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MULTIPLIERS WITH RESPECT TO ORTHOGONAL EXPANSIONS IN BANACH SPACES

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Abstract. A necessary and sufficient condition for multipliers of generalized Lipschitz classes in certain Banach spaces is obtained. The majorant of the modulus of continuity is supposed to be slowly decreasing. For the one-dimensional periodic case the result was known earlier.

Key words: Fourier multipliers, Banach spaces.

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1. Introduction

The purpose of this paper is to extend some results concerning multipliers that preserve the modulus of continuity of integrable functions to the setting of abstract Fourier expansions in Banach spaces.

Let L be the space of 2π-periodic integrable functions f with the norm

$$\|f\|_L = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|\,dt$$

and let A and $B \subset L$ be any subsets. Let

$$f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \tag{1.1}$$

be the Fourier series of f. The classical problem of multipliers asks for properties that an arbitrary sequence of complex numbers $\lambda = \{\lambda_k\}$ should satisfy to ensure that $f \in A$ always implies that the series

$$\sum_{k=-\infty}^{\infty} \lambda_k \hat{f}(k) e^{ikx} \tag{1.2}$$

is a Fourier series of a function $f_\lambda \in B$. In that case we say that λ is a multiplier from A to B. If $A = B$ we say that λ is a multiplier of A. The answers to our problem we shall search in terms of the behaviour of the means of the series

$$\sum_{k=-\infty}^{\infty} \lambda_k e^{ikx}. \tag{1.3}$$

Let $\omega(\delta)$ be an abstract modulus of continuity, that is, $\omega(\delta)$ is nondecreasing, $\omega(0)=0$ and $\omega(\delta_1+\delta_2)\leqslant\omega(\delta_1)+\omega(\delta_2)$. Let $Lip(\omega; L)$ denote the set of all integrable functions for the moduli of continuity of which we have the estimate $\omega(f,\delta)_L=O(\omega(\delta))$, where

$$\omega(f,\delta)=\sup_{|h|\leqslant\delta}\|f(x+h)-f(x)\|_L.$$

In the case $\omega(\delta)=\delta^\alpha$ $(0<\alpha<1)$ the problem of multipliers of $Lip(\omega; C)$ was solved by A. Zygmund in 1959 [1]. He proved that the integrated kernel of the multiplier

$$\sum_{k\neq 0}\frac{\lambda_k}{ik}e^{ikx} \tag{1.4}$$

should belong to the Zygmund class in integral metrics L_*. Later this result was extended to more general classes of moduli of continuity and different metrics. An important development was the characterization of the properties of the modulus of continuity via suitably defined sequences (see [2]). Namely, with a given $\omega(\delta)$ let us associate a sequence $D(\omega)=\{\delta_k\}$ defined by induction

$$\delta_0=2\pi,$$

$$\delta_{k+1}=\min\left\{\delta:\max\left(\frac{\omega(\delta)}{\omega(\delta_k)},\ \frac{\delta\omega(\delta_k)}{\delta_k\omega(\delta)}\right)=\frac{1}{2}\right\}. \tag{1.5}$$

The latest result in this direction was (see [3], also [4,5]).

Theorem A. *Let $D(\omega)=\{\delta_k\}$ be defined by* (1.5) *and let $n_k=[1/\delta_k]$. Then a sufficient condition for $\lambda=\{\lambda_k\}$ to be a multiplier of $Lip(\omega; L)$ is*

$$\|v_{n_{k+1}}\Lambda-v_{n_k}\Lambda\|_L=O(1)\qquad(k\to\infty). \tag{1.6}$$

If $\omega(\delta)$ is slowly decreasing, then this condition is also necessary.

Here $v_n\Lambda$ denotes the n, $2n$ Vallée Poussin means of the series (1.3) ($s_n\Lambda$ is the symmetrical partial sum of (1.3))

$$v_n\Lambda=\frac{1}{n}(s_n\Lambda+s_{n+1}\Lambda+\ldots+s_{2n-1}\Lambda).$$

In the present paper we intend to give an equivalent of Theorem A in the setting of abstract Banach spaces. In doing that we follow the ideas of the Aachen school [6, 7].

Section 2 is devoted to the exposition of the concept of biorthogonal expansions in Banach spaces. For details we refer to [6; 7] and the literature cited there. In Section 3 we prove the main result of this paper and Section 4 is devoted to discussion and possible further applications.

