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NORMALIZATION FOR ARITHMETICAL COMPREHENSION WITH RESTRICTED OCCURRENCES OF HILBERT'S EPSILON-SYMBOL

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Abstract. We present a normalization proof for the second order arithmetic with arithmetical comprehension and Hilbert's epsilon-axiom $F[T] \to F[\varepsilon XFX]$ which represents a kind of choice principle. The proof is carried out by transfinite induction up to ε_0.

Key words: second order arithmetic, normalization of proof, axiom of choice, Hilbert's epsilon-axiom.

INTRODUCTION

Since the existence of a normal form for the second order classical logic with the axiom of choice was proved in [1] there has been no progress in establishing a normalization theorem for that theory or even in finding a promising set of reductions. It became possible to look at this problem from another point of view after it was noted in [2] that the (0,1)-Axiom of Choice is derivable in Hilbert's epsilon-calculus (the derivation in fact is contained in [3], pp. 467—469). This gave an opportunity to apply normalization techniques to systems with epsilon-symbol, developed, for instance, in [3], [4] and [5] for normalizing theories with various kinds of the axiom of choice (AC).

Analysis of the derivation of AC in [3] shows that in the presence of quantifiers only epsilon-terms of a special kind are needed for deriving AC: we can assume that epsilon-terms do not contain secnod order variables bound by exterior quantifiers or epsilon-symbols (though they may contain second order variables bound inside them). This restriction allows to avoid the problems connected with the absence of a notion of rank in the second order logic and can be kept under some reasonable sequence of reductions [6].

In this paper we examine the sequence of reductions [6] for a weak subsystem of analysis, second order arithmetic with arithmetical comprehension and corresponding choise principle $FT \to F[\varepsilon XFX]$ with arithmetical T, and prove its convergence by induction up to ε_0. Note that, as it follows from [7], one cannot add an unrestricted axiom of choice to weak predicative subsystems without increasing the proof-theoretical strength of the theory.

By arithmetical analysis **AA** we mean the second order arithmetic with arithmetical comprehension (without parameters). As it is noted in [7], this system is conservative over Peano Arithmetic. By arithmetical analysis with epsilon-symbol **AA**ε we mean the following extension of **AA**:

(1) an additional item in the definition of terms and formulas is allowed: if FA is a formula and A does not occur in the scope of any epsilon-symbol of FA then εXFX is a 1-term (predicator);

(2) second order quantification is restricted with respect to epsilon-symbol: if FA is a formula and A does not occur in the scope of any epsilon-symbol in FA then $\exists XFX$ and $\forall XFX$ are formulas;

(3) we have additional axioms $FT \rightarrow F[\varepsilon XFX]$ for T being arithmetic lambda-terms.

We provide the embedding of **AA**ε into its ω-version **AA**$\omega\varepsilon$, where positive occurences of first order quantifiers are introduced by ω-rules and Hilbert's epsilon-axiom is taken in the form of epsilon-rule

$$\frac{\Gamma \rightarrow \Theta, FT \quad F[\varepsilon XFX], \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta}\varepsilon,$$

and normalization of **AA**$\omega\varepsilon$. Two types of reductions are developed. One is standard cutelimination as presented in [8], [9] (see Lemma 4.1, Theorem 4.2 of this paper). Another type of reductions is elimination of epsilon-rules in a way similar to [5] (Lemma 4.3, Theorem 4.4).

1. DESCRIPTION OF THE SYSTEM AA$\omega\varepsilon$

1.1. The language

Let us use function constants 0 (nil), $'$ (next), and possibly other constants for computable functions; equality $=$; bound individual variables; free and bound predicate variables; logical connectives $\neg$, $\vee$, $\wedge$, $\exists$, $\forall$, λ (lambda-symbol); ε (second order epsilon-symbol); subsidiary symbols (,),, $\rightarrow$.

By denotation $e[s]$, or shortly es, we will distinguish some occurrences of a subword s in a word e.

1.2. 0-terms

0-terms are built from function constants.

Note that all 0-terms have their values calculated via interpretations of function constants.

1.3. 1-terms and formulas

1-terms and *formulas* are defined simultaneously.

