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DECOUPLING OF DISCRETE-TIME NONLINEAR SYSTEMS BY MINIMAL ORDER DYNAMIC STATE FEEDBACK

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Abstract. The paper studies the dynamic input-output decoupling problem for discrete-time nonlinear systems. Using the linear-algebraic framework and inversion algorithm for this class of systems, the dynamic state feedback compensator of minimal dimension that solves the problem is obtained.

Key words. Discrete-time system, nonlinear control system, dynamic feedback, input-output decoupling.

1. INTRODUCTION

The system is said to be input-output decoupled if each scalar control variable (input) affects one and only one scalar output variable. If the given system does not possess such a property, then one may try to compensate the original system in order to achieve a decoupled system. Depending on the permitted control laws, either static or dynamic decoupling problems can be formulated. Our interest here is in the dynamic input-output decoupling problem in which one achieves decoupling by the dynamic state feedback.

During the last decade there has been much interest in the problem of dynamic input-output decoupling of nonlinear control systems, both continuous-time [1–7] and discrete-time [8–10]. For continuous-time case, the algorithms base either on the state-dependent transformations of the output [3, 5, 6], or on the state-dependent transformations of the input [1, 2, 4, 7]. However, all the algorithms suggest different feedback control laws. For the comparison of these algorithms, except [7], see [4]. Let us also note that the algorithm by Xia [4] can be considered as a combination of [1] and [4]. In [11] the authors have focused on the problem of finding the decoupling compensator of minimal dimension: they have proved that a Singh compensator [6] is a decoupling compensator of minimal order.

All the papers considering the problem of dynamic input-output decoupling for discrete-time nonlinear systems prefer the approach which bases on the input transformations. The first attempt has been proposed in [9] where a solution to the problem only for a fixed time instant is given. However, [9] forms the starting point of the paper [10], where the problem has been studied locally in the neighbourhood of an equilibrium point of the system. The results of [8] are valid on a larger subset than the neighbourhood of the equilibrium point, and besides that the problem of the computation of the feedback is addressed in the paper. The latter is achieved via the application of the theory multiple Lie series to the solution of the system on nonlinear equations. However, the fact that the results are valid on the larger subset than the neighbourhood of the equilibrium point make the algorithm more complicated: at every step of the algorithm one must take care of the transformation of the subset where the results hold.

In this paper we shall also consider the dynamic input-output decoupling problem for discrete-time nonlinear system. Unlike [8–10] we shall make use of the dual approach which bases on the inversion (structure) algorithm, allowing the output transformations. Our approach, therefore, can be considered as the modification of the algorithms by Singh [6], and Li and Feng [3] to discrete-time systems. We show how to construct the dynamic state feedback compensator via the inversion algorithm that solves the input-output decoupling problem locally around an equilibrium point of the system. Furthermore, it will be shown that the obtained compensator has a minimal order among all other regular dynamic compensators that solve the input-output decoupling problem for discrete-time systems. The last result can be considered as a discrete-time counterpart of the results presented in [11].

The paper is organized as follows. In Section 2, the problem formulation is given. In Section 3, we briefly review the linear algebraic framework [12] as well as the properties of essential orders [13, 14] for discrete-time nonlinear control system. Then, in the next section, the inversion (structure) algorithm for discrete-time nonlinear system and some of its properties are presented. In Section 5, via the inversion algorithm, the compensator is obtained that solves the input-output decoupling problem. In Secion 6, the linear algebraic framework allows us to prove the minimal dimensionality of the decoupling compensator obtained via the inversion algorithm.

2. PROBLEM STATEMENT

Consider a discrete-time nonlinear plant P described by equations of the form

$$x(t+1)=f(x(t),u(t)),\quad x(0)=x_0,$$
$$y(t)=h(x(t)),\tag{1}$$

where the states $x(t)$, $t=0, 1, \ldots$, belong to an open subset X of R^n, the controls $u(t)$, $t=0, 1, \ldots$, belong to an open subset U of R^m, and the outputs $y(t)$, $t=0, 1, \ldots$, belong to an open subset Y of R^p, $p\leqslant m$. The mappings f and h are supposed to be analytic.

We are assumed to work in a neighbourhood of an equilibrium point of system (1), that is around $(x^0, u^0)\in X\times U$, such that $f(x^0, u^0)=x^0$. From the fact that $f(x^0, u^0)=x^0$ it follows that using the control sequence $u(0), u(1), \ldots$ with each $u(t)$ sufficiently close to u^0 provided the disturbance sequence $w(0), w(1), \ldots$ is such that each $w(t)$ is sufficiently close to w^0, we can assure that the states $x(t)$ are sufficiently close to x^0, and the outputs $y(t)$ are sufficiently close to $y^0=h(x^0)$.

The system (1) is said to be input-output decoupled if the first p components of the control u, i.e $u_1, \ldots, u_p$, independently influence the p outputs $y_1, \ldots, y_p$, and all other components of the control u, i.e. $u_{p+1}, \ldots, u_m$, affect none of the outputs.

