical frame field. Let further $m=2$. The derivation formulae are now

$$dx = e_i \omega^i, \quad de_i = e_j \omega^j_i + h_{ij} \omega^j,$$

where $h_{ij} = e_{\alpha} h^{\alpha}_{ij}$ and i, j etc. run $\{1, 2\}$. After the transformation

$$e'_1 = e_1 \cos\varphi + e_2 \sin\varphi, \quad e'_2 = -e_1 \sin\varphi + e_2 \cos\varphi,$$

one has

$$\omega^1 = \omega^{1'} \cos\varphi - \omega^{2'} \sin\varphi, \quad \omega^2 = \omega^{1'} \sin\varphi + \omega^{2'} \cos\varphi$$

and thus

$$h'_{11} = h_{11} \cos^2\varphi + 2h_{12} \sin\varphi \cos\varphi + h_{22} \sin^2\varphi,$$

$$h'_{12} = (h_{22} - h_{11}) \sin\varphi \cos\varphi + h_{12} (\cos^2\varphi - \sin^2\varphi),$$

$$h'_{22} = h_{11} \sin^2\varphi - 2h_{12} \sin\varphi \cos\varphi + h_{22} \cos^2\varphi.$$

Hence the vectors

$$A = \frac{1}{2} (h_{11} - h_{22}), \quad B = h_{12}, \quad H = \frac{1}{2} (h_{11} + h_{22})$$

transform according to the formulae

$$A' = A\cos 2\varphi + B\sin 2\varphi,\quad B' = -A\sin 2\varphi + B\cos 2\varphi,\quad H' = H,$$

which show that H is an invariant vector (the mean curvature vector) and the span$\{A, B\}$ is an invariant 2-dimensional subspace of $T_x^\perp M^2$.

The normal curvature ellipse (see Section 1.4.) of M^2 at x lies on a 2-plane trough w with $\overrightarrow{xw} = H$ and with the 2-direction span$\{A, B\}$, provided that $A \nparallel B$; here w is the centre of this ellipse. To see it one must take $X = e_1\cos\alpha + e_2\sin\alpha$ and calculate $h(X, X)$; the vectors A and B are the conjugate radius vectors of two points of this ellipse.

A simple calculation shows that

$$\langle A', B'\rangle = \frac{1}{2}(B^2 - A^2)\sin 4\varphi + \langle A, B\rangle\cos 4\varphi,$$

$$\frac{1}{2}(B'^2 - A'^2) = \frac{1}{2}(B^2 - A^2)\cos 4\varphi + \langle A, B\rangle\sin 4\varphi.$$

Hence the pair of conditions $\langle A, B\rangle = B^2 - A^2 = 0$ is invariant and characterizes the case when the normal curvature ellipse is a circle.

Otherwise there exists a φ_0 so that $\langle A', B'\rangle = 0$ and thus A' and B' are in principal directions of the ellipse. Let this transformation be done already so that in the following let $\langle A, B\rangle = 0$. Note that by $\varphi = \frac{\pi}{4}$ the roles of A and B can be interchanged.

In general $A \nparallel B$, $A^2 \neq B^2$ on an open part of M^2. At every point x of this part the frame can be partly canonized in $T_x^\perp M^2$, so that $A = ae_3$, $B = be_4$, $H = \alpha e_3 + \beta e_4 + \gamma e_5$, $a > b > 0$. Then

$$\omega_1^3 = (\alpha + a)\omega^1,\quad \omega_2^3 = (\alpha - a)\omega^2,$$

$$\omega_1^4 = \beta\omega^1 + b\omega^2,\quad \omega_2^4 = b\omega^1 + \beta\omega^2,$$

$$\omega_1^5 = \gamma\omega^1,\qquad \omega_2^5 = \gamma\omega^2, \tag{3}$$

$$\omega_1^\varrho = \omega_2^\varrho = 0;\quad \varrho, \sigma \text{ etc. run } \{6, \ldots, n\}. \tag{4}$$

The curvature 2-forms of this part are

$$-\Omega_2^1 = \Omega_1^2 = (a^2 + b^2 - H^2)\omega^1 \wedge \omega^2,\quad -\Omega_4^3 = \Omega_3^4 = -2ab\omega^1 \wedge \omega^2;$$

all other Ω_i^j, Ω_α^β are zero.

