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MASSLESS INTEGER-HELICITY GAUGE FIELDS

Using the formalism of spinprojectors the general theory of massless integer-helicity gauge fields corresponding to the Pauli-Fierz program is analysed. The general realizations of helicities 2, 3 and 4 for symmetrical tensor fields are considered.

1. Introduction

The study of the Lagrangian formulation for arbitrary spin was started by M. Fierz and W. Pauli [1]. The Pauli-Fierz program implied that all field equations and subsidiary conditions should be derived from an action principle. C. Fronsdal [2] and J. Fang and C. Fronsdal [3] obtained the massless Lagrangians for arbitrary helicity, using the symmetrical tensors $h^{\mu_1\cdots\mu_n}$ to describe helicity $\lambda=n$, and symmetrical tensor-bispinors $\psi_{\alpha}^{\mu_1\cdots\mu_n}$ to describe helicity $\lambda=n+1/2$. The vierbein description of massless gauge fields was proposed by C. Aragone and S. Deser [4], and M. A. Vasiliev [5]. However, in the higher-helicity case, the proposed wave equations and Lagrangians do not follow the Pauli-Fierz program, since there exists too many additional restrictions on fields and gauge parameters.

B. de Wit and D. Z. Freedman [6] used a hierarchy of generalized Christoffel symbols with simple gauge transformation properties to demonstrate the systematics of higher-helicity gauge fields. The wave equations are expressed simply in terms of generalized Christoffel symbols. Also, the method of de Wit and Freedman works in the case when the algebraic constraints on fields and gauge parameters are added. The method which reverses the analysis of B. de Wit and D. Z. Freedman was discussed in [7–9].

The question whether consistent theories exist describing non-trivial interactions of massless higher-helicity fields among themselves and with the lower-spin fields, has become one of the principial questions of the modern field theory. The considerable progress in the description of some higher-spin interactions has been achieved in [10–13]. Due to the universality of the gravitational interaction, the question of the existence of a consistent gravitational interaction of massless fields is of great importance. A cubic interaction of all massless higher-helicity fields was recently constructed in [14]. This interaction incorporates gravitational interactions of massless higher-helicity fields.

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In the search for consistent interactions of massless gauge fields correct wave equations and Lagrangians are needed. In this paper we give a general form of arbitrary-helicity wave equations using the formalism of spin-projection operators in the form given in [15–17]. The formalism of spin-projectors was previously developed in [18, 19]. Our projection operators are connected with the fixed representations of the Lorentz group and have fixed non-localities which uniquely depend on the representations used. This fact allows to construct the operators of needed order without the knowledge of explicit expressions of spin-projectors and give the general structure of arbitrary-helicity wave equations. Likewise, it is possible to verify that the higher-helicity wave equations and Lagrangians must have the proposed structure.

In [20] it was stated that the construction of spin-projectors becomes increasingly complicated, and for that reason it is not applicable in the higher-helicity case. It is in some sense true, but the calculation of spin-projectors is rather a technical than a principal problem. We think that the formalism of spin-projectors is necessary in the case of higher-spin fields too, since it allows to obtain correct general expressions of field equations, bilinear forms and Lagrangians.

2. Helicity $\lambda=n$ Lagrangian wave equations

The helicity $\lambda=n$ $(n\geqslant 2)$ is described by two irreducible Lorentz fields ψ_1 and ψ_2 which correspond to the representations $1=\left(\frac{1}{2}n, \frac{1}{2}n\right)$ and $2=\left(\frac{1}{2}(n-2), \frac{1}{2}(n-2)\right)$. The gauge parameter ε_3 corresponds to the irreducible representation $3=\left(\frac{1}{2}(n-1), \frac{1}{2}(n-1)\right)$.

The general gauge-invariant wave equation is the following [21] —

$$\Box\begin{pmatrix} P_{11} & aP_{12} \\ bP_{21} & cP_{22}\end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}=0, \tag{2.1}$$

where

$$P_{11}=\sum_{s=0}^{n}\alpha_{11}(s)P^s_{11}, \quad P_{22}=\sum_{s=0}^{n-2}\alpha_{22}(s)P^s_{22},$$
$$P_{12}=\sum_{s=0}^{n-2}\alpha_{12}(s)P^s_{12}, \quad P_{21}=\sum_{s=0}^{n-2}\alpha_{12}(s)P^s_{21}. \tag{2.2}$$

P^s_{ij} are spin-projectors satisfying $P^s_{ij}P^{s'}_{kl}=\delta_{jk}\delta_{ss'}P^s_{il}$. The coefficients $\alpha_{ij}(s)$ are

$$\alpha_{11}(s)=-\frac{(n+s)(n-s-1)}{2n}, \quad s=0, \ldots, n,$$
$$\alpha_{22}(s)=\frac{(n-s)(n+s+1)}{2(2n-1)}, \quad s=0, \ldots, n-2, \tag{2.3}$$
$$\alpha_{12}(s)=\left[\frac{(n+s)(n-s)(n+s+1)(n-s-1)}{2(2n-1)(2n-2)}\right]^{1/2}, \quad s=0, \ldots, n-2.$$

Eq. (2.1) is gauge-invariant iff det $\beta_s=0$ $(s=0, \ldots, n-2)$ [22], where β_s are the reduced spin-matrices formed from the parameters a, b and c, and $\alpha_{ij}(s)$. The coefficients $\alpha_{ij}(s)$ in (2.3) are so chosen that while demanding

$$ab=-\frac{n-1}{n}c, \tag{2.4}$$

the conditions det $\beta_s=0$ are fulfilled.