If the system (1) is not input-output decoupled, we may try to satisfy this property via feedback, that is to find a state feedback compensator such that the closed-loop system is input-output decoupled.

We are looking for an analytic compensator C (dynamic state feedback) with a μ-dimensional state z, a new m-dimensional control v, described by equations of the form

$$z(t+1)=f^C(z(t),x(t),v(t)),\quad z(0)=z_0,$$
$$u(t)=h^C(z(t),x(t),v(t)),\tag{2}$$

defined locally around (to be found) a point (z^0,x^0,v^0,u^0) that satisfies the equalities $z^0=f^C(z^0,x^0,v^0)$ and $u^0=h^C(x^0,u^0,v^0)$. The point (z^0,x^0,v^0,u^0) is the equilibrium point of the compensator C, corresponding to the equilibrium point (x^0,u^0,y^0) of the plant P.

We call the compensator C described by the equation (2) regular if the dynamic system

$$x(t+1)=f(x(t),h^C(z(t),x(t),v(t))),$$
$$z(t+1)=f^C(z(t),x(t),v(t)),\tag{3}$$
$$u(t)=h^C(z(t),x(t),v(t)),$$

with inputs $v(t)$ and outputs $u(t)$ is invertible (see [15] for details about the notion of invertibility) around the point (x^0,z^0,v^0,u^0).

The closed-loop system (1), (3), initialized at (x^0,z^0), that is the system

$$x(t+1)=f(x(t),h^C(z(t),x(t),v(t))),\quad x(0)=x_0,\quad z(0)=z_0,$$
$$z(t+1)=f^C(z(t),x(t),v(t)),\tag{4}$$
$$y(t)=h(x(t))$$

is denoted by $P\circ C$.

Definition 1. Local dynamic input-output decoupling problem. *Given the system P together with an initial state x_0, described by equations* (1) *around an equilibrium point (x^0,u^0), find, if possible, a regular analytic compensator C defined by equations of the form* (2) *together with an initial state z_0, the equilibrium point (z^0,x^0,v^0,u^0), and the neighbourhoods $Z^0\times X^0\times V^0$ of (z^0,x^0,v^0) and U^0 of u^0, being domain and range of C, so that the closed-loop system $P\circ C$ described by* (4) *and initialized at (z_0,x_0), is input-output decoupled on $Z^0\times X^0\times V^0\times U^0$.*

Remark 1. The static input-output linearization problem is obtained when $\mu=\dim z=0$.

3. LINEAR ALGEBRAIC TOOLS

We briefly review a linear algebraic framework introduced by Grizzle [12] for the analysis of discrete-time nonlinear control systems. This framework will be employed later on in our paper.

Consider a discrete-time nonlinear system described by equations

$$x(t+1)=f(x(t),u(t)),$$
$$y(t)=h^*(x(t),u(t)),\tag{5}$$

where x, u, y, f and h^* are defined as in (1). Note that both the equations of the plant (1) and the compensator (2) can be given in this form.

Recall that a meromorphic function η is a function of the form $\eta=\pi/\Theta$, where π and Θ are analytic functions with Θ not the zero function. View $x,u(0),\ldots,u(n)$ as variables and let $\mathcal{K}$ denote the field of meromorphic functions in the variables $(x,u(0),\ldots,u(n))$.

A system (5) is said to be generically submersive if the rank of the Jacobian matrix of the function $f(x, u)$ over the field $\mathcal{K}$ of meromorphic functions is equal to the dimension n of the system, i. e. if

$$\operatorname{rank}_{\mathcal{K}}\left[\frac{\partial f}{\partial x}(x, u), \frac{\partial f}{\partial u}(x, u)\right]_{u=u(0)}=n.$$

Note that many systems of the form (5) are generically submersive, since this is a necessary condition for accessibility.

For the system (5) we define in a natural way

$$y(0)=h^*(x_0, u(0)):=\xi_0(x_0, u(0)),$$
$$y(1)=h^*(f(x_0, u(0)), u(1)):=\xi_1(x_0, u(0), u(1)),$$
$$\cdots$$
$$y(t)=h^*(f(\ldots f(f(x_0, u(0)), u(1)), \ldots), u(t)):=$$
$$:=\xi_t(x_0, u(0), \ldots, u(t)).$$

Note that $y(0), y(1), y(2), \ldots, y(t)$ so defined have components in the field $\mathcal{K}$.

Let $\mathcal{E}$ denote the vector space over $\mathcal{K}$ spanned by $\{dx, du(0), \ldots, du(n)\}$. Observe that $dy_i(k) \in \mathcal{E}$ for all $1 \leqslant i \leqslant p$ and $0 \leqslant k \leqslant n$, since

$$dy_i(k)=\sum_{j=1}^{n} \frac{\partial y_i(k)}{\partial x_j} dx_j+\sum_{l=0}^{k} \sum_{j=1}^{m} \frac{\partial y_i(k)}{\partial u_j(l)} du_j(l).$$

Define a chain of subspaces $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n$ of $\mathcal{E}$ by

$$\mathcal{E}_k:=\operatorname{span}_{\mathcal{K}}\{dx, dy(0), \ldots, dy(k)\}.$$

Definition 2 [8–10]. *The delay order d_i corresponding to the ith $(i=1, \ldots, p)$ output y_i of* (5) *is defined as the smallest nonnegative integer k for which*

$$dy_i(k) \notin \operatorname{span}_{\mathcal{K}}\{dx\}.$$

If such a k does not exist, one sets $d_i=\infty$.