1) A free predicate variable is a *1-term*;

2) if s and t are 0-terms and T is a *1-term* then $s=t$ and $T(t)$ are *formulas*;

3) if F and G are *formulas* then $\neg F$, $F \vee G$, and $F \wedge G$ are *formulas*;

4) if $F0$ is a *formula* then $\exists xFx$ and $\forall xFx$ are *formulas*;

5) if FA is a *formula* and A does not occur in the scope of any epsilon-symbol in FA then $\exists XFX$ and $\forall XFX$ are *formulas*;

6) if FA is a *formula* and A does not occur in the scope of any epsilon-symbol in FA then εXFX is a *1-term*;

7) if $F0$ is a *formula* then λxFx is a *1-term*.

A formula is *elementary* iff it is of a form $s=t$, $A(t)$ or $\varepsilon XFX(t)$.

1.4. Quasiterms and quasiformulas

Quasiterms and *quasiformulas* are obtained from terms and formulas by replacement of some occurrences of numerals and free predicate variables by bound variables. An occurrence of a bound variable is called *principal* in a quasiterm or quasiformula E iff it is not bound by a quantifier or epsilon- or lambda-symbol in E.

An *expression* is a quasiterm or quasiformula.

An expression is *arithmetical* iff it does not contain predicate variables.

Let e be a quasiterm and E be an expression. *Epsilon-degree* of an occurrence of e in E is the number of epsilon-symbols in E to the scopes of which this particular occurrence of e belongs. e is a *quasisubterm* of E iff its principal variables are principal in E.

Note 1. If E is an expression and e is its quasisubterm then the result of substituting terms for all principal variables of e is a subterm of the result of the same substitution in E.

Note 2. If F is a formula and T is an epsilon-quasiterm occurring in F then T contains no principal predicate variables.

Note 3. If F is a formula and $T1$ and $T2$ are occurrences of epsilon-quasiterms in F then one of the following holds:

(1) $T2$ is a subword of $T1$;

(2) $T1$ is a subword of $T2$;

(3) no occurrences of letters in $T1$ belong to $T2$.

1.5. Matrix

The *matrix* of an epsilon-quasiterm is obtained by replacement of its exterior 0-quasisubterms by free individual variables. Two epsilon-quasiterms are *congruent* iff their matrices coincide (up to names of variables).

Two formulas or epsilon-terms are *similar* iff they have the same expression after the replacement of exterior 0-subterms by their values.

1.6. Rules of inference

● Axioms:

$$D, \Gamma \to \Theta, D,$$

where D is elementary;

● Rules for the introduction of logical connectives: usual Gentzen-type rules for ω-system preserving main formula in premises, for instance:

$$\frac{\Gamma \to \Theta, F \wedge G, F \quad \Gamma \to \Theta, F \wedge G, G}{\Gamma \to \Theta, F \wedge G} \to \wedge;$$

$$\frac{\ldots \Gamma \to \Theta, \forall x F x, Fn; \ \ldots \ (n < \omega)}{\Gamma \to \Theta, \forall x F x} \to \forall;$$

$$\frac{FT, \forall X F X, \Gamma \to \Theta}{\forall X F X, \Gamma \to \Theta} \forall\forall \to,$$

where T is an arithmetical 1-term or a free predicate variable;

$$\frac{\Gamma \to \Theta, \exists XFX, FT}{\Gamma \to \Theta, \exists XFX} \to \exists,$$

where T is an arithmetical 1-term or a free predicate variable;

$$\frac{\Gamma \to \Theta, \forall XFX, FA}{\Gamma \to \Theta, \forall XFX} \to \forall$$

(A does not occur in the conclusion);

$$\frac{\Gamma \to \Theta, \lambda x Fx(t), Ft}{\Gamma \to \Theta, \lambda x Fx(t)} \to \lambda;$$

- Epsilon-rule:

$$\frac{\Gamma \to \Theta, FT \quad F[\varepsilon XFX], \Gamma \to \Theta}{\Gamma \to \Theta} \varepsilon,$$

where T is an artihmetical 1-term or a free predicate variable;

- Equality rule:

$$\frac{s=t, \Gamma[s] \to \Theta[s]}{s=t, \Gamma[t] \to \Theta[t]} \mathrm{Eq};$$

- Mathematical rules:

$$\frac{s=t, \Gamma \to \Theta}{\Gamma \to \Theta} \to M,$$

where $s=t$ is a true equality;

$$\frac{\Gamma \to \Theta, s=t}{\Gamma \to \Theta} M \to,$$

where $s=t$ is a false equality;

- Cut:

$$\frac{\Gamma \to \Theta, F \quad F, \Gamma \to \Theta}{\Gamma \to \Theta} \mathrm{cut}.$$

Lists of formulas are treated up to permutations of their members. Cuts and epsilon-rules will be jointly called *cut-epsilon-rules*.

