https://doi.org/10.3176/phys.math.1993.4.07

Proc. Estonian Acad. Sci. Phys. Math., 1993, 42, 4, 347—349

ON LOOP SEMIRINGS

W. B. VASANTHA KANDASAMY

Department of Mathematics, Indian Institute of Technology, Madras-600 036, India

Presented by G. Liidja

Received November 9, 1992; revised April 19, 1993; accepted June 17, 1993

POOLRINGIDE LUUPIDEST. W. B. VASANTHA KANDASAMY

О ЛУПАХ ПОЛУКОЛЕЦ. В. Б. ВАСАНТА КАНДАСАМИ

Key words: loop, semiring.

In this note we initiate the study of loops over semirings, which we call loop semirings analogously to group semirings. Loop semirings are nonassociative semirings. We study some properties of loop semirings.

The author in [1] calls a nonempty set of elements of L a loop if in L there is defined a binary operation called the product and denoted by $\cdot$ such that

(i) $a, b \in L$ implies $a \cdot b \in L$;

(ii) for every pair of elements a, b in L there is one and only one x in L such that $a \cdot x = b$ and one and only one y in L such that $y \cdot a = b$ in L;

(iii) there exists an element $e \in L$ such that $a \cdot e = e \cdot a = a$ for all $a \in L$, e called the identity element of L.

Usually a loop is denoted by $(L, \cdot, e)$. For further properties refer to [1].

Louis Dale in [2] calls a nonempty set S to be a semiring if in S there are defined two operations, denoted by $+$ and $\cdot$, such that for all a, b, c in S

(i) $a+b$ is in S;

(ii) $a+b=b+a$;

(iii) $(a+b)+c=a+(b+c)$;

(iv) there is an element 0 in S such that $a+0=a$ (for every a in S);

(v) $a \cdot b$ is in S;

(vi) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

(vii) $a \cdot (b+c) = a \cdot b + a \cdot c$ and
$(b+c) \cdot a = b \cdot a + c \cdot a$.

A semiring S is said to be commutative if $a\cdot b=b\cdot a$ for all a, b in S. A semiring S has zero divisors if $ab=0\,(a\neq 0, b\neq 0)$, $a, b\in S$. A semiring S is said to have identity if there exists $1\in S$ such that $1\cdot x=x\cdot 1=x$ for all x in S. A semiring S is said to be a strict semiring if a, $b\in S$, and $a+b=0$ implies $a=b=0$. For more properties refer to [2].

Definition 1. *Let S be a semiring with identity 1 and let L be a loop. We let SL denote the loop semiring of L over S; that is, SL consists of all finite formal sums of $\alpha=\sum\limits_{m\in L}\alpha(m)m$ with $\alpha(m)\in S$ and $m\in L$.* (That is, we assume that in $\alpha=\sum\limits_{m}\alpha(m)m$ only finitely many $\alpha(m)$ in S are different from zero.)

The elements of SL satisfy the following operational rules:

(i) $\sum\limits_{m\in L}\alpha(m)m=\sum\limits_{m\in L}\mu(m)m\Leftrightarrow\alpha(m)=\mu(m)$ for all $m\in L$

$(\alpha(m)\in S,\ m\in L$ and $\sum\limits_{m\in L}\alpha(m)m$ and $\sum\limits_{m\in L}\mu(m)m$ are in $SL)$;

(ii) $\sum\limits_{m\in L}\alpha(m)m+\sum\limits_{m\in L}\mu(m)m=\sum\limits_{m\in L}(\alpha(m)+\mu(m))m$;

(iii) $(\sum\limits_{m\in L}\alpha(m)m)(\sum\limits_{m\in L}\mu(m)m)=\sum\limits_{m\in L}\nu(m)m$,

where $\nu(m)=\sum\alpha(x)\mu(y)$ with $xy=m$.

Dropping the zero components of the formal sum we may write

$$\sum_{m\in L}\alpha(m)m=\sum_{i=1}^{n}\alpha_i m_i,\quad n \text{ finite.}$$

If we replace e in L by 1 then we have a natural embedding of S in SL given by $s\to s\cdot 1$. (That is, after the identification of S with $S\cdot e=S\cdot 1$ we shall assume $S\leqslant SL$). Clearly $mr=rm$ for all $m\in L$ and $r\in S$.