The gauge transformation of Eq. (2.1) is the following —

$$\delta\psi=\alpha\sqrt{\Box}\left[\begin{array}{c} P_{13}\varepsilon_3 \\ \dfrac{n-1}{na}\sqrt{\dfrac{2n-1}{n}}P_{23}\varepsilon_3 \end{array}\right], \tag{2.5}$$

where α is some nonzero coefficient, and

$$P_{13}=\sum_{s=0}^{n-1}\alpha_{13}(s)P^s_{13}, \quad P_{23}=\sum_{s=0}^{n-2}\alpha_{23}(s)P^s_{23},$$
$$\alpha_{13}(s)=\left[\frac{(n-s)(n+s+1)}{2n}\right]^{1/2}, \quad \alpha_{23}(s)=\left[\frac{(n+s)(n-s-1)}{2(n-1)}\right]^{1/2}. \tag{2.6}$$

The source constraint $Q^z J=0$ is given by the operator

$$Q^z=\sqrt{\Box}\left(P_{31}\quad \frac{n-1}{nb}\sqrt{\frac{2n-1}{n}}P_{32}\right), \tag{2.7}$$

where

$$P_{31}=\sum_{s=0}^{n-1}\alpha_{13}(s)P^s_{31}, \quad P_{32}=\sum_{s=0}^{n-2}\alpha_{23}(s)P^s_{32}. \tag{2.8}$$

Our operators P_{ij} are so constructed that $\Box P_{11}$, $\Box P_{22}$, $\Box P_{12}$, and $\Box P_{21}$ are second-order differential operators, $\sqrt{\Box}P_{13}$, $\sqrt{\Box}P_{23}$, $\sqrt{\Box}P_{31}$, and $\sqrt{\Box}P_{32}$ are first-order differential operators. The gauge invariance and source constraint follow from the operator identities $\pi Q^g=0$ and $Q^z\pi=0$, where (2.1) is denoted by $\pi\psi=0$, the gauge transformation (2.5) by $\delta\psi=Q^g\varepsilon$ and the source constraint by $Q^zJ=0$. The derivation of source constraints from $Q^z\pi=0$ was previously discussed in [13, 17, 23].

In our approach we first derive the wave equation, then obtain the invariant bilinear form consistent with a given equation, and the last step consists in writing down the corresponding Lagrangian. As we shall see below, in the massless case the Lagrangian does not determine the equation uniquely, it must be coupled with the knowledge of invariant bilinear form.

Eq. (2.1) is consistent with the following invariant bilinear form —

$$\tilde{\psi}\psi=\psi_1^+\Lambda_{11}\psi_1+\frac{a}{b}\psi_2^+\Lambda_{22}\psi_2\equiv\tilde{\psi}_1\psi_1+\frac{a}{b}\tilde{\psi}_2\psi_2, \tag{2.9}$$

where

$$\Lambda_{11}=t^{n}_{11}-t^{n-1}_{11}+\ldots+(-1)^{n}t^{0}_{11},$$
$$\Lambda_{22}=t^{n-2}_{22}-t^{n-3}_{22}+\ldots+(-1)^{n}t^{0}_{22}, \tag{2.10}$$

and $t^{s}_{ij}=P^{s}_{ij|\partial_k=0}$ are noncovariant spin-projectors.

The Lagrangian is obtained after the partial integration of the following expression —

$$L=\tilde{\psi}_1\bar{P}_{11}\psi_1+a(\tilde{\psi}_1\bar{P}_{12}\psi_2+\tilde{\psi}_2\bar{P}_{21}\psi_1)-\frac{na^2}{n-1}\tilde{\psi}_2\bar{P}_{22}\psi_2, \tag{2.11}$$

where $\bar{P}_{ij}=\Box P_{ij}$. Variation of (2.11) with respect to the conjugated wave function $\tilde{\psi}=\left(\tilde{\psi}_1\frac{a}{b}\tilde{\psi}_2\right)$ gives Eq. (2.1).

From (2.11) it follows that the Lagrangian depends only on one parameter a. The equations corresponding to the same gauge transformation but of different source constraints and bilinear forms, are obtained from the same Lagrangian. For that reason the Lagrangian (2.11) is invariant under the following transformation of parameters —

$$a\rightarrow a,\quad b\rightarrow \varkappa b,\quad c\rightarrow \varkappa c, \tag{2.12}$$

where $\varkappa\neq 0$ is an arbitrary coefficient. The transformation (2.12) is equivalent to the following redefinition of Eq. (2.1). If Eq. (2.1) is denoted by $W=0$, then the transformation (2.12) leads to the equation

$$W+(\varkappa-1)\Pi_{22}W=0, \tag{2.13}$$

where Π_{22} is the Lorentz projector which extracts the representation 2. The other transformation of parameters

$$b\rightarrow b,\quad a\rightarrow \varkappa a,\quad c\rightarrow \varkappa c, \tag{2.14}$$

where $\varkappa\neq 0$ is an arbitrary coefficient, extracts the subset of equations corresponding to the same source constraint. This transformation is equivalent to the following field redefinition

$$\psi\rightarrow\psi+(\varkappa-1)\Pi_{22}\psi. \tag{2.15}$$

In F. A. Berends et al. [18] and F. A. Berends and J. C. J. M. van Reisen [19], the redefinition of field variables (2.15) is used to obtain the Lagrangian corresponding to a given wave equation. That procedure is indeed possible, because the Lagrangian can be obtained from the equation corresponding to the symmetrical choice of parameters ($a=b$) by the multiplication to conjugated wave function $\tilde{\psi}=(\tilde{\psi}_1\,\tilde{\psi}_2)$ in its simplest form. The field redefinition corresponds to $\varkappa=b/a$.