In an exceptional case, when $A \nparallel B$, $A^2 = B^2$, and so $a = b > 0$, the normal curvature ellipse is a circle and the frame cannot be canonized in this way, but the above equations still hold.

Another exceptional case, when $A \parallel B$ and thus $B = 0$, leads to $b = 0$. Then the normal curvature ellipse degenerates, in general if $A \neq 0$, into a segment and e_4, e_3 become free. If $H \nparallel A$, the frame can be partly canonized further, so that $\gamma = 0$; if $H \parallel A \neq 0$ and thus $\beta = 0$, the frame vectors in $T_x^\perp M$, except e_3, remain free. The particular case, when $A = B = 0$, leads to $a = b = 0$, the ellipse degenerates into a point, and if $H \neq 0$ by $H = \alpha e_3$, it can be made $\beta = \gamma = 0$; if here $H = 0$, then $a = b = \alpha = \beta = \gamma = 0$.

These considerations show that the above equations hold for a surface M^2 in E^n in all possible cases.

Now it is easy to prove the assertion in Remark 1. One has to take (2) by all values of indices α, β, i, j, to make the substitutions from the expressions of Ω_i^j and Ω_α^β to control that the results are identities.

3. PROOF OF THEOREM A

Let a surface M^2 in E^n have the parallel normal curvature tensor $R^{\perp}$, i.e. $\nabla R^{\perp}=0$ or, componentwise, $\nabla R^{\beta}_{\alpha,ik}=0$. Since all $R^{\beta}_{\alpha,ij}$ are zero, except maybe

$$R^4_{3,12}=-R^3_{4,12}=-2ab,$$

this condition due to (1) yields

$$d(ab)=0,\quad ab\omega^{\zeta}_3=ab\omega^{\zeta}_4=0;\quad \zeta,\ \eta \text{ etc. run } \{5,\ldots,n\}.$$

Thus either

(i) $b=0$, or

(ii) $ab=k=\text{const.}>0,\ \ \omega^{\zeta}_3=\omega^{\zeta}_4=0.$

Conversely, (i) or (ii) yields $\overline{\nabla} R^{\perp}=0$.

If (i) holds on an open part, then $\nabla^{\perp}$ is flat on this part.

Let the conditions of (ii) hold on some open part. Then after exterior differentation Eqs. (3) give $d\gamma\wedge\omega^1=d\gamma\wedge\omega^2$, thus $\gamma=\text{const.}$, but Eqs. (4) give in the same way $\gamma\omega^1\wedge\omega^{\rho}_5=\gamma\omega^2\wedge\omega^{\rho}_5=0$.

Let $\gamma=0$ on an open part. Then $\omega^{\zeta}_1=\omega^{\zeta}_2=0$ and this part lies in a $E_4\subset E_n$.

Let $\gamma\neq 0$ on some open part. Then $\omega^{\rho}_5=0$, hence this part lies in a $E^5\subset E^n$. On the other hand, for $z=x+\gamma^{-1}e_5$ it follows that $dz=$ $=dx+\gamma^{-1}(-\gamma dx)=0$. Thus the point z with this radius vector in E^5 is fixed and the considered part lies in a sphere $S^4(r)$ around this point with the radius $r=\gamma^{-1}$. The condition $ab=\text{const.}$ of (ii) means geometrically that the normal curvature ellipses have a constant area.

Conversely, for a surface in E^4 or $S^4(r)$, whose normal curvature ellipses have the same constant area, conditions (ii) are satisfied, thus $R^{\perp}$ is parallel. □

4. PROOF OF THEOREM B

Let a surface M^2 of Theorem A be minimal, i.e. $H=0$.