Remark. Clearly the loop semiring SL is a nonassociative semiring as L is a nonassociative structure under multiplication.

Example. Z^+, the set of all positive integers with 0 under usual addition, is a strict semiring.

Theorem 2. *Let L be a loop and Z^+ be the strict semiring of positive integers with zero. The loop semiring Z^+L is a strict nonassociative semiring.*

Proof. The loop semiring Z^+L is a nonassociative semiring by definition. Let $\alpha=\sum\limits_{i=1}^{n}\alpha_i m_i$ and $\beta=\sum\limits_{i=1}^{m}\beta_i m_i$ be in Z^+L. If $\alpha+\beta=0$ then $\sum\limits_{i=1}^{k}(\alpha_i+\beta_i)m_i=0$. By (ii) of Definition 1, this is possible only when $\alpha_i+\beta_i=0$, as $\alpha_i, \beta_i\in Z^+$ and Z^+ is a strict semiring; consequently, $\alpha_i=\beta_i=0$. Hence the loop semiring Z^+L is a strict nonassociative semiring.

Theorem 3. *The loop semiring SL is a strict semiring if and only if S is a strict semiring for any loop L.*

Proof. If SL is a strict semiring as we have $S\leqslant SL$, so is S. Conversely, if S is a strict semiring by Theorem 2, SL is a strict semiring.

Theorem 4. *Let L be a finite loop and let S be a semiring without nontrivial divisors of zero. Then SL has nontrivial divisors of zero if and only if S is not a strict semiring.*

Proof. Let $L=\{m_1=1, m_2, m_3, \ldots, m_n\}$ be a finite loop of the order n and let S be a semiring without nontrivial divisors of zero. Let $\alpha=\sum_{i=1}^{n}\alpha_i m_i$ and $\beta=\sum_{j=1}^{t}\beta_j m_j$ be in SL with $\alpha\beta=0$.

Now $\alpha\beta=\sum_{k=1}^{s}\gamma_k m_k=0$ implies that $\gamma_k=0$. But by (iii) of Definition 1, we have $\gamma_k=\sum\alpha_i\beta_j$, $m_i m_j=m_k$ for $k=1, 2, \ldots, n$. Since S is a semiring without nontrivial divisors of zero we see S is not a strict semiring.

Conversely, if S is not a strict semiring, choose α, β in SL such that $\alpha\beta=0$, by using the fact that S can contain elements a, b such that $a+b=0$.

Definition 5. *A loop L is an ordered loop if it admits a strict linear ordering $<$ such that $x<y$ implies $xz<yz$ and $zx<zy$ for all $z\in L$.*

Theorem 6. *Let S be a semiring which has no nontrivial divisors of zero and let L be an ordered loop. The loop semiring SL has no divisors of zero even if S is not a strict ring.*

Proof. Let $\alpha=\sum_{i=1}^{n}\alpha_i m_i$ and $\beta=\sum_{j=1}^{m}\beta_j h_j$ be in SL. To prove that $\alpha\beta\neq0$ it is enough if we prove that $\alpha\beta=0$ implies $\alpha_i\beta_j=0$; which will contradict the fact that S has no nontrivial divisors of zero. Suppose $m=n=1$: we have nothing to prove. Suppose $n\geqslant2$, $m\geqslant2$. Since L is ordered and $m_1, m_2, \ldots, m_n$ and $h_1, \ldots, h_m$ are mutually distinct, we may assume that $m_1<m_2<\ldots<m_n$, $h_1<h_2<\ldots<h_m$. We have $\alpha\beta=\sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}}\alpha_i\beta_j m_i h_j$. In $\alpha\beta=0$, m_1h_1 is the 'smallest among m_ih_j', i.e. we have $m_1h_1<m_ih_j$ for any i, j with $1<i$, $1<j$. Thus, if $\alpha\beta=0$, we should have $\alpha_1\beta_1=0$, which is a contradiction. Hence SL has no divisors of zero.

Theorem 7. *Let S be a semiring in which $ab=0$ for every a, b in S (a, b distinct or otherwise)*, and L be any loop. *Then the loop semiring SL is such that $\alpha\beta=0$ for every $\alpha, \beta\in SL$ (α, β distinct or otherwise).*

Proof. Clearly S does not contain the identity 1; further, in SL we have $\alpha\beta=0$ by (iii) of Definition 1. Theorem is proved.

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