1.7. Height h of a derivation

Let d be a derivation. We define $h(d)$ by induction on d.
If d is an axiom with main formula $s=t$ then $h(d) := 0$;
if d is an axiom with main formula $A(t)$ or $\varepsilon XFX(t)$ then $h(d) := \omega$;
if d ends in an equality or mathematical rule and h_0 is the height of the derivation of its premise then $h(d) := h_0$;

if d ends in any other rule and h_i are the heights of the derivations of its premises then $h(d) := \sup_i (h_i + 1)$.

1.8. Embedding of AAε into AAωε

Arithmetical analysis with epsilon-symbol is obviously embeddable into AAωε: first, comprehension axiom is cut-epsilon-free derivable by a derivation of finite height using $\to\lambda$, $\lambda\to$-rules introducing $\lambda x Fx$ and $\to\exists\exists$-rule with $\lambda x Fx$ as side term: we derive $\to \forall y (\lambda x Fx(y) \leftrightarrow Fy)$

and then $\rightarrow \exists X \forall y(X(y) \leftrightarrow Fy)$; second, Hilbert's epsilon-axiom $FT \rightarrow F[\varepsilon XFX]$ is cutfree derivable by a derivation of the height $< \omega * 2$ using epsilon-rule with side term T and main term εXFX: we apply it to derivations of $FT \rightarrow FT, F[\varepsilon XFX]$ and $FT, F[\varepsilon XFX] \rightarrow F[\varepsilon XFX]$; third, induction-rule is derivable via cuts and ω-rule similarly to [9], Theorem 20.13: its translation increases the height up to the first limit ordinal greater than heights of translations of premises.

1.9. Lemma

For each epsilon-rule with side formula FT and main formula $F[\varepsilon XFX]$ the following holds:

1) each occurrence of εXFX in $F[\varepsilon XFX]$ shown explicitly has epsilon-degree 0 *in it;*

2) each epsilon-quasiterm occurring in FT, $F[\varepsilon XFX]$ either is the term εXFX shown explicitly or occurs inside such εXFX;

3) formulas FT and $F[\varepsilon XFX]$ contain no epsilon-quasiterms congruent with εXFX except εXFX shown explicitly.

Proof.

1) Suppose that some occurrence of εXFX shown explicitly has epsilon-degree > 0 in $F[\varepsilon XFX]$. Then there is an epsilon-quasiterm R in FA containing A. But it is impossible due to definition 1.3, 6).

2) Let R be an epsilon-quasiterm in FT or $F[\varepsilon XFX]$ distinct from εXFX shown explicitly. According to 1.4, Note 3 either one of R, εXFX occurs inside the other or they do not intersect. T and εXFX cannot be subterms of R due to 1). R cannot occur inside T since T contains no bound predicate variables. If R and εXFX do not intersect the R occurs in FA and A does not occur in R and hence R occurs inside εXFX.

3) Immediate from 2) in view of the observation that congruent epsilon-quasiterms cannot occur inside each other. □

2. RANK

2.1. Rank of artihmetical expressions

1) Rank of a 0-term is 0;

2) rank of $s=t$ is 0; rank of $\lambda xFx(t)$ is $\text{rank}(\lambda xFx)$;

3) rank of $\neg F$ is $\text{rank}(F)+1$; rank of $F \vee G, F \wedge G$ is $\max(\text{rank}(F), \text{rank}(G))+1$;

4) rank of $\exists xFx$, $\forall xFx$ is $\text{rank}(F0)+1$;

5) rank of λxFx is $\text{rank}(F0)+1$.

2.2. *R*-rank

Let R be an integer. If the opposite is not stated explicitly, everywhere below "rank" means R-rank.