2. ADMISSIBLE BANACH SPACES

Let **Z** be the set of all integers, **C** the set of complex numbers, and for some natural number N let $\mathbf{Z}^N$ denote the N-fold Cartesian product of $\mathbf{Z}$. Let $[\alpha]$ denote the integer part of a real number α and $|k|$ the Educlidean norm of a multiindex k.

Let X and Y be complex Banach spaces with the norm $\|\cdot\|_X$ and dual X^*. Then $[X, Y]$ denotes the set of all bounded linear operators of X into Y. We shall write $[X]$ for $[X, X]$.

For an arbitrary Hilbert space H with inner product $(\cdot,\cdot)$ let $\{f_k; k\in \mathbf{Z}^N\}$ be an orthonormal sequence with respect to that inner product. Let

$$\Pi=\Pi(\{f_k\})=\Big\{p\in H; p=\sum_{\text{finite}}\alpha_k f_k,\ \alpha_k\in \mathbf{C}\Big\}$$

be the set of polynomials generated by $\{f_k\}$, and

$$\Pi_\rho=\Big\{p\in H; p=\sum_{|k|\leqslant\rho}\alpha_k f_k,\ \alpha_k\in \mathbf{C}\Big\}$$

be the subset of polynomials of (radial) degree of ϱ.

Let the pair H, $\{f_k\}$ be fixed. The Banach spaces we shall study will be constructed via the following procedure.

A Banach space is called *admissible* (with respect to the given orthonormal structure H, $\{f_k\}$) if

$$\{f_k\}\subset X,\quad \text{and}\quad \Pi \text{ is dense in } X,$$

$$|(p, f_k)|\leqslant A_k\|p\|_X\quad (p\in\Pi,\ k\in \mathbf{Z}^N),$$

$$\{f_k^*\}\ \text{is total on } X.$$

Here $f_k^*\in X^*$ denotes the unique bounded linear extension of the functional that is generated by f_k via $f_k^*(p)=(p,f_k)$ on $\Pi\subset X$. The sequence $\{f_k^*\}$ is said to be total on X if $f_k^*(f)=0$ for all $k\in\mathbf{Z}^N$ and some $f\in X$ necessarily implies $f=0$.

Let s be the set of sequences $\lambda=\{\lambda_k;\ k\in\mathbf{Z}^N\}$ of complex numbers. Let X and Y be admissible. A sequence $\lambda\in s$ is called a multiplier of the type (X, Y) if to each $f\in X$ there corresponds an $f_\lambda\in Y$ such that

$$f_k^*(f_\lambda)=\lambda_k f_k^*(f)\quad (k\in\mathbf{Z}^N).$$

The set of all multipliers of the type (X,Y) is denoted by $M(X,Y)$. We also use the notation $M(X)=M(X,X)$.

With each $\lambda\in M(X,Y)$ we may associate the multiplier operator $\Lambda\in[X,Y]$ with $\Lambda f=f_\lambda$. The set of all multiplier operators is denoted by $[X,Y]_M$. With the natural vector operations $M(X,Y)$ becomes a Banach space in respect to the operator norm

$$\|\lambda\|_{M(X,Y)}=\sup_{\|f\|_X=1}\|f_\lambda\|_Y=\|\Lambda\|_{[X,Y]}.$$

With any element f of an admissible Banach space we may associate its Fourier expansion

$$f\sim\sum_{k\in\mathbf{Z}^N}f_k^*(f)f_k. \tag{2.1}$$

The (radial) partial sums of (2.1) are defined by

$$S_\rho f=\sum_{|k|\leqslant\rho}f_k^*(f)f_k\quad (f\in X,\ \varrho\geqslant 0)$$

and the Riesz means of the order $\alpha\geqslant 0$ by

$$(R,\alpha)_\rho f=\sum_{k\in\mathbf{Z}^N}r_\alpha(|k|/\varrho)f_k^*(f)f_k,$$

where

$$r_\alpha(t)=[(1-t)_+]^\alpha=\begin{cases}(1-t)^\alpha, & \text{if}\quad 0\leqslant t\leqslant 1;\\ 0, & \text{if}\quad t\geqslant 1.\end{cases}$$

We say that an admissible Banach space is *regular* if there exists some $\alpha\geqslant0$ such that

$$\|r_\alpha(|k|/\varrho\|_{M(X)}\leqslant C_\alpha, \tag{2.2}$$

the constant C_α being independent of $\varrho\geqslant0$.