3. Symmetrical realization of a gauge field

The integer-helicity $\lambda=n$ is usually described by the symmetrical tensor field $h^{\mu_1\cdots\mu_n}$. The field $h^{\mu_1\cdots\mu_n}$ corresponds to the representation $\left(\frac{1}{2}n,\frac{1}{2}n\right)\oplus\left(\frac{1}{2}(n-2),\frac{1}{2}(n-2)\right)\oplus\left(\frac{1}{2}(n-4),\frac{1}{2}(n-4)\right)\oplus\ldots$

$\ldots \oplus (0,0)$ $\left(\text{or} \left(\frac{1}{2}, \frac{1}{2}\right)\right)$. The gauge invariant Eq. (2.1) uses only two first representations — 1 and 2. The components of $h^{\mu_1 \cdots \mu_n}$ that correspond to the lower representations $\left(\frac{1}{2}(n-4), \frac{1}{2}(n-4)\right), \ldots$ $\ldots, (0,0)$ $\left(\text{or} \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ are free. Therefore Eq. (2.1) for $h^{\mu_1 \cdots \mu_n}$ has a form

$$\Box \begin{pmatrix} P_{11} & aP_{12} & 0 \\ bP_{21} & cP_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_a \end{pmatrix} = 0, \qquad (3.1)$$

where the field ψ_a corresponds to all lower representations. Since the lower representations are free, Eq. (3.1) admits an extra gauge invariance $\delta\psi = \psi_a$.

Concerning the additional restrictions, such as the doubletracelessness $h^{\rho\sigma}{}_{\rho\sigma}{}^{\mu_5 \cdots \mu_n} = 0$ $(n \geqslant 4)$, it should be mentioned that these restrictions are not needed. The existence of additional restrictions means that the equation for $h^{\mu_1 \cdots \mu_n}$ has not the needed structure (3.1) and contains operators P_{2a} and P_{aa}, which connect lower representations. The wave equation for $h^{\mu_1 \cdots \mu_n}$ has, in that case, the structure

$$\Box \begin{pmatrix} P_{11} & aP_{12} & 0 \\ bP_{21} & cP_{22} & dP_{2a} \\ 0 & 0 & eP_{aa} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_a \end{pmatrix} = 0. \qquad (3.2)$$

Eq. (3.2) is gauge invariant with respect to the gauge transformation (2.7), yet it has no consistent bilinear form or Lagrangian. In order to have the consistent bilinear form the term $dP_{2a}\psi_a$ must be excluded. The extra gauge invariance $\delta\psi = \psi_a$ needs the elimination of $eP_{aa}\psi_a$. Therefore the equation with additional restrictions does not have the right structure and should be modified.

The massless theories proposed so far do not have the needed structure (3.1). From the well-known theories, the one given in [2] corresponds to $d=0$, $e \neq 0$. Now, there exists a correct Lagrangian theory, but with lower spins. The theory given in [6] corresponds to $d \neq 0$, $e \neq 0$. There will be no Lagrangians if we start directly from Eq. (3.2). In both cases the additional restriction $h^{\rho\sigma}{}_{\rho\sigma}{}^{\mu_5 \cdots \mu_n} = 0$ is needed to eliminate superfluous representations in (3.2).

Similar considerations are applicable in the case of gauge parameter $\varepsilon^{\mu_1 \cdots \mu_{n-1}}$ too. The gauge parameter $\varepsilon^{\mu_1 \cdots \mu_{n-1}}$ corresponds to the representation $\left(\frac{1}{2}(n-1), \frac{1}{2}(n-1)\right) \oplus \left(\frac{1}{2}(n-3), \frac{1}{2}(n-3)\right) \oplus \ldots$ $\ldots \oplus \left(\frac{1}{2}, \frac{1}{2}\right)$ (or $(0,0)$). If the gauge transformation is presented in the correct form (2.5), only the first representation $\left(\frac{1}{2}(n-1), \frac{1}{2}(n-1)\right)$ will be used and there will be no need for additional restrictions. In theories used so far the gauge transformation has a form $\delta h^{\mu_1 \cdots \mu_n} = \sum \partial^{\mu_1} \varepsilon^{\mu_2 \cdots \mu_n}$, where $\varepsilon^{\rho}{}_{\rho}{}^{\mu_4 \cdots \mu_n} = 0$ $(n \geqslant 3)$. The latter restriction means that the gauge transformation does not have the right form (2.5).