Definition 3 [13]. *The essential order ε_i corresponding to the ith $(i=1, \ldots, p)$ output y_i of* (5) *is defined as the smallest nonnegative integer k for which*

$$dy_i(k) \notin \operatorname{span} \mathcal{K}\{dx, dy(0), \ldots, dy(k-1), dy_{j \neq i}(k), dy(k+1), \ldots, dy(n)\}.$$

If such a k does not exist, one sets $\varepsilon_i=\infty$.

Lemma 1 [13]. *The essential orders ε_i, $i=1, \ldots, p$ cannot decrease under the action of a static or dynamic compensator.*

Lemma 2 [13]. *Consider a right invertible nonlinear system* (1) *with equal number of inputs and outputs. Then for all $1 \leqslant i \leqslant m$*

(i) $\varepsilon_i, d_i<\infty$,

(ii) $\varepsilon_i \geqslant d_i$,

(iii) $\varepsilon_i=d_i$ *if and only if the input-output decoupling problem around the equilibrium point (x^0, u^0) is locally solvable via a regular static state feedback.*

Lemma 3 [12]. Suppose that (5) is submersive and that $I_0 \subset I_1 \subset \ldots \subset I_n \subset\{1, \ldots, p\}$ are index sets such that for $0 \leqslant k \leqslant n$ $\mathcal{E}_k= \operatorname{span}_{\mathcal{K}}\{dx, dy_{i_0}(0), \ldots, dy_{i_k}(k) \mid i_j \in I_j, 0 \leqslant j \leqslant k\}$. *Moreover, suppose that I_n does not equal $\{1, \ldots, p\}$, i. e. there exists $j \in\{1, \ldots, p\}$ such that $j \notin I_n$. Then for each $j \notin I_n$, $1 \leqslant j \leqslant p$, exists an integer N, $1 \leqslant N \leqslant n$ such that*

$$dy_j(N) \in \operatorname{span}_{\mathcal{K}}\{dy_j(0), \ldots, dy_j(N-1), dy_{i_0}(0), \ldots, dy_{i_N}(n) \mid$$
$$i_k \in I_k,\ 0 \leqslant k \leqslant N\}$$

and

$$dy_j(k) \in \operatorname{span}_{\mathcal{K}}\{dy_j(0), \ldots, dy_j(N-1), dy_{i_0}(0), \ldots, dy_{i_k}(k) \mid$$
$$i_s \in I_s,\ \textit{for}\ 0 \leqslant s \leqslant N\ \textit{and}\ i_s \in I_n,\ \textit{for}\ s > N\},$$
$$N+1 \leqslant k \leqslant N+r,\ r \geqslant 1.$$

4. INVERSION ALGORITHM

In this section, for completeness, we recall an inversion algorithm for discrete-time nonlinear systems [15] in a form [16], and some of its properties that will be employed in the sequel. Denote $\hat{y}_0(t) = h(x(t))$ and $\varrho_0 = 0$.

Step 1. Calculate $y(t+1) = h(f(x(t), u(t)))$, and define

$$\varrho_1 = \operatorname{rank} \frac{\partial}{\partial u} h(f(x, u)) \Big|_{x=x^0,\ u=u^0}$$

Let us assume that $\varrho_1 = \text{const}$ in some neighbourhood O_1 of (x^0, u^0). Permute, if necessary, the components of the output so that the first ϱ_1 rows of the matrix $\partial h(f(x, u))/\partial u$ are linearly independent. Decompose $y(t+1)$ and $h(f(x, u))$ according to

$$y(t+1) = \begin{bmatrix} \tilde{y}_1(t+1) \\ \hat{y}_1(t+1) \end{bmatrix}, \qquad h(f(x, u)) = \begin{bmatrix} \tilde{a}_1(x, u) \\ a_1(x, u) \end{bmatrix},$$

where $\tilde{y}_1(t+1)$ and $\tilde{a}_1(x, u)$ consist of the first ϱ_1 components of $y(t+1)$ and $h(f(x, u))$, respectively. Since the last $p - \varrho_1$ rows of the matrix $\partial h(f(x, u))/\partial u$ are linearly dependent on the first ϱ_1 rows, we can write

$$\tilde{y}_1(t+1) = \tilde{a}_1(x(t), u(t)),$$
$$\hat{y}_1(t+1) = \hat{a}_1(x(t), u(t)) = \psi_1(x(t), \tilde{y}_1(t+1)).$$

Denote $\tilde{a}_1(x, u)$ by $A_1(x, u)$.