If (i) holds on an open part, then one can make $\gamma=0$ on this part and minimality means that $\alpha=\beta=0$. Thus $\omega^3_1=a\omega^1$, $\omega^3_2=-a\omega^2$, $\omega^{\varphi}_1=\omega^{\varphi}_2=0$, where φ, ψ etc. run $\{4,\ldots,n\}$. After exterior differentiation one obtains

$$da\wedge\omega^1+2a\omega^2_1\wedge\omega^2=0,\quad -2a\omega^2_1\wedge\omega^1+da\wedge\omega^2=0,$$

$$a\omega^1\wedge\omega^{\varphi}_3=0,\quad -a\omega^2\wedge\omega^{\varphi}_3=0.$$

If $a=0$ on an open part, this part is an open domain of a plane $E^2\subset E^n$. Otherwise on the open part with $a\neq 0$ one has $\omega^{\varphi}_3=0$ and this part is a minimal surface in a $E^3\subset E^n$.

In the case (ii) let M^2 be minimal in a $S^4(r)\subset E^5\subseteq E^n$. Then $\alpha=\beta=0$ and

$$\omega^3_1=a\omega^1,\quad \omega^3_2=-a\omega^2,\quad \omega^4_1=\frac{k}{a}\omega^2,\quad \omega^4_2=\frac{k}{a}\omega^1,$$

$$\omega_1^5=\gamma\omega^1, \quad \omega_2^5=\gamma\omega^2 \quad (\gamma=\text{const.}), \quad \omega_1^\rho=\omega_2^\rho=0,$$
$$\omega_3^5=\omega_4^5=\omega_3^\rho=\omega_4^\rho=0;$$

the case of minimal M^2 of (ii) in a E^4 is included here with $\gamma=0$. The equations of the first row yield after exterior differentiation

$$da\wedge\omega^1+\left(2a\omega_1^2-\frac{k}{a}\omega_3^4\right)\wedge\omega^2=0,$$

$$-\left(2a\omega_1^2-\frac{k}{a}\omega_3^4\right)\wedge\omega^1+da\wedge\omega^2=0,$$

$$da\wedge\omega^1-\left(2a\omega_1^2-\frac{a^3}{k}\omega_3^4\right)\wedge\omega^2=0,$$

$$\left(2a\omega_1^2-\frac{a^3}{k}\omega_3^4\right)\wedge\omega^1+da\wedge\omega^2=0;$$

the other equations give identities. Thus

$$\left[4a\omega_1^2-\left(\frac{k}{a}+\frac{a^3}{k}\right)\wedge\omega_3^4\right]\wedge\omega^2=0, \quad \left[4a\omega_1^2-\left(\frac{k}{a}+\frac{a^3}{k}\right)\wedge\omega_3^4\right]\wedge\omega^1=0.$$

So

$$4ka^2\omega_1^2=(k^2+a^4)\omega_3^4.$$

This, by exterior differentiation, gives

$$da\wedge\left(2a\omega_1^2-\frac{a^3}{k}\omega_3^4\right)=\left[\gamma^2a^2-\frac{3}{2}(k^2+a^4)\right]\omega^1\wedge\omega^2,$$

where, recall, γ and k are some constants.

On the other hand, the exterior equations above yield due to Cartan lemma

$$da=a_1\omega^1+a_2\omega^2,$$

$$2a\omega_1^2-\frac{a^3}{k}\omega_3^4=-a_2\omega^1+a_1\omega^2,$$

hence

$$a_1^2+a_2^2+\frac{3}{2}(k^2+a^4)=\gamma^2a^2.$$

If $\gamma=0$, this is a contradiction with $a>0$, thus in E^4 such a M^2 does not exist.

In $S^4(r)$, $r=\gamma^{-1}$, there must be

$$\gamma^2a^2-\frac{3}{2}(k^2+a^4)\geqslant 0.$$

Here in the case "=" one has $\gamma^2=\frac{3}{2}\left(\frac{k^2}{a^2}+a^2\right)$, $a_1=a_2=0$, thus $2a\omega_1^2-\frac{a^3}{k}\omega_3^4=0$. But on the other hand, from the two first exterior equations above $2a\omega_1^2 - \frac{k}{a}\omega_3^4 = 0$, hence $\left(\frac{a^3}{k} - \frac{k}{a}\right)\omega_3^4 = 0$. If $\omega_3^4 = 0$ on an open part, the exterior differentiation gives a contradiction $2k\omega^1 \wedge \omega^2 = 0$. So it must be $a^2 = k$, thus $2\omega_1^2 = \omega_3^4$, $\gamma = a\sqrt{3}$. These relations characterize the Veronese surface $V(\tilde{r}) \subset S^4(r)$, $\tilde{r} = r\sqrt{3} = a^{-1} = \sqrt{k}$.