4. Vierbein realization of a gauge field

In the vierbein case [5], the tensor field $h^{\mu_1\bar{\nu}_2\ldots\bar{\nu}_n}$ is used. The vierbein field is symmetrical in indices $\bar{\nu}_2, \ldots, \bar{\nu}_n$, and it satisfies $h^{\mu_1\bar{\nu}_2\ldots\bar{\nu}_{n-2}\bar{\nu}}{}_{\bar{\nu}}=0$. The gauge transformation is $\delta h^{\mu_1\bar{\nu}_2\ldots\bar{\nu}_n}=\partial^{\mu_1}\varepsilon^{\bar{\nu}_2\ldots\bar{\nu}_n}$, where the gauge parameter satisfies $\varepsilon^{\bar{\nu}_2\ldots\bar{\nu}_{n-2}\bar{\nu}}{}_{\bar{\nu}}=0$.

In the higher-helicity case a vierbein realization is more economical since a vierbein field corresponds to the representation $\left(\frac{1}{2}n, \frac{1}{2}n\right)\oplus$ $\oplus\left(\frac{1}{2}(n-2), \frac{1}{2}(n-2)\right)\oplus\left(\frac{1}{2}n, \frac{1}{2}(n-2)\right)\oplus\left(\frac{1}{2}(n-2), \frac{1}{2}n\right)$. The helicity $\lambda=n$ is described with the help of two representations — 1 and 2. The corresponding equation has a general structure (3.1), where ψ_a corresponds to the representation $\left(\frac{1}{2}n, \frac{1}{2}(n-2)\right)\oplus$ $\oplus\left(\frac{1}{2}(n-2), \frac{1}{2}n\right)$. Therefore, the vierbein realization of helicity $\lambda=n$ is equivalent to the symmetrical one.

It should be mentioned that the correct form (3.1) does not follow directly from the usually demanded gauge transformation $\delta h^{\mu_1\bar{\nu}_2\ldots\bar{\nu}_n}=$ $=\partial^{\mu_1}\varepsilon^{\bar{\nu}_2\ldots\bar{\nu}_n}$. If the components corresponding to ψ_a are not eliminated, the lower spins are present. In [24] it is demonstrated that ψ_a allows a set of gauge-invariant wave equations describing $\lambda=0$.

In the following sections we consider the equations for helicities 2, 3 and 4 in the symmetrical realization and we illustrate the general considerations given in previous sections.

5. Helicity 2

The massive spin 2 case in a form presented here was analysed in [16]. The massless $\lambda=2$ equation can be obtained from the massive one by setting $m=0$ and demanding that the parameters a, b and c satisfy (2.4).

The general gauge invariant equation for $h^{\mu\nu}$ is the following —

$$\Box h^{\mu\nu}-\partial^{\mu}\partial_{\rho}h^{\rho\nu}-\partial^{\nu}\partial_{\rho}h^{\rho\mu}+\left(\frac{1}{2}-\frac{a}{\sqrt{3}}\right)\partial^{\mu}\partial^{\nu}h^{\rho}{}_{\rho}+$$

$$+\left(\frac{1}{2}-\frac{b}{\sqrt{3}}\right)\eta^{\mu\nu}\partial_{\rho}\partial_{\sigma}h^{\rho\sigma}-\frac{1}{2}\left(\frac{3}{4}+ab-\frac{a+b}{2\sqrt{3}}\right)\eta^{\mu\nu}\Box h^{\rho}{}_{\rho}=0. \quad (5.1)$$

Choosing $\alpha=\sqrt{2}$ we obtain, from (2.5), the following gauge transformation

$$\delta h^{\mu\nu}=\partial^{\mu}\varepsilon^{\nu}+\partial^{\nu}\varepsilon^{\mu}-\frac{1}{2}\left(1+\frac{\sqrt{3}}{2a}\right)\eta^{\mu\nu}\partial_{\rho}\varepsilon^{\rho} \quad (5.2)$$

and from (2.7) the source constraint

$$\partial_{\mu}J^{\mu\nu}-\frac{1}{2}\left(1+\frac{\sqrt{3}}{2b}\right)\partial^{\nu}J^{\mu}{}_{\mu}=0. \quad (5.3)$$

The invariant bilinear form is

$$\tilde{h}_{\mu\nu}h^{\mu\nu}=h^{+}{}_{\mu\nu}h^{\mu\nu}-\frac{1}{4}\left(1-\frac{a}{b}\right)h^{+\mu}{}_{\mu}h^{\nu}{}_{\nu}, \tag{5.4}$$

giving the following Lagrangian

$$L=-\partial^{\rho}h^{+}{}_{\mu\nu}\partial_{\rho}h^{\mu\nu}+2\partial^{\rho}h^{+}{}_{\rho\nu}\partial_{\sigma}h^{\sigma\nu}+\left(\frac{a}{\sqrt{3}}-\frac{1}{2}\right)(\partial_{\rho}h^{+\mu}{}_{\mu}\partial_{\sigma}h^{\rho\sigma}+$$

$$+\partial^{\mu}h^{+}{}_{\mu\nu}\partial^{\nu}h^{\sigma}{}_{\sigma})+\frac{1}{2}\left(\frac{3}{4}+a^{2}-\frac{a}{\sqrt{3}}\right)\partial^{\rho}h^{+\mu}{}_{\mu}\partial_{\rho}h^{\nu}{}_{\nu}. \tag{5.5}$$

Eq. (5.1) admits the covariant gauge

$$\partial_{\mu}h^{\mu\nu}-\frac{1}{4}\left(1-\frac{2a}{\sqrt{3}}\right)\partial^{\nu}h^{\mu}{}_{\mu}=0. \tag{5.6}$$

Depending on the choice of either a and b, we obtain the particular examples.