Step $k+1$ $(k \geqslant 1)$. Suppose that in Steps 1 through k, $\tilde{y}_1(t+1)$, $\tilde{y}_2(t+2), \ldots, \tilde{y}_k(t+k)$, $\hat{y}_k(t+k)$ have been defined so that

$$\tilde{y}_1(t+1) = \tilde{a}_1(x(t), u(t)),$$
$$\tilde{y}_2(t+2) = \tilde{a}_2(x(t), u(t), \tilde{y}_1(t+2)),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\tilde{y}_k(t+k) = \tilde{a}_k(x(t), u(t), \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k-1,\ i+1 \leqslant j \leqslant k\}),$$
$$\hat{y}_k(t+k) = \psi_k(x(t), \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant k\}).$$

Suppose also that the matrix $\dfrac{\partial}{\partial u} A_k = \dfrac{\partial}{\partial u} [\tilde{a}_1^{\mathrm{T}} \ldots \tilde{a}_k^{\mathrm{T}}]^{\mathrm{T}}$ has full rank equal to ϱ_k in some neighbourhood O_k of (x^0, u^0).

Compute

$$\hat{y}_k(t+k+1) = \psi_k(f(x(t), u(t)),\ \{\tilde{y}_i(t+j+1),\ 1 \leqslant i \leqslant k,\ i \leqslant j \leqslant k\}) =$$
$$= a_{k+1}(x(t), u(t),\ \{\tilde{y}_i(t+j),\ 1 \leqslant i \leqslant k,\ i+1 \leqslant j \leqslant k+1\})$$

and define

$$\varrho_{k+1}=\operatorname{rank}\frac{\partial}{\partial u}\begin{bmatrix} A_k(\cdot) \\ a_{k+1}(\cdot) \end{bmatrix}_{x=x^0,\ u=u^0,\ y=y^0=h(x^0)}$$

Let us assume that $\varrho_{k+1}=\text{const}$ in some neighbourhood O_{k+1} of (x^0, u^0). Permute, if necessary, the components of $\hat{y}_k(t+k+1)$ so that the first ϱ_{k+1} rows of the matrix $\partial[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}/\partial u$ are linearly independent. Decompose $\hat{y}_k(t+k+1)$ and a_{k+1} according to

$$\hat{y}_k(t+k+1)=\begin{bmatrix} \tilde{y}_{k+1}(t+k+1) \\ \hat{y}_{k+1}(t+k+1) \end{bmatrix}, \qquad a_{k+1}=\begin{bmatrix} \tilde{a}_{k+1} \\ \hat{a}_{k+1} \end{bmatrix},$$

where $\tilde{y}_{k+1}(t+k+1)$ and $\tilde{a}_{k+1}$ consist of the first $\varrho_{k+1}-\varrho_k$ components of $\hat{y}_k(t+k+1)$ and a_{k+1} respectively. Since the last $p-\varrho_{k+1}$ rows of the matrix $\partial[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}/\partial u$ are linearly dependent on the first ϱ_{k+1} rows, we can write

$$\begin{aligned}
&\tilde{y}_1(t+1)=\tilde{a}_1(x(t), u(t)),\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\tilde{y}_{k+1}(t+k+1)=\tilde{a}_{k+1}(x(t), u(t), \{\tilde{y}_i(t+j),\ 1\leqslant i\leqslant k,\ i+1\leqslant j\leqslant k+1\}),\\
&\hat{y}_{k+1}(t+k+1)=\psi_{k+1}(x(t), \{\tilde{y}_i(t+j),\ 1\leqslant i\leqslant k+1,\ i\leqslant j\leqslant k+1\}).
\end{aligned}$$

Denote $A_{k+1}=[A^{\mathrm{T}}_k, \tilde{a}^{\mathrm{T}}_{k+1}]^{\mathrm{T}}$. End of step $k+1$.

Note that we can apply the inversion algorithm not necessarily in a unique way. There exist, in general, different permutations of output components $\hat{y}_k(t+k+1)$ at step $k+1$, $k\geqslant 0$, so that the first ϱ_{k+1} rows of the matrix $\partial[A^{\mathrm{T}}_k, a^{\mathrm{T}}_{k+1}]^{\mathrm{T}}/\partial u$ are linearly independent. Different permutations of output components, that is, different selections of $\tilde{y}_{k+1}(t+k+1)$ in each step result in different functions $A_{k+1}(\cdot)$; see [16] for a relation between such different selections.

In the inversion algorithm certain constant rank conditions have been imposed to ensure that the algorithm can be carried out on system (1) locally around an equilibrium point. We shall summarize these conditions in the definition of regularity of an equilibrium point.

Definition 4. *We call the equilibrium point* (x^0, u^0) *of the system* (1) *regular with respect to the inversion algorithm if in case of some specific application of the inversion algorithm for all* $k\geqslant 1$, $\operatorname{rank}\partial A_k(\cdot)/\partial u$ *is constant in some neighbourhood of* (x^0, u^0). *We call* (x^0, y^0) *strongly regular if above holds for each application of the algorithm.*

It has been shown that around a regular equilibrium point the inversion algorithm terminates in at most n steps [17].