1) rank of a 0-term is 0;

2) rank of $s=t$ is 0;

3) rank of a free predicate variable is R;

4) rank of $A(t)$, $\lambda xFx(t)$, $\varepsilon XFX(t)$ is $\text{rank}(A)$, $\text{rank}(\lambda xFx)$, $\text{rank}(\varepsilon XFX)$, respectively;

5) rank of $\neg F$ is $\text{rank}(F)+1$; rank of $F \vee G$, $F \wedge G$ is $\max(\text{rank}(F), \text{rank}(G))+1$;

6) rank of $\exists xFx$, $\forall xFx$, λxFx is $\text{rank}(F0)+1$; rank of $\exists XFX$, $\forall XFX$, εXFX is $\text{rank}(FA)+1$.

Rank of an expression E is the rank of the term or formula from which E is obtained.

εR-rank of a formula or sequent is the maximum R-rank of epsilon-quasiterms occurring in it.

2.3. Rank of a cut

Rank of a cut is the rank of its cut-formula; rank of an epsilon-rule is the rank of its main term.

2.4. Lemma

1) Congruent epsilon-quasiterms have the same rank;

2) if e is an epsilon-quasiterm occurring in an epsilon-term or formula E then $\mathrm{rank}(e) \leqslant \mathrm{rank}(E)$; *if additionally e is not a term then* $\mathrm{rank}(e) < \mathrm{rank}(E)$.

Immediate from the definition of rank.

2.5. Lemma

S, T being 1*-terms and EA being a term or a formula, if* $\mathrm{rank}(S) \leqslant \mathrm{rank}(T)$ *then* $\mathrm{rank}(ES) \leqslant \mathrm{rank}(ET)$.

Proof is by induction on the expression EA.

If EA is an atomic formula not containing A then the assertion is evident. If EA is a formula $A(t)$ then the assertion follows from item 4 of the definition of R-rank. If EA is a formula $\varepsilon YGY(t)$ or $\lambda yGy(t)$ then the assertion follows from the induction hypothesis for εYGY or λyGy.

If EA is a formula $\neg F$, $F \vee G$ or $F \wedge G$ then the assertion follows from the hypotheses for F and G. If EA is a formula $\exists yGy$, $\forall yGy$ or a term λyGy then the assertion follows from the hypothesis for the formula $G0$. Finally, if EA is a formula $\exists YGY$, $\forall YGY$ or a term εYGY then the assertion follows from the hypothesis for GB.

1) EA is a 0-term. Then
$\mathrm{rank}(ES) = \mathrm{rank}(EA) = \mathrm{rank}(ET) = 0$.

2) EA is an atomic formula not containing A. Then
$\mathrm{rank}(ES) = \mathrm{rank}(EA) = \mathrm{rank}(ET)$.

3) EA is a formula $A(t)$. Then
$\mathrm{rank}(ES) = \mathrm{rank}(S(t)) = \mathrm{rank}(S) \leqslant \mathrm{rank}(T) = \mathrm{rank}(T(t)) = \mathrm{rank}(ET)$.

4) EA is a formula $\lambda yG[y, A](t)$. Then
$\mathrm{rank}(ES) = \mathrm{rank}(\lambda yG[y, S](t)) = \mathrm{rank}(\lambda yG[y, S]) \leqslant$
/* induction hypothesis */ $\leqslant \mathrm{rank}(\lambda yG[y, T]) = \mathrm{rank}(\lambda yG[y, T](t)) = \mathrm{rank}(ET)$.

5) EA is a formula $\varepsilon YG[Y, A](t)$. Then
$\mathrm{rank}(ES) = \mathrm{rank}(\varepsilon YG[Y, S](t)) = \mathrm{rank}(\varepsilon YG[Y, S]) \leqslant$
/* induction hypothesis */ $\leqslant \mathrm{rank}(\varepsilon YG[Y, T]) = \mathrm{rank}(\varepsilon YG[Y, T](t)) = \mathrm{rank}(ET)$.

6) EA is a formula $\neg FA$, $FA \vee GA$ or $FA \wedge GA$. Then
$\mathrm{rank}(\neg FS) = \mathrm{rank}(FS) + 1 \leqslant \mathrm{rank}(FT) + 1 = \mathrm{rank}(\neg FT)$,
$\mathrm{rank}(ES) = \max(\mathrm{rank}(FS), \mathrm{rank}(GS)) + 1 \leqslant \max(\mathrm{rank}(FT), \mathrm{rank}(GT)) + 1 = \mathrm{rank}(ET)$.

7) EA is a formula $\exists yG[y, A]$, $\forall yG[y, A]$ or a term $\lambda yG[y, A]$. Then
$\mathrm{rank}(ES) = \mathrm{rank}(G[0, S]) + 1 \leqslant \mathrm{rank}(G[0, T]) + 1 = \mathrm{rank}(ET)$.