For our approach it is essential that in regular spaces we can use the Vallée Poussin means. Let $\theta(t)$ be an arbitrarily often differentiable function satisfying $0\leqslant\theta(t)\leqslant1$,

$$\theta(t)=\begin{cases}1, & \text{if } 0\leqslant t\leqslant1;\\ 0, & \text{if } t\geqslant2.\end{cases}$$

The Vallée Poussin means of (2.1) are defined by $(v_0f=0)$

$$v_\varrho f=\sum_{k\in Z^N}\theta(|k|/\varrho)f_k^*(f)f_k \quad (f\in X,\ \varrho\geqslant0). \tag{2.3}$$

By $E_\varrho(f;X)$ we denote the best approximation of $f\in X$ by polynomials of the degree $\varrho\geqslant0$,

$$E_\varrho(f;X)=\inf\{\|f-p\|_X:\ p\in\Pi_\varrho\}.$$

The following lemma summarizes some well-known properties of the Vallée Poussin means in the present setting.

Lemma 1 (see [7]). *For an admissible Banach space X let* (2.2) *be satisfied for some $\alpha\geqslant0$. Then the means* (2.3) *possess the properties*

(*i*) $\quad v_\varrho f\in\Pi_{2\varrho}\subset H\cap X$ *for each* $f\in X$;

(*ii*) $\quad v_\varrho p=p$ *for each* $p\in\Pi_\varrho$;

(*iii*) $$\|v_\varrho f\|_X\leqslant D_\alpha\int_0^2 t^{[\alpha]+1}\left|\theta^{([\alpha]+2)}(t)\right|dt\|f\|_X,$$

the constant D_α being independent of $f\in X$ and $\varrho\geqslant0$;

(*iv*) $$\|v_\varrho f-f\|_X\leqslant CE_\varrho(f;X),$$

the constant C being independent of $f\in X$, $\varrho\geqslant0$.

Next we introduce the generalized Lipschitz classes via the Peetre's K-functional. This is a well-known way to use some measure of smoothness in abstract spaces. The K-functional is defined for $f\in X$, and $t\geqslant0$ by

$$K(X,Z;f,t)=\inf_{g\in Z}(\|f-g\|_X+t|g|_Z), \tag{2.4}$$

where $Z\subset X$ is a subspace with the seminorm $|\cdot|_Z$. In many cases (2.4) is equivalent to some standard modulus of continuity. For a given modulus of continuity $\omega(t)$ we define the generalized Lipschitz class

$$Lip(\omega,X)=\{f\in X;\ K(X,Z;f,t)=O(\omega(t)),\ t\to0+\}.$$

We also need two inequalities connecting the best approximations to the properties of smoothness. These are the so-called Jackson-type inequality $(A\geqslant1)$

$$E_\varrho(g;X)\leqslant(A/\varrho)|g|_Z \quad (g\in Z) \tag{2.5}$$

and a Bernstein-type inequality

$$|p_\rho|_Z \leqslant \rho \|p_\rho\|_X \quad (p_\rho \in \Pi_\rho), \tag{2.6}$$

where we suppose that $\Pi \subset Z \subset X$.

As to the modulus of continuity, we exclude the class $Lip(\delta)$ supposing that

$$\sup_{t>0} \frac{\omega(t)}{t} = \infty. \tag{2.7}$$

3. MAIN THEOREM

Suppose that the seminorm $|\cdot|_Z$ and the multiplier operator are connected by the condition

$$|\Lambda p|_Z \leqslant \|\Lambda\|_{M(X)} |p|_Z \quad (p \in \Pi). \tag{3.1}$$

Also we say that $\omega(t)$ is slowly decreasing if

$$\frac{\omega(t)}{\sqrt{t}} \uparrow \infty \quad (t \to 0+).$$

Then we have

Theorem 1. *Let X be regular and suppose that the Jackson-type and the Bernstein-type inequalities* (2.5) *and* (2.6) *and* (3.1) *are satisfied. Let $\omega(t)$ be slowly decreasing and let the sequence $D(\omega)=\{\delta_j\}$ be defined by* (1.5). *Let $\rho_j = 1/\delta_j$. Then a necessary and sufficient condition for a sequence $\lambda \in s$ to be a multiplier of $Lip(\omega, X)$ is that*

$$\|w_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\|_{M(X)} = O(1) \quad (j \to \infty). \tag{3.2}$$

Proof:

Sufficiency: Suppose (3.2) holds. Let $f \in Lip(\omega, X)$. Consider the decomposition $(v_{\rho_0} f = 0)$

$$f = \sum_j \{v_{\rho_{j+1}} f - v_{\rho_j} f\}. \tag{3.3}$$

As v_ρ is a bounded linear operator from X into Π_ρ we have $\Lambda(v_\rho f) = v_\rho f_\lambda$. Thus in view of part (iv) of Lemma 1 we have

$$\sum_j \|\Lambda\{v_{\rho_{j+1}} f - v_{\rho_j} f\}\|_X \leqslant \sum_j \|\{v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\}\{v_{2\rho_{j+1}} f - v_{\rho_j/2} f\}\|_X$$

$$= O\Big(\sum_j \|v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\|_{M(X)} E_{\rho_j/2}(f; X) \Big)$$

$$= O\Big(\sum_j \|v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\|_{M(X)} \omega(\delta_j) \Big)$$

$$= O\Big(\sum_j \omega(\delta_j) \Big). \tag{3.4}$$

In view of (1.5) the last series converges. Hence this series corretly defines an element $f_\lambda \in X$.

Next we have to demonstrate that f_λ is an element of $Lip(\omega, X)$. Let us estimate the corresponding K-functional. We need the basic property of the sequences $D(\omega)$: they are maximal ω-lacunary sequences in the following sense.

Lemma 2 (see [2]). *Let* $\omega(\delta)$ *satisfy* (2.7) *and let the sequence* $D(\omega)=\{\delta_j\}$ *be defined by* (1.5). *Then*

$$(i) \qquad \delta_{j+1}/\delta_j \leqslant 1/4 \quad (j=0,1,\ldots);$$

$$(ii) \qquad (1/c)\,\omega(\delta) \leqslant \sum_{j=0}^{\infty} \omega(\delta_j) \min(1, \delta/\delta_j) \leqslant c\omega(\delta) \quad (\delta>0).$$

As $v_\rho f_\lambda \in \Pi_\rho \subset Z$, we have for $\delta_{l+1} \leqslant t \leqslant \delta_l$

$$K(X,Z;f_\lambda,t) = \inf_{g\in Z} (\|f_\lambda - g\|_X + t|g|_Z)$$
$$\leqslant \|f_\lambda - v_{\rho_{l+1}} f_\lambda\|_X + t|v_{\rho_{l+1}} f_\lambda|_Z. \tag{3.5}$$

Using the decomposition (3.3) we obtain, as we did in (3.4), the standard estimate for the first term in (3.5):

$$\|f_\lambda - v_{\rho_{l+1}} f_\lambda\|_X = O\Big(\sum_{j=l+1}^{\infty} \omega(\delta_j)\Big). \tag{3.6}$$

As to the second term, we apply (3.1) to a similar decomposition. Hence by Lemma 1

$$t|v_{\rho_{l+1}} f_\lambda|_Z = t\Big|\sum_{j=0}^{l} \{v_{\rho_{j+1}} f_\lambda - v_{\rho_j} f_\lambda\}\Big|_Z$$
$$\leqslant t\sum_{j=0}^{l} |\{v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\}\{v_{2\rho_{j+1}} f - v_{\rho_j/2} f\}|_Z$$
$$\leqslant \sum_{j=0}^{l} \|v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\|_{M(X)} \|v_{2\rho_{j+1}} f - v_{\rho_j/2}\|_X$$
$$= O\Big(\sum_{j=0}^{l} \omega(\delta_j)\Big). \tag{3.7}$$

Combining (3.6) and (3.7) with Lemma 2 and (3.5) we get the final estimate

$$K(X,Z;f_\lambda,t) = O(\omega(t)).$$

Thus we have proved the sufficiency part of the theorem.

Necessity: To prove this part we use the following

Proposition 1 (see [7]). *Let X be regular and* ω *be such that* (2.7) *holds true. Suppose that the Bernstein-type inequality* (2.6) *is satisfied. If* $\{\varrho_j\}$ *is a sequence of positive numbers monotonically increasing to infinity such that*

$$\sum_{j=0}^{l-1} \omega(\varrho_j)/\varrho_j \leqslant \omega(\varrho_l)/\varrho_l,$$
$$\omega(\varrho_l) \leqslant (1/2)\,\omega(\varrho_{l-1}),$$

then the element

$$\omega = \sum_{j=0}^{\infty} h_l, \quad h_l = \omega(1/\varrho_l)\,\omega_{\rho_l},$$

belongs to $Lip(\omega, X)$ *for any* $\omega_\rho \in \Pi_{2\rho}$ *with* $\|\omega_\rho\|_X \leqslant B$.