From the following lemma, proved by Grizzle [12], it is evident that around a regular equilibrium point the inversion algorithm defines the basis for vector spaces $\mathscr{E}_k$, $1\leqslant k\leqslant n$.

Lemma 4. *Apply the inversion algorithm to submersive nonlinear system* (1). *Then for each* $1\leqslant k\leqslant n$

(i) $\{dx, \{d\tilde{y}_i(j)\mid 1\leqslant i\leqslant k,\ i\leqslant j\leqslant k\}\}$ *is a basis for* $\mathscr{E}_k$.
(ii) $\dim_{\mathcal{K}}\mathscr{E}_k=n+\varrho_1+\ldots+\varrho_k$.

Though the result of the inversion algorithm is not unique, it has been proved [16] that the integers $\varrho_1, \ldots, \varrho_k$ do not depend on the particular permutation of the components of $\hat{y}_k(t+k+1)$. Thus, using this algorithm around a strongly regular equilibrium point we obtain a uniquely defined sequence of integers $0 \leqslant \varrho_1 \leqslant \ldots \leqslant \varrho_k \leqslant \ldots \leqslant \leqslant \min(p, m)$. Let $\varrho^* = \max\{\varrho_k, k \geqslant 1\}$ and let α be defined as the smallest $k \in N$ such that $\varrho_k = \varrho^*$. On the analogy with Moog [18], the ϱ_k's are called the invertibility indices of the system (1).

5. INPUT-OUTPUT DECOUPLING COMPENSATOR

In this section we shall show that using the inversion algorithm, we can, for locally right invertible systems, construct a regular dynamic compensator that will locally solve the input-output decoupling problem. Note that right invertibility is necessary and sufficient condition for the solvability of a input-output decoupling problem [10].

Suppose that a system (1) is locally right invertible around the regular equilibrium point. This means that applying the inversion algorithm to (1) we obtain, at the αth step,

$$\begin{aligned} &\tilde{y}_1(t+1) = \tilde{a}_1(x(t), u(t)), \\ &\tilde{y}_2(t+1) = \tilde{a}_2(x(t), u(t), \tilde{y}_1(t+2)), \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\tilde{y}_\alpha(t+\alpha) = \tilde{a}_\alpha(x(t), u(t), \{\tilde{y}_i(t+j), 1 \leqslant i \leqslant \alpha-1, i+1 \leqslant j \leqslant \alpha\}), \end{aligned} \tag{6}$$

where the Jacobian matrix of the right-hand side of (6) with respect to u around the equilibrium point has a full row rank $\varrho_\alpha = p$. For $i = 1, \ldots, p$, denote by $t+\gamma_i$ the smallest time instant and by $t+\delta_i$ the greatest time instant in which the ith scalar component y_i of the output y appears in (6), and rewrite (6) as

$$\begin{bmatrix} y_{\varrho_{k-1}+1}(t+k) \\ \cdot \quad \cdot \quad \cdot \\ y_{\varrho_k}(t+k) \end{bmatrix} = \tilde{a}_k(x(t), u(t), \{y_i(t+j), 1 \leqslant i \leqslant \varrho_{k-1}, \gamma_i+1 \leqslant j \leqslant \min(k, \delta_i)\}), \tag{7}$$

$k = 1, \ldots, \alpha$.

After a possible permutation of inputs we may assume that the Jacobian matrix of the right-hand side of (7) with respect to $u^1 = (u_1, \ldots, u_p)^{\mathrm{T}}$ around the equilibrium point has a full row rank p. Moreover, at the equilibrium point the value of the vector function $\tilde{a}_k(\cdot)$ is equal to $(y^0_{\varrho_{k-1}+1}, \ldots, y^0_{\varrho_k})^{\mathrm{T}}$.

Therefore, Eq. (7) can be solved for $u^1(t)$ uniquely around the equilibrium point by applying Implicit Function Theorem. Define $u^2 = (u_{p+1}, \ldots, u_m)^{\mathrm{T}}$. Then, from (7), we obtain

$$u^1(t) = \varphi(x(t), \{y_i(t+j), 1 \leqslant i \leqslant p, \gamma_i \leqslant j \leqslant \delta_i\}, u^2(t)) \tag{8}$$

which is such that for $k = 1, 2, \ldots, \alpha$

$$\begin{aligned} &[y_{\varrho_{k-1}+1}(t+k), \ldots, y_{\varrho_k}(t+k)]^{\mathrm{T}} = \tilde{a}_k(x(t), \varphi(x(t), \{y_i(t+j), 1 \leqslant i \leqslant \varrho^*, \\ &\gamma_i+1 \leqslant j \leqslant \delta_i\}, u^2(t)), \{y_i(t+j), 1 \leqslant i \leqslant \varrho_{k-1}, \gamma_i+1 \leqslant j \leqslant \min(k, \delta_i)\}). \end{aligned} \tag{9}$$

Notice that $\varphi: M_1 \to M_2$ is defined for some (possible small) neighbourhoods M_1 and M_2 of $(x^0, y^0, \ldots, y^0, u^{20})$ in $X^0 \times (Y^0)^r \times U^{20}$ and of u^{10} in U^{10}.