It remains to show that the case of "$>$" leads to a contradiction. This is the case of $a_1^2 + a_2^2 > 0$ or, equivalently, $a \neq \text{const}$.

Denoting $\left[\gamma^2 a^2 - \frac{3}{2}(k^2 + a^4)\right]^{\frac{1}{2}} = h(a)$, one has

$$a_1 = h(a)\cos\alpha, \quad a_2 = h(a)\sin\alpha.$$

If to substitute this into the expressions of da and $2a\omega_1^2 - \frac{a^3}{k}\omega_3^4 = \frac{1}{2ka}(k^2 - a^4)\omega_3^4$ and then to differentiate exteriorily, the results are

$$(d\alpha + \omega_1^2) \wedge (-\omega^1 \sin\alpha + \omega^2 \cos\alpha) = 0,$$

$$-\frac{1}{a}\left[h^2(a)\frac{k^2 + 3a^4}{k^2 - a^4} + (k^2 - a^4)\right]\omega^1 \wedge \omega^2 =$$

$$= h(a)\left[\frac{dh(a)}{da}\omega^1 \wedge \omega^2 - (d\alpha + \omega_1^2) \wedge (\omega^1 \cos\alpha + \omega^2 \sin\alpha)\right]$$

and yield

$$d\alpha + \omega_1^2 = p(a)(-\omega^1 \sin\alpha + \omega^2 \cos\alpha),$$

where

$$p(a) = -\frac{1}{ah(a)}\left[h^2(a)\frac{k^2 + 3a^4}{k^2 - a^4} + (k^2 - a^4)\right] - \frac{dh(a)}{da}.$$

Now the exterior differentiation gives

$$-\gamma^2 + a^2 + \frac{k^2}{a^2} = \frac{dp(a)}{da}h(a) - p^2(a),$$

but this is not an identity and thus is a contradiction to $a \neq \text{const}$. □

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PARALLEELSE NORMAALKÕVERUSE TENSORIGA PINNAD

Ülo LUMISTE

On tõestatud, et kui pind M^2 eukleidilises ruumis E^n rahuldab tingimust $\bar{\nabla} R^{\perp}=0$, kus $R^{\perp}$ on normaalkõveruse tensor ja $\bar{\nabla}$ on van der Waerdeni—Bortolotti seostus, siis kas (i) $R^{\perp}=0$ või (ii) pinna M^2 normaalkõveruse ellipsid on konstantse pindalaga ning $M^2 \subset E^4 \subset E^n$ või $M^2 \subset S^4(r) \subset E^5 \subset E^n$. On leitud kõik minimaalpinnad selliste pindade seas. Klassikalistele minimaalpindadele $M^2 \subset E^3 \subset E^n$ lisanduvad vaid Veronese pinnad $V^2(\tilde{r}) \subset S^4(r) \subset E^5 \subset E^n$ (kui sfääri $S^4(r)$ minimaalpinnad; $\tilde{r}=r\sqrt{3}$).

ПОВЕРХНОСТИ С ПАРАЛЛЕЛЬНЫМ ТЕНЗОРОМ НОРМАЛЬНОЙ КРИВИЗНЫ

Юло ЛУМИСТЕ

Доказано, что если поверхность M^2 в евклидовом пространстве E^n удовлетворяет условию $\bar{\nabla} R^{\perp}=0$, где $R^{\perp}$ есть тензор нормальной кривизны и $\bar{\nabla}$ есть связность ван дер Вардена—Бортолотти, то либо (i) $R^{\perp}=0$, либо (ii) M^2 обладает эллипсами нормальной кривизны постоянной площади и лежит или в $E^4 \subset E^n$, или в $S^4(r) \subset E^5 \subset E^n$. Найдены все минимальные M^2 среди таких поверхностей. К классическим минимальным поверхностям $M^2 \subset E^3 \subset E^n$ прибавляются лишь поверхности Веронезе $V^2(\tilde{r}) \subset S^4(r) \subset E^5 \subset E^n$ (последние как минимальные в $S^4(r)$; $\tilde{r}=r\sqrt{3}$).