1. The linearised Einstein equation used in supergravity [25] corresponds to $a=b=-\sqrt{3}/2$

$$\Box h^{\mu\nu}-\partial^{\mu}\partial_{\rho}h^{\rho\nu}-\partial^{\nu}\partial_{\rho}h^{\rho\mu}+\partial^{\mu}\partial^{\nu}h^{\rho}{}_{\rho}+\eta^{\mu\nu}\partial_{\rho}\partial_{\sigma}h^{\rho\sigma}-\eta^{\mu\nu}\Box h^{\rho}{}_{\rho}=0. \tag{5.7}$$

The given Eq. (5.7) has the simplest gauge transformation and source constraint

$$\delta h^{\mu\nu}=\partial^{\mu}\varepsilon^{\nu}+\partial^{\nu}\varepsilon^{\mu},\quad \partial_{\mu}J^{\mu\nu}=0, \tag{5.8}$$

(5.6) gives the de Donder gauge

$$\partial_{\mu}h^{\mu\nu}-\frac{1}{2}\partial^{\nu}h^{\mu}{}_{\mu}=0. \tag{5.9}$$

2. In the relativistic theory of gravitation [26, 27], the equation corresponding to $a=b=\sqrt{3}/2$ is used —

$$\Box h^{\mu\nu}-\partial^{\mu}\partial_{\rho}h^{\rho\nu}-\partial^{\nu}\partial_{\rho}h^{\rho\mu}-\frac{1}{2}\eta^{\mu\nu}\Box h^{\rho}{}_{\rho}=0. \tag{5.10}$$

The gauge transformation and source constraint are the following

$$\delta h^{\mu\nu}=\partial^{\mu}\varepsilon^{\nu}+\partial^{\nu}\varepsilon^{\mu}-\eta^{\mu\nu}\partial_{\rho}\varepsilon^{\rho},\quad \partial_{\mu}J^{\mu\nu}-\frac{1}{2}\partial^{\nu}J^{\mu}{}_{\mu}=0. \tag{5.11}$$

Eq. (5.10) has the simplest gauge

$$\partial_{\mu}h^{\mu\nu}=0. \tag{5.12}$$

This particular equation must be used in the covariant theory of closed bose strings [28] where the field $h^{\mu\nu}$ satisfying $\partial_{\mu}h^{\mu\nu}=0$ is needed. Here the condition $\partial_{\mu}h^{\mu\nu}=0$ does not follow from an equation but on the choice of covariant gauge.

This Section we conclude with two remarks:

(1). In the symmetrical case ($a=b$), we have the simplest conjugated wave function $\tilde{h}_{\mu\nu}=h^{+}{}_{\mu\nu}$, in the nonsymmetrical case $\tilde{h}_{\mu\nu}=h^{+}{}_{\mu\nu}-\frac{1}{4}(1-a/b)h^{+\rho}{}_{\rho}\eta_{\mu\nu}$. For that reason one must be careful in considering the massless limit of massive spin-2 wave equations. The single-particle spin-2 equation corresponds to a nonsymmetrical choice of parameters, as a and b must satisfy, in addition to (2.4), $ab=-1/4$ [16].

(2). In [6] the following $\lambda=2$ equation was proposed

$$\Box h^{\mu\nu}-\partial^{\mu}\partial_{\rho}h^{\rho\nu}-\partial^{\nu}\partial_{\rho}h^{\rho\mu}+\partial^{\mu}\partial^{\nu}h^{\rho}{}_{\rho}=0. \tag{5.13}$$

Eq. (5.13) corresponds to the choice of parameters $a=-\sqrt{3}/2$, $b=\sqrt{3}/2$. Eq. (5.13) has the gauge transformation (5.8) and source constraint (5.11). The linearised Einstein Eq. (5.7), corresponding to the same gauge transformation as (5.13), is obtained from (5.13) via the transformation (2.15) where $\varkappa=-1$

$$W^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}W^{\rho}{}_{\rho}=0. \tag{5.14}$$

6. Helicity 3

The massless $\lambda=3$ case is analysed similarly to the $\lambda=2$ case. The new moment is that the gauge parameter $\varepsilon^{\mu\nu}$ is usually constrained to $\varepsilon^{\rho}{}_{\rho}=0$, but here we give the gauge transformation in the form (2.5), and for that reason the constraint $\varepsilon^{\rho}{}_{\rho}=0$ is not needed.