Now construct the compensator for (1) in the following way. Let $z_i=(z_{i1},\dots,z_{i,\,\delta_i-\gamma_i})^{\mathrm{T}}$, $i=1,\dots,p$ be a vector of dimension $\delta_i-\gamma_i$, v^2 — a vector of dimension $m-p$, and consider the system

$$
\begin{aligned}
&z_{i1}(t+1)=z_{i2}(t),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&z_{i,\,\delta_i-\gamma_i-1}(t+1)=z_{i,\,\delta_i-\gamma_i}(t),\quad i=1,\dots,p,\\
&z_{i,\,\delta_i-\gamma_i}(t+1)=v_i(t),\\
&u^1(t)=\varphi(x(t),\{z_{ij}(t),\ 1\leqslant j\leqslant\delta_i-\gamma_i,v_i(t),\ 1\leqslant i\leqslant p\},v^2(t)),\\
&u^2(t)=v^2(t)
\end{aligned}
\tag{10}
$$

with controls $v^1(t)=(v_1,\dots,v_p)^{\mathrm{T}}$ and v^2, outputs u^1 and u^2.

Moreover, in accordance with (8) and (10) define $z^0_i=y^0_i$, $i=1,\dots,p$, $v^{10}=y^0$, $v^{20}=u^{20}$.

Denote the dimension of the compensator (10) obtained via the application of the inversion algorithm in one specific way, by σ. Then obviously $\sigma=\sum\limits_{i=1}^{p}(\delta_i-\gamma_i)$.

It has been shown in [17] that the compensator (10) is regular on a neighbourhood of an equilibrium point.

Now, it is easy to see that the compensator (10) with an arbitrary initial state, applied to (1) yields locally around the equilibrium point for $i=1,\dots,p$

$$
\begin{aligned}
&y_i(\gamma_i+j-1)=z_{ij}(0),\quad j=1,\dots,\delta_i-\gamma_i,\\
&y_i(t+\delta_i)=v_i(t),\quad 0\leqslant t\leqslant t_F.
\end{aligned}
\tag{11}
$$

Moreover, the inspection of the inversion algorithm gives that for the compensated system (1), (10) we have that $y_i(0),\dots,y_i(\gamma_i-1)$, $i=1,\dots,p$ depend only on x_0, and are therefore independent of the new controls. Hence any compensator (10) obtained via the inversion algorithm, solves the input-output decoupling problem locally around the strongly regular equilibrium point (x^0,u^0).

6. MINIMALITY OF DYNAMIC INPUT-OUTPUT DECOUPLING COMPENSATOR

In this section we shall prove that the decoupling compensator (10) which is actually a discrete-time counterpart of the so-called Singh compensator is of minimal order. The proof is quite a straightforward generalization of a proof for continuous-time systems and consists of two parts. We first prove that any decoupling compensator obtained via the inversion algorithm has the same dimension σ around a strongly regular equilibrium point. Then we prove that for any dynamic decoupling feedback of the form (2) with a μ-dimensional state space we have that $\mu\geqslant\sigma$.

For this we need the following Definition and Lemma.

Definition 5 [19]. *Let* V *be a given vector space over a field* $\mathcal{F}$. *Let* $\Lambda=\{\lambda_1,\dots,\lambda_r\}$ *be a family of vectors in* V. *Then* λ_i *is called an essential vector of* Λ *if*

$$\nexists\,\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_r\in\mathcal{F}:\lambda_i=\sum_{j\neq1}\alpha_j\lambda_j.$$

The above definition means that an essential vector of Λ is linearly independent of all other vectors of Λ. This implies that every subset of Λ that forms a basis of span $\{\lambda_1, \ldots, \lambda_r\}$ necessarily contains the essential vectors of Λ.

Lemma 5 [11]. *Let* V *be a given vector space over a field* $\mathcal{F}$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ *be a family of vectors in* V. *Let* $s := \dim \operatorname{span}\{\lambda_1, \ldots, \lambda_r\}$ *and assume that* $\{\lambda_1, \ldots, \lambda_s\}$ *is a set of linearly independent vectors. Then* λ_i, $i=1, \ldots, s$, *is an essential vector of* Λ *if and only if for all* $j=s+1, \ldots, r$

$$\lambda_j = \sum_{k=1}^{s} \alpha_{jk} \lambda_k \to \alpha_{ji} = 0.$$

In general, we can apply the inversion algorithm to (1) in several specific ways. We prove the following Lemma.