Observe that if $D(\omega)$ is defined by (1.5) and $\varrho_j = 1/\delta_j$, then in view of Lemma 2 the sequence $\{\varrho_j\}$ satisfies the conditions of Proposition 1.

Suppose (3.2) does not hold. Then there exists a sequence of indices $\{j(l)\}$ such that

$$\|v_{\rho_{j(l)+1}}\Lambda - v_{\rho_{j(l)}}\Lambda\|_{M(X)} > l \quad (l \to \infty). \tag{3.8}$$

In the following we shall write j for $j(l)$. Let $\varepsilon > 0$ be fixed. For every j let $\omega_{\rho_{j+1}}$ be an element of X with $\|\omega_{\rho_{j+1}}\|_X \leqslant 1$ such that

$$\begin{aligned}\|(v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda)\,\omega_{\rho_{j+1}}\|_X &= \sup_{\|f\|_X \leqslant 1} \|(v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda) f\|_X - \varepsilon \\ &= \|v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\|_{M(X)} - \varepsilon.\end{aligned} \tag{3.9}$$

Since the supports of the operators $v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda$ are by definition concentrated on $\Pi_{2\rho_{j+1}} \setminus \Pi_{\rho_j}$ we may also suppose that $\omega_{\rho_{j+1}} \in \Pi_{2\rho_{j+1}} \setminus \Pi_{\rho_j}$. Apply Proposition 1 to the sum

$$\omega = \sum_{j=0}^{\infty} h_j, \quad \text{with} \quad h_j = \omega(\delta_{j+1})\,\omega_{\rho_{j+1}}.$$

We get that $\omega \in Lip(\omega, X)$.

Next let us consider the difference $v_{\rho_{j+1}}\omega_\lambda - v_{\rho_j}\omega_\lambda$. If ω_λ is to be in $Lip(\omega, X)$ we should have by part (iv) of Lemma 1

$$\|v_{\rho_{j+1}}\omega_\lambda - v_{\rho_j}\omega_\lambda\|_X \leqslant 2E_{\rho_j}(\omega_\lambda; X) = O(\omega(\delta_j)).$$

On the other hand, we have by (3.8) ant (3.9) (taking into account that $\omega(t)$ is slowly decreasing, thus $\omega(\delta_{j+1}) = (1/2)\,\omega(\delta_j)$)

$$\begin{aligned}\|v_{\rho_{j+1}}\omega_\lambda - v_{\rho_j}\omega_\lambda\|_X &= \|\{v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\} h_j\|_X \\ &= \omega(\delta_{j+1})\,(\|v_{\rho_{j+1}}\Lambda - v_{\rho_j}\Lambda\|_{M(X)} - \varepsilon) \\ &\geqslant \omega(\delta_j)\, l \neq O(\omega(\delta_j)).\end{aligned}$$

This contradiction proves the theorem.

4. DISCUSSION

First of all we would like to note that the proof of Theorem 1 is actually simpler in this abstract setting than it was in the one-dimensional periodic case. This is mainly due to the definition of the modulus of continuity directly via approximations rather than via the norms of the first differences. So we can skip some sequels of the proofs. On the other hand, this work is now incorporated into condition (3.1), which in the classical case is superfluous.

The present approach in addition to the well-known one-dimensional periodic case also covers the radial multivariate periodic case. As to possible further applications we would like to point to Jacobi expansions with suitably defined shift operators and also Legendre expansions, but we postpone detailed investigations in these directions to later dates. The main problem arising in this context is the interpretation of condition (3.1) and its implications in these settings.

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ORTOGONAALSETE ARENDUSTE MULTIPLIKAATORID BANACHI RUUMIDES

Jüri LIPPUS

On leitud tarvilikud ja piisavad tingimused Fourier' multiplikaatoritele üldistatud Lipschitzi klassidel teatavate regulaarsuse omadustega Banachi ruumides. Ühemõõtmelise perioodilise juhu puhul olid tulemused varem teada.

МУЛЬТИПЛИКАТОРЫ ОРТОГОНАЛЬНЫХ РАЗЛОЖЕНИЙ В НЕКОТОРЫХ БАНАХОВЫХ ПРОСТАНСТВАХ

Юри ЛИППУС

Находятся необходимые и достаточные условия для мультипликаторов обобщенных классов Липшица в определенном смысле регулярных банаховых пространствах. Для одномерного периодического случая эти условия были известны ранее.