The general gauge-invariant equation for $h^{\mu_1\mu_2\mu_3}$ is the following [29] —

$$\begin{aligned}\Box h^{\mu_1\mu_2\mu_3}&-\sum\partial^{\mu_1}\partial_{\rho}h^{\rho\mu_2\mu_3}+\left(\frac{1}{3}-\frac{a}{\sqrt{5}}\right)\sum\partial^{\mu_1}\partial^{\mu_2}h^{\mu_3\rho}{}_{\rho}+\\&+\left(\frac{1}{3}-\frac{b}{\sqrt{5}}\right)\sum\eta^{\mu_1\mu_2}\partial_{\rho}\partial_{\sigma}h^{\rho\sigma\mu_3}+\\&+\left(\frac{a+b}{6\sqrt{5}}-\frac{ab}{4}-\frac{2}{9}\right)\Box\sum\eta^{\mu_1\mu_2}h^{\mu_3\rho}{}_{\rho}+\\&+\left(\frac{a+b}{3\sqrt{5}}-\frac{ab}{20}+\frac{1}{18}\right)\sum\partial^{\mu_1}\eta^{\mu_2\mu_3}\partial_{\rho}h^{\rho\sigma}{}_{\sigma}=0,\end{aligned} \tag{6.1}$$

where $\sum$ denotes the sum of terms symmetrical in free indices μ_1, μ_2, μ_3.

The general gauge transformation is

$$\begin{aligned}\delta h^{\mu_1\mu_2\mu_3}=&\sum\partial^{\mu_1}\varepsilon^{\mu_2\mu_3}-\frac{1}{3}\left(1+\frac{2\sqrt{5}}{3a}\right)\sum\eta^{\mu_1\mu_2}\partial_{\rho}\varepsilon^{\rho\mu_3}-\\&-\frac{1}{6}\left(1-\frac{\sqrt{5}}{3a}\right)\sum\partial^{\mu_1}\eta^{\mu_2\mu_3}\varepsilon^{\rho}{}_{\rho},\end{aligned} \tag{6.2}$$

and the general source constraint is

$$\partial_{\rho}J^{\rho\mu_2\mu_3}-\frac{1}{6}\left(1+\frac{2\sqrt{5}}{3b}\right)\sum\partial^{\mu_2}J^{\mu_3\rho}{}_{\rho}-\frac{1}{6}\left(1-\frac{\sqrt{5}}{3b}\right)\eta^{\mu_2\mu_3}\partial_{\rho}J^{\rho\sigma}{}_{\sigma}=0. \tag{6.3}$$

The invariant bilinear form

$$\tilde{h}_{\mu_1\mu_2\mu_3}h^{\mu_1\mu_2\mu_3}=h^{+}{}_{\mu_1\mu_2\mu_3}h^{\mu_1\mu_2\mu_3}+\frac{1}{2}\left(1-\frac{a}{b}\right)h^{+}{}_{\mu_1}{}^{\rho}{}_{\rho}h^{\mu_1\sigma}{}_{\sigma} \quad (6.4)$$

leads to the Lagrangian

$$L=-\partial^{\rho}h^{+}{}_{\mu_1\mu_2\mu_3}\partial_{\rho}h^{\mu_1\mu_2\mu_3}+3\partial^{\rho}h^{+}{}_{\rho\mu_2\mu_3}\partial_{\sigma}h^{\sigma\mu_2\mu_3}+$$

$$+\left(\frac{3a}{\sqrt{5}}-1\right)(\partial^{\rho}h^{+}{}_{\rho\sigma\mu_3}\partial^{\sigma}h^{\mu_3\mu}{}_{\mu}+\partial_{\rho}h^{+\mu}{}_{\mu\mu_3}\partial_{\sigma}h^{\rho\sigma\mu_3})+$$

$$+\left(\frac{2}{3}+\frac{3a^2}{4}-\frac{a}{\sqrt{5}}\right)\partial^{\rho}h^{+\mu}{}_{\mu\mu_3}\partial_{\rho}h^{\mu_3\nu}{}_{\nu}+ \quad (6.5)$$

$$+\left(\frac{3a^2}{20}-\frac{2a}{\sqrt{5}}-\frac{1}{6}\right)\partial^{\mu}h^{+}{}_{\mu}{}^{\nu}{}_{\nu}\partial_{\rho}h^{\rho\sigma}{}_{\sigma}.$$

Eq. (6.1) admits the covariant gauge

$$\partial_{\mu}h^{\mu\mu_2\mu_3}-\frac{1}{6}\left(1+\frac{3\sqrt{5}\,a}{10}\right)\eta^{\mu_2\mu_3}\partial_{\rho}h^{\rho\sigma}{}_{\sigma}-$$

$$-\frac{1}{6}\left(1-\frac{3a}{\sqrt{5}}\right)\sum\partial^{\mu_2}h^{\mu_3\rho}{}_{\rho}=0. \quad (6.6)$$

Now we demonstrate that the gauge transformation (6.2) corresponds to the representation $3=(1,1)$. If we denote $\varepsilon_3{}^{\mu_1\mu_2}$ by $\bar{\varepsilon}^{\mu_1\mu_2}$, we have

$$\bar{\varepsilon}^{\mu_1\mu_2}=\varepsilon^{\mu_1\mu_2}-\frac{1}{4}\eta^{\mu_1\mu_2}\varepsilon^{\rho}{}_{\rho}. \quad (6.7)$$

The gauge transformation (6.2) may be rewritten in the form

$$\delta h^{\mu_1\mu_2\mu_3}=\sum\partial^{\mu_1}\bar{\varepsilon}^{\mu_2\mu_3}-\frac{1}{3}\left(1+\frac{2\sqrt{5}}{3a}\right)\sum\eta^{\mu_1\mu_2}\partial_{\rho}\bar{\varepsilon}^{\rho\mu_3}. \quad (6.8)$$

This form of gauge transformation is useful, since it indicates that the simplest gauge transformation is $\delta h^{\mu_1\mu_2\mu_3}=\sum\partial^{\mu_1}\bar{\varepsilon}^{\mu_2\mu_3}$ and corresponds to the choice $a=-2\sqrt{5}/3$. The symmetrical choice $a=b=-2\sqrt{5}/3$ gives the simplest source constraint and bilinear form.