Lemma 6. *Any decoupling compensator* (10), *obtained via the inversion algorithm, has the following properties*

(i) $\delta_i = \varepsilon_i$, $i=1, \ldots, p$,

where by ε_i *are denoted the essential orders of the system.*

(ii) $\sum_{i=1}^{p} \gamma_i = \sum_{k=1}^{\alpha} k s_k$,

where $s_k := \varrho_k - \varrho_{k-1}$, $1, \ldots, \alpha$.

Remark 2. From (11) and Lemma 6 it is not difficult to see that for any decoupling compensator (10) obtained via the inversion algorithm the essential orders are not increased.

Proof (i). By definition of essential orders and by Lemma 4 we have that $dy_i(k)$ is not an essential vector of $\mathcal{E}_k$ for $k=1, \ldots, \varepsilon_i - 1$. This implies by Lemma 5 that

$$\frac{\partial \hat{y}_{\varepsilon_i - 1}(\varepsilon_i - 1)}{\partial y_i(\varepsilon_i - 1)} \neq 0$$

and, hence, $\delta_i \geqslant \varepsilon_i$.

Moreover, by a definition of the essential orders and essential vectors as well as Lemma 4 we have that $dy_i(\varepsilon_i)$ is an essential vector of $\mathcal{E}_k$ for $k=\varepsilon_i, \ldots, n$. Again, by Lemma 5, this implies that $\partial \hat{y}_k(k)/\partial y_i(r) = 0$ for $k=\varepsilon_i, \ldots, n$ and $r=\varepsilon_i, \ldots, n$. This means that $\delta_i \leqslant \varepsilon_i$. Hence $\delta_i = \varepsilon_i$.

(ii) Note that γ_i is the smallest $k \in N$ for which y_i is an entry of $\tilde{y}_k$. Inspection of the inversion algorithm gives that the set $\{y_i \mid i \in \{1, \ldots, p\}, \gamma_i = k\}$ has $s_k = \varrho_k - \varrho_{k-1}$ elements. Therefore

$$\sum_{i=1}^{p} \gamma_i = \sum_{k=1}^{\alpha} k s_k.$$

The consequence of Lemma 6 is that around a strongly regular equilibrium point (x^0, u^0) any decoupling compensator obtained via the inversion algorithm, has the same dimension

$$\sigma = \sum_{i=1}^{p} \varepsilon_i - \sum_{k=1}^{\alpha} k s_k.$$

Our main result can be stated as follows.

Theorem. *Consider the submersive nonlinear system* (1) *around a strongly regular equilibirium point* (x^0, u^0), *and consider a regular dynamic state feedback* (2) *around the equilibrium point* (z^0, x^0, v^0, u^0) *corresponding to* (x^0, u^0). *Assume that the compensator* (2) *of dimension* μ *solves the input-output decoupling problem locally around* (z^0, x^0, v^0, u^0). *Then* $\mu \geqslant \sigma$.

Proof. Consider a regular dynamic state-feedback C described by equations (2) that solves the input-output decoupling problem locally around (z^0, x^0, v^0, u^0). Then, by Lemmas 1 and 2, we have

$$\varepsilon_i(P \circ C) = d_i(P \circ C) \geqslant \varepsilon_i(P), \quad i = 1, \ldots, p.$$

It is known [8, 9] that for closed-loop system $P \circ C$ described by (4) the differentials $dy_i^{P \circ C}(k)$, $i = 1, \ldots, p$, $k = 0, \ldots, d_i(P \circ C) - 1$ are linearly independent (over $\tilde{\mathcal{K}}^{P \circ C}$, the subfield of $\mathcal{K}^{P \circ C}$ consisting of the meromorphic functions of x and z). By Lemmas 4 and 6 we can find a reordering of the outputs of (1) and integers $\gamma_1, \ldots, \gamma_p$ satisfying $\sum_{i=1}^{p} \gamma_i = \sum_{i=1}^{p} \varepsilon_i - \sigma$ such that for (1) the differentials

$$\{dx, \{dy_i(j),\ 1 \leqslant i \leqslant p,\ \gamma_i \leqslant j \leqslant \varepsilon_i - 1\}\}$$

are linearely independent over $\mathcal{K}$. Assume that for closed-loop system $P \circ C$ described by Eq. (4) these differentials are not linearly independent over $\tilde{\mathcal{K}}^{P \circ C}$. Note that for the closed-loop system (4) these differentials can be expressed in the form

$$dy_i^{P \circ C}(j) = \frac{\partial y_i(j)}{\partial x} dx + \sum_{s=0}^{j-1} \frac{\partial y_i(j)}{\partial u(s)} du(s), \tag{12}$$

where $du(s)$ depends on (x, z). Linear dependence over $\tilde{\mathcal{K}}^{P \circ C}$ implies that there exist ψ_{ik}, $i = 1, \ldots, p$, $k = \gamma_i, \ldots, \varepsilon_i - 1$ and ψ_0 in $\tilde{\mathcal{K}}^{P \circ C}$ (not all identically zero) such that

$$\psi_0 dx + \sum_{i=1}^{p} \sum_{k=\gamma_i}^{\varepsilon_i - 1} \psi_{ik} dy_i^{P \circ C}(k) = 0. \tag{13}$$