8) EA is a formula $\exists YG[Y, A]$, $\forall YG[Y, A]$ or a term $\varepsilon YG[Y, A]$. Then $\mathrm{rank}(ES) = \mathrm{rank}(G[B, S]) + 1 \leqslant \mathrm{rank}(G[B, T]) + 1 = \mathrm{rank}(ET)$. □

2.6. Lemma

$\mathrm{Rank}(\varepsilon XFX) = \mathrm{rank}(\exists XFX) = \mathrm{rank}(\forall XFX) > \mathrm{rank}(FT)$ *for arithmetical* T *of* $\mathrm{rank} \leqslant R$ *or a free variable.*

Proof. By the definition $\mathrm{rank}(\varepsilon XFX) = \mathrm{rank}(\exists XFX) = \mathrm{rank}(\forall XFX) = \mathrm{rank}(FA) + 1 > \mathrm{rank}(FT)$ due the previous Lemma. □

2.7. Symbols $d(R, \alpha, r1, r2) \vdash$ and $(R, \alpha, r1, r2) \vdash$

Let R, $r1$ and $r2$ be integers, α be an ordinal $< \varepsilon_0$ and S be a sequent. Denotation $(R, \alpha, r1, r2) \vdash S$ means that there is a derivation d of S such that:

1) arithmetical ranks of arithmetical side terms of rules $\forall\forall\to$, $\to\exists\exists$ and ε in d are $\leqslant R$;
2) $h(d) \leqslant \alpha$;
3) R-ranks of all cuts in d are $< r1$;
4) R-ranks of all epsilon-rules in d are $\leqslant r2$.

Denotation $d(R, \alpha, r1, r2) \vdash S$ means that d is such a derivation of S.

2.8. Embedding Lemma

If a sequent S *is derivable in* $\mathrm{AA}\varepsilon$ *then there are* R *and* r *such that* $(R, \omega^2, r, r) \vdash S$.

Proof. Let d be a derivation of S in $\mathrm{AA}\varepsilon$. We set $R := \max(\mathrm{rank}(T), \mathrm{rank}(\lambda xFx) \mid T$ to be a side term of epsilon-axioms and $F0$ to be an arithmetical formula of comprehension axioms in $d)$; $r :=$ the maximum R-rank of cuts, induction formulas and main terms of epsilon-axioms in $d+1$.

The translation $d\omega$ of d according to 1.8 satisfies $d\omega(R, \omega^2, r, r) \vdash S$. □

3. SUBSIDIARY OPERATIONS

3.1. Cleaning

A derivation in $\mathrm{AA}\omega\varepsilon$ is cleaned iff its main equality is true for each equality rule in it.

Any derivation can be turned into a cleaned derivation of the same sequent. To ensure this we eliminate all equality rules

$$\frac{s=t, \Gamma[s] \to \Theta[s]}{s=t, \Gamma[t] \to \Theta[t]}\,\mathrm{Eq}$$

with false main equalities $s=t$, deriving their conclusions by

$$\frac{s=t, \Gamma[t] \to \Theta[t], s=t}{s=t, \Gamma[t] \to \Theta[t]} \to M.$$

From now on we will assume all derivations to be cleaned.

Further, we will not distinguish similar formulas and epsilon-terms in derivations, since equal 0-terms can always be replaced one by another by the use of $M{\to}{-}$ and Eq-rules:

$$\frac{\dfrac{s=t, \Gamma[s] \to \Theta[s]}{s=t, \Gamma[t] \to \Theta[t]}}{\Gamma[t] \to \Theta[t]}.$$

Note 1. These transformations do not change the parameters of a derivation. That means that if S' is obtained from S by replacement of some 0-terms by equal 0-terms and $d(R, \alpha, r1, r2) \vdash S$ then $d'(R, \alpha, r1, r2) \vdash S'$ for some cleaned d'.

Note 2. Normalization steps described below in section 4 transform noncleaned derivations into cleaned ones. Normalization of a translation $d\omega$ of a derivation d in **AA**ε begins with cleaning $d\omega$.

3.2. Weakening

The transformation described here is similar to that in [8], Lemma 2.3.