Due to the complicated form of the general massless $\lambda=3$ wave equation and Lagrangian only special cases were previously treated. The symmetrical choice $a=b=-2\sqrt{5}/3$ gives the Lagrangian given by C. Fronsdal [2]. The $\lambda=3$ equation given by B. de Wit and D. Z. Freedman [6]

$$\Box h^{\mu_1\mu_2\mu_3}-\sum\partial^{\mu_1}\partial_{\rho}h^{\rho\mu_2\mu_3}+\sum\partial^{\mu_1}\partial^{\mu_2}h^{\mu_3\rho}{}_{\rho}=0 \quad (6.9)$$

corresponds to $a=-2\sqrt{5}/3$ and $b=\sqrt{5}/3$. The equation corresponding to $a=b=-2\sqrt{5}/3$ can be obtained from (6.9) via the transformation (2.15) where $\varkappa=-2$

$$W^{\mu_1\mu_2\mu_3}-\frac{1}{2}\sum\eta^{\mu_1\mu_2}W^{\mu_3\rho}{}_{\rho}=0. \quad (6.10)$$

7. Helicity 4

The general gauge-invariant equation for $h^{\mu_1\mu_2\mu_3\mu_4}$ is the following [30] —

$$\Box h^{\mu_1\mu_2\mu_3\mu_4} - \sum \partial^{\mu_1}\partial_\rho h^{\rho\mu_2\mu_3\mu_4} + \left(\frac{1}{4} - \frac{a}{\sqrt{7}}\right) \sum \partial^{\mu_1}\partial^{\mu_2} h^{\mu_3\mu_4\rho}{}_\rho +$$

$$+ \left(\frac{1}{4} - \frac{b}{\sqrt{7}}\right) \sum \eta^{\mu_1\mu_2}\partial_\rho\partial_\sigma h^{\rho\sigma\mu_3\mu_4} +$$

$$+ \left(\frac{a+b}{8\sqrt{7}} - \frac{ab}{6} - \frac{5}{32}\right) \Box \sum \eta^{\mu_1\mu_2} h^{\mu_3\mu_4\rho}{}_\rho +$$

$$+ \left(\frac{1}{16} + \frac{a+b}{4\sqrt{7}} - \frac{ab}{21}\right) \sum \partial^{\mu_1}\eta^{\mu_2\mu_3}\partial_\rho h^{\rho\mu_4\sigma}{}_\sigma + \quad (7.1)$$

$$+ \left(\frac{ab}{21} - \frac{a}{12\sqrt{7}} + \frac{b}{4\sqrt{7}} - \frac{1}{16}\right) \sum \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\partial_\rho\partial_\sigma h^{\rho\sigma\varkappa}{}_\varkappa +$$

$$+ \left(\frac{ab}{42} + \frac{a}{8\sqrt{7}} - \frac{b}{24\sqrt{7}} - \frac{1}{32}\right) \sum \partial^{\mu_1}\partial^{\mu_2}\eta^{\mu_3\mu_4} h^{\rho\sigma}{}_{\rho\sigma} +$$

$$+ \left(\frac{1}{32} - \frac{a-b}{24\sqrt{7}} + \frac{ab}{14}\right) \Box \sum \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} h^{\rho\sigma}{}_{\rho\sigma} = 0.$$

The general gauge transformation is

$$\delta h^{\mu_1\mu_2\mu_3\mu_4} = \sum \partial^{\mu_1}\varepsilon^{\mu_2\mu_3\mu_4} - \frac{1}{24}\left(1 - \frac{9\sqrt{7}}{4a}\right) \sum \partial^{\mu_1}\eta^{\mu_2\mu_3}\varepsilon^{\mu_4\rho}{}_\rho -$$

$$- \left(1 + \frac{3\sqrt{7}}{4a}\right) \sum \left(\frac{3}{4}\eta^{\mu_1\mu_2}\partial_\rho\varepsilon^{\rho\mu_3\mu_4} - \frac{1}{4}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}\partial_\rho\varepsilon^{\rho\sigma}{}_\sigma\right), \quad (7.2)$$

and the general source constraint is

$$\partial_\rho J^{\rho\mu_2\mu_3\mu_4} - \frac{1}{24}\left(1 - \frac{9\sqrt{7}}{4b}\right) \sum \eta^{\mu_2\mu_3}\partial_\rho J^{\rho\mu_4\sigma}{}_\sigma -$$

$$- \left(1 + \frac{3\sqrt{7}}{4b}\right) \sum \left(\frac{3}{4}\partial^{\mu_2} J^{\mu_3\mu_4\rho}{}_\rho - \frac{1}{4}\partial^{\mu_2}\eta^{\mu_3\mu_4} J^{\rho\sigma}{}_{\rho\sigma}\right). \quad (7.3)$$