Combining (12) and (13) we obtain

$$\left(\psi_0 + \sum_{i=1}^{p} \sum_{k=\gamma_i}^{\varepsilon_i - 1} \psi_{ik} \frac{\partial y_i(k)}{\partial x} \right) dx + \sum_{i=1}^{p} \sum_{k=\gamma_i}^{\varepsilon_i - 1} \sum_{s=0}^{k-1} \psi_{ik} \frac{\partial y_i(k)}{\partial u(s)} du(s) = 0. \tag{14}$$

The invertibility of the plant (1) implies that there must be at least one $du_i(j)$ that appears in the left-hand side of (14). Choose $r \in \{1, \ldots, m\}$, $s \in N$ such that $du_r(s)$ will appear on the left-hand side of (14), and s will be as large as possible. Then, from (14), it follows taht we can find a function $\Phi_{rs}(\cdot)$ such that

$$u_r(s) = \Phi_{rs}(x, \{u_i(j),\ i \neq r,\ 0 \leqslant j \leqslant s\}, \{u_r(j),\ 0 \leqslant j \leqslant s - 1\}). \tag{15}$$

By Lemma 3 this means that for all $k \geqslant s$ there exists a function $\Phi_{rk}(x, \{u_i(j),\ i \neq r,\ 0 \leqslant j \leqslant k\}, \{u_r(j),\ 0 \leqslant j \leqslant s - 1\})$

such that

$$u_r(k) = \Phi_{rs}(x, \{u_i(j),\ i \neq r,\ 0 \leqslant j \leqslant k\}, \{u_r(j),\ 0 \leqslant j \leqslant s - 1\}). \tag{16}$$

This implies that, applying the inversion algorithm to (3), we obtain $\varrho_{n+\mu} < m$ which means that the compensator (2) is not a regular dynamic state feedback and it gives a contradiction. Hence, for the system (4) the differentials $\{dx, \{dy_i(j),\ 1 \leqslant i \leqslant p,\ \gamma_i \leqslant j \leqslant \varepsilon_i - 1\}\}$ are linearely independent over $\tilde{\mathcal{K}}^{P \circ C}$. In particular this implies that

$$\operatorname{rank}_{\tilde{\mathcal{K}}^{P\circ C}}\begin{pmatrix}\frac{\partial x}{\partial x} & 0\\ \frac{\partial y_i(j)}{\partial x} & \frac{\partial y_i(j)}{\partial z}\end{pmatrix}_{i=1,\ldots,p,\ \gamma_i\leqslant j\leqslant \varepsilon_i-1}=$$

$$=n+\sum_{i=1}^{p}(\varepsilon_i-\gamma_i)=n+\sigma$$

and hence we must necessarily have that

$$\operatorname{rank}_{\tilde{\mathcal{K}}^{P\circ C}}\left(\frac{\partial y_i(j)}{\partial z}\right)_{i=1,\ldots,p,\ \gamma_i\leqslant j\leqslant \varepsilon_i-1}=\sum_{i=1}^{p}(\varepsilon_i-\gamma_i)=\sigma.$$

Obviously,

$$\operatorname{rank}_{\tilde{\mathcal{K}}^{P\circ C}}\left(\frac{\partial y_i(j)}{\partial z}\right)_{i=1,\ldots,p,\ \gamma_i\leqslant j\leqslant \varepsilon_i-1}\leqslant \dim z=\mu$$

and so $\mu\geqslant\sigma$, which establishes our claim.

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DISKREETSE AJAGA MITTELINEAARSETE SÜSTEEMIDE DEKOMPONEERIMINE MINIMAALSET JÄRKU DÜNAAMILISE OLEKUTAGASISIDE ABIL

Ülle KOTTA

On vaadeldud diskreetsete mittelineaarsete süsteemide klassi jaoks sisend—väljund-kujutise dekomponeerimise ülesannet. Süsteemi tasakaalupunkti ümbruses on otsitud dünaamilise olekutagasisidekujulist lokaalset lahendit. Otsitav kompensaator (tagasiside) on leitud pööramisalgoritmi abil ja näidatud, et saadud kompensaatori järk on väikseim võimalikest.

РАСЩЕПЛЕНИЕ НЕЛИНЕЙНЫХ СИСТЕМ С ДИСКРЕТНЫМ ВРЕМЕНЕМ С ПОМОЩЬЮ МИНИМАЛЬНОГО ПОРЯДКА ДИНАМИЧЕСКОЙ ОБРАТНОЙ СВЯЗИ ПО СОСТОЯНИЮ

Юлле КОТТА

Решена задача расщепления вход—выходного отображения нелинейной системы с дискретным временем. Локальное решение в виде динамической обратной связи по состоянию найдено в окрестности точки равновесия системы. Искомый компенсатор (обратная связь) определен с помощью алгоритма обращения. Показано, что из всех компенсаторов, решающих рассматриваемую задачу, найденный имеет минимальную размерность.