If $\Gamma \subseteq \Gamma'$, $\Theta \subseteq \Theta'$ *and* $(R, \alpha\, r1, r2) \vdash \Gamma \to \Theta$ *then* $(R, \alpha, r1, r2) \vdash \Gamma' \to \Theta'$:

after renaming variables, missing members of Γ' and Θ' are added to all sequents of the derivation $\Gamma \to \Theta$.

3.3. Contraction

If $(R, \alpha, r1, r2) \vdash F, F, \Gamma \to \Theta$ *or* $(R, \alpha, r1, r2) \vdash \Gamma, \Theta, F, F$ *then* $(R, \alpha, r1, r2) \vdash F, \Gamma \to \Theta$ *or* $(R, \alpha, r1, r2) \vdash \Gamma \to \Theta, F$ respectively: all pairs (F, F), which predecessors of (F, F) in the final sequent, are replaced by F.

3.4. Inversions

Standard inversions of the rules $\to \neg$, $\neg \to$, $\vee \to$, $\to \wedge$, $\exists \to$, $\to \forall$, $\exists\exists \to$, $\to \forall\forall$, $\to \lambda$, $\lambda \to$ *hold in our system as well* (cf. [8], Lemma 2.5).

Here, as an example, we describe inversions of the rules $\to \forall$ and $\to \forall\forall$.

Let H be a derivation of either $\Gamma \to \Theta$, $\forall x Fx$ or $\Gamma \to \Theta$, $\forall XFX$ and let n be a numeral and T be an arithmetical lambda-term of rank $\leqslant R$ or a free predicate variable. A derivation of $\Gamma \to \Theta$, Fn or $\Gamma \to \Theta$, FT, respectively, is obtained in the following way:

1) eigenvariables of H are renamed so that none of them occurs in T;
2) all predecessors of $\forall x Fx$, $\forall XFX$ are replaced by Fn, FT, respectively;
3) superfluous premises of damaged rules $\to \forall$ are pruned;
4) T is substituted for eigenvariables of damaged rules $\to \forall\forall$;
5) sequents $\lambda x Fx(t)$, $\Gamma \to \Theta$, $\lambda x Fx(t)$ which appeared in place of axioms are derived without cut-epsilon-rules in a standard way;
6) contraction rules which appeared in place of former $\to \forall$, $\to \forall\forall$ are eliminated from the tops to the bottom by the contraction operation.

Note that this operation does not change the parameters as well: if $(R, \alpha, r1, r2) \vdash \Gamma \to \Theta, \neg F$ then $(R, \alpha, r1, r2) \vdash F, \Gamma \to \Theta$ etc.

4. NORMALIZATION

4.1. Lemma

If $(R, \alpha, r1, r2) \vdash$ *and* $(R, \beta, r1, r2) \vdash$

$\Gamma \to \Theta, \neg F$	$\neg F, \Gamma \to \Theta$
$\Gamma \to \Theta, F \vee G$	$F \vee G, \Gamma \to \Theta$
$F \wedge G, \Gamma \to \Theta$	$\Gamma \to \Theta, F \wedge G$
$\Gamma \to \Theta, \exists x F x$	$\exists x F x, \Gamma \to \Theta$
$\forall x F x, \Gamma \to \Theta$	$\Gamma \to \Theta, \forall x F x$
$\Gamma \to \Theta, \exists X F X$	$\exists X F X, \Gamma \to \Theta$
$\forall X F X, \Gamma \to \Theta$	$\Gamma \to \Theta, \forall X F X$
$\Gamma \to \Theta, \lambda x F x(t)$	$\lambda x F x(t), \Gamma \to \Theta$

or

$\Gamma \to \Theta, F$	$F, \Gamma \to \Theta$

for elementary F,

and $\text{rank}(\neg F, F \vee G, F \wedge G, \exists x F x, \forall x F x, \exists X F X, \forall X F X, \lambda x F x, F) = r1$ *then* $(R, \beta + \alpha, r1, r2) \vdash \Gamma \to \Theta$.

Proof is standard (cf. [8], Lemma 2.6 and [9], 22.4, Lemma 3). New cuts of ranks $< r1$ are introduced in the places of introduction of formulas $\neg F, \ldots$ from the left column. □

4.2. Cutelimination Theorem

If $(R, \alpha, r1 + 1, r2) \vdash S$ *then* $(R, 2^{\alpha}, r1, r2) \vdash S$.

Proof is by induction on α.

If S is an axiom then the assertion is trivial.

If S is the conclusion of mathematical or equality rule then the assertion follows from the inductive hypothesis by the definition of height.

If S is the conclusion of any rule except cut of rank $r1$, mathematical or equality rules then by the induction hypothesis for the premises of that rule $(R, 2^{\alpha_i}, r1, r2) \vdash$ holds for some $\alpha_i < \alpha$. Hence $(R, \sup_i(2^{\alpha_i} + 1), r1, r1) \vdash$ holds for the conclusion. The assertion follows from the fact that $\sup_i(2^{\alpha_i} + 1) \leq 2^{\alpha}$.

If S is the conclusion of a cut of rank $r1$ then by the induction hypothesis $(R, 2^{\alpha_i}, r1, r2) \vdash$ holds for its premises for some $\alpha_i < \alpha$, $i = 1, 2$. By the previous Lemma $(R, 2^{\alpha_1} + 2^{\alpha_2}, r1, r2) \vdash$ holds for the conclusion. The assertion follows from the fact that $2^{\alpha_1} + 2^{\alpha_2} \leq 2^{\alpha}$. □

4.3. Lemma (substitution for an epsilon-term)

If $(R, \alpha, r+1, r) \vdash F[\varepsilon X F X], \Gamma \to \Theta$, $\text{rank}(\varepsilon X F X) > r$, $\Gamma \to \Theta$ *contains no* $\varepsilon X F X$ *and quasiterms congruent with* $\varepsilon X F X$ *which are not terms, and* T *is an arithmetical* 1*-term of* $\text{rank} \leq R$ *or a free variable then* $(R, \alpha, r+1, r) \vdash FT, \Gamma \to \Theta$.

Proof. Let $d(R, \alpha, r+1, r) \vdash F[\varepsilon X F X], \Gamma \to \Theta$. Condition $d(R, \alpha, r+1, r) \vdash$ implies that there are no cut-epsilon-rules whose main formulas (terms) contain epsilon-quasiterms congruent with $\varepsilon X F X$ in d. It means that there are no epsilon-quasiterms congruent with $\varepsilon X F X$ which are not terms in d at all, and all occurrences of $\varepsilon X F X$ in d are predecessors of $\varepsilon X F X$ from the formula $F[\varepsilon X F X]$ shown explicitly.

Thus substitution of T for $\varepsilon X F X$ preserves all rules of inference except axioms and does not change formulas from Γ, Θ. Sequents $T(t), \Delta \to \Lambda, T(t)$ which replace axioms are cut-epsilon-free, derived in a standard way. □

4.4. Epsilon-elimination Theorem

If $(R, \alpha, r+1, r+1) \vdash \Gamma \to \Theta$ *and for each occurrence of an epsilon-quasiterm* εXFX *of rank* $> r$ *in* $\Gamma \to \Theta$ *there is a formula* $F[\varepsilon XFX]$ *in* Γ *then* $(R, \alpha, r+1, r) \vdash \Gamma \to \Theta$.

Proof is by induction on α.

If $\Gamma \to \Theta$ is an axiom then the assertion is trivial.

If $\Gamma \to \Theta$ is the conclusoin of any other rule except an epsilon-rule of rank $r+1$ then the assertion follows immediately from the induction hypothesis.

Suppose that $\Gamma \to \Theta$ is the conclusion of an epsilon-rule with main term εXFX of rank $r+1$.

If εXFX occurs in $\Gamma \to \Theta$ then $\Gamma \to \Theta$ can be presented in the form $F[\varepsilon XFX], \Gamma' \to \Theta$. Hence the right premise of this epsilon-rule is $F[\varepsilon XFX], F[\varepsilon XFX], \Gamma' \to \Theta$ and by the induction hypothesis $(R, \alpha_1, r+1, r) \vdash$ holds for it for some $\alpha_1 < \alpha$. Thus by contraction 3.3 $(R, \alpha, r+1, r) \vdash \Gamma \to \Theta$.

If εXFX does not occur in $\Gamma \to \Theta$ then by the induction hypothesis $(R, \alpha_1, r+1, r) \vdash \Gamma \to \Theta, FT$ for arithmetical T of rank$\leqslant R$ or a free variable and $(R, \alpha_2, r+1, r) \vdash F[\varepsilon XFX], \Gamma \to \Theta$ for some $\alpha_1, \alpha_2 < \alpha$. By Lemma 4.3 $(R, \alpha_2, r+1, r) \vdash FT, \Gamma \to \Theta$. Since $\text{rank}(FT) < \text{rank}(\varepsilon XFX) = r+1$, we have $(R, \alpha, r+1, r) \vdash \Gamma \to \Theta$. □

4.5. Normalization Theorem

If $(R, \alpha, r, r) \vdash S$ *and* $\varepsilon R - \text{rank}(S) = \mathbf{R}$ *then* $(R, 2^\alpha_r, 0, \mathbf{R}) \vdash S$.

Proof. If $(R, \alpha, r+1, r+1) \vdash S$ for $\mathbf{R} < r+1$ then by Epsilon-elimination Theorem $(R, \alpha, r+1, r) \vdash S$ and by Cutelimination Theorem $(R, 2^\alpha, r, r) \vdash S$. If $(R, \alpha, r+1, \mathbf{R}) \vdash S$ then by Cutelimination Theorem $(R, 2^\alpha, r, \mathbf{R}) \vdash S$.

Applying this argument r times we obtain $(R, 2^\alpha_r, 0, \mathbf{R}) \vdash S$. □

4.6. Corollary

If a sequent S *is derivable in* AAε *then* $(R, \beta, 0, \mathbf{R}) \vdash S$ *for some integer* R, *ordinal* $\beta < \varepsilon_0$ *and* $\mathbf{R} = \varepsilon R - \text{rank}(S)$.

Immediate from Embedding Lemma 1.8 and Normalization Theorem 4.5.

4.7. Corollary

If an epsilon-free sequent is derivable in AAε *then it is cut-epsilon-free derivable in* AAωε.

Immediate from Corollary 4.6.

4.8. Corollary about consistency

AAε *is consistent.*

Immediate from Corollary 4.7.

4.9. Corollary. Herbrand's theorem

If $\exists xFx$ is derivable in $AA\varepsilon$ for propositional F then $Fn_1 \vee Fn_2 \vee \ldots \vee Fn_k$ is derivable for some numerals $n_1, n_2, \ldots, n_k$.

Proof. Let d be a derivation of $\exists xFx$ in $AA\varepsilon$. Then by Corollary 4.7 $\exists xFx$ is cut-epsilon-free derivable in $AA\omega\varepsilon$. Since the latter derivation contains no ω-rules, it is a cutfree derivation in the first order predicate calculus. □

REFERENCES

1. Минц Г. Изв. АН СССР. Сер. матем., 1989, 32, 3, 587—605.
2. Simpson, S. Second Order Arithmetic and the Epsilon-Calculus. Talk at the University of Illinois, Urbana, March 1992.
3. Hilbert, D., Bernays, P. Grundlagen der Mathematik. II. Springer, 1970.
4. Leisenring, A. Mathematical Logic and Hilbert's Epsilon-Symbol. New York, 1969.
5. Минц Г. Зап. науч. семинаров ЛОМИ, 1975, 49, 67—122.
6. Tupailo, S. Research Report CS 44/92. Institute of Cybernetics, Estonian Acad. Sci., Tallinn, 1992.
7. Feferman, S. In: Handbook of Mathematical Logic, Amsterdam, 1977, 913—971.
8. Schwichtenberg, H. In: Handbook of Mathematical Logic, Amsterdam, 1977.
9. Schütte, K. Proof Theory. Springer, 1977.

NORMALISEERIMINE HILBERTI EPSILONSÜMBOLIGA ARITMEETILISE ANALÜÜSI PUHUL

Sergei TUPAILO

On esitatud teatavat valikuprintsiipi esitava Hilberti epsilonaksioomiga täiendatud aritmeetilise analüüsi normaliseeritavuse tõestus. Tõestamisel on kasutatud transfiniitset induktsiooni kuni epsilon—0-ni.

НОРМАЛИЗАЦИЯ ДЛЯ АРИФМЕТИЧЕСКОГО АНАЛИЗА С ГИЛЬБЕРТОВСКИМ ЭПСИЛОН-СИМВОЛОМ

Сергей ТУПАЙЛО

Приведено доказательство нормализуемости для арифметики второго порядка с арифметическим свертыванием и Гильбертовской эпсилон-аксиомой, представляющей некоторый принцип выбора. Доказательство проведено индукцией до эпсилон-0.