The invariant bilinear form

$$\tilde{h}_{\mu_1\mu_2\mu_3\mu_4} h^{\mu_1\mu_2\mu_3\mu_4} = h^{+}{}_{\mu_1\mu_2\mu_3\mu_4} h^{\mu_1\mu_2\mu_3\mu_4} - \frac{3}{4}\left(1 - \frac{a}{b}\right) h^{+}{}_{\mu_1\mu_2}{}^{\rho}{}_{\rho} h^{\mu_1\mu_2\sigma}{}_\sigma \quad (7.4)$$

leads to the Lagrangian

$$
\begin{aligned}
L= & -\partial^{\rho}h^{+}{}_{\mu_1\mu_2\mu_3\mu_4}\partial_{\rho}h^{\mu_1\mu_2\mu_3\mu_4}+4\partial^{\rho}h^{+}{}_{\rho\mu_2\mu_3\mu_4}\partial_{\sigma}h^{\sigma\mu_2\mu_3\mu_4}+\\
& +\left(\frac{6a}{\sqrt{7}}-\frac{3}{2}\right)(\partial^{\rho}h^{+}{}_{\rho\sigma\mu_3\mu_4}\partial^{\sigma}h^{\mu_3\mu_4\varkappa}{}_{\varkappa}+\partial_{\rho}h^{+\varkappa}{}_{\varkappa\mu_3\mu_4}\partial_{\sigma}h^{\rho\sigma\mu_3\mu_4})+\\
& +\left(\frac{4a^2}{7}-\frac{6a}{\sqrt{7}}-\frac{3}{4}\right)\partial^{\rho}h^{+}{}_{\rho\mu_2}{}^{\varkappa}{}_{\varkappa}\partial_{\sigma}h^{\sigma\mu_2\lambda}{}_{\lambda}+ \qquad (7.5)\\
& +\left(\frac{3}{16}-\frac{a}{2\sqrt{7}}-\frac{a^2}{7}\right)(\partial_{\rho}h^{+\varkappa\lambda}{}_{\varkappa\lambda}\partial_{\sigma}h^{\rho\sigma\mu}{}_{\mu}+\partial^{\rho}h^{+}{}_{\rho\sigma}{}^{\mu}{}_{\mu}\partial^{\sigma}h^{\varkappa\lambda}{}_{\varkappa\lambda})+\\
& +\left(a^2-\frac{3a}{2\sqrt{7}}+\frac{15}{16}\right)\partial^{\rho}h^{+}{}_{\mu_1\mu_2}{}^{\varkappa}{}_{\varkappa}\partial_{\rho}h^{\mu_1\mu_2\lambda}{}_{\lambda}+\\
& +\left(\frac{a}{4\sqrt{7}}-\frac{3a^2}{14}-\frac{3}{32}\right)\partial^{\rho}h^{+\varkappa\lambda}{}_{\varkappa\lambda}\partial_{\rho}h^{\mu\nu}{}_{\mu\nu}.
\end{aligned}
$$

The general transformation of parameters (2.12) leads to the redefinition of Eq. (7.1)

$$W^{\mu_1\mu_2\mu_3\mu_4}+\frac{\varkappa-1}{8}\sum\eta^{\mu_1\mu_2}W^{\mu_3\mu_4\rho}{}_{\rho}-\frac{\varkappa-1}{16}\sum\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}W^{\varkappa\lambda}{}_{\varkappa\lambda}=0, \qquad (7.6)$$

where (7.1) is denoted by $W^{\mu_1\mu_2\mu_3\mu_4}=0$.

Due to the complicated form of the general massless $\lambda=4$ theory, only special cases were previously treated. In [2] the Lagrangian corresponds to $a=-3\sqrt{7}/4$, but due to the additional restriction $h^{\rho\sigma}{}_{\rho\sigma}=0$ some terms of the correct Lagrangian (7.5) are absent. The correct theory corresponding to $a=b=-3\sqrt{7}/4$ was first derived in [31]. In [20] the equation corresponding to the same symmetrical choice was treated, yet the term $\square\,\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}h^{\rho\sigma}{}_{\rho\sigma}$ has a wrong coefficient.

The $\lambda=4$ equation given in [6]

$$\square\, h^{\mu_1\mu_2\mu_3\mu_4}-\sum\partial^{\mu_1}\partial_{\rho}h^{\rho\mu_2\mu_3\mu_4}+\sum\partial^{\mu_1}\partial^{\mu_2}h^{\mu_3\mu_4\rho}{}_{\rho}=0 \qquad (7.7)$$

is invariant with respect to the gauge transformation $\delta h^{\mu_1\mu_2\mu_3\mu_4}=\sum\partial^{\mu_1}\varepsilon^{\mu_2\mu_3\mu_4}$ if $\varepsilon^{\mu_2\mu_3\mu_4}$ is restricted to $\varepsilon^{\mu_2\rho}{}_{\rho}=0$. As we have mentioned above, Eq. (7.7) does not admit any consistent bilinear form and is not derivable from a Lagrangian. Eq. (7.7) is not invariant with respect to the gauge transformation (7.2). In [31] Eq. (7.7) is modified and it leads to the theory which corresponds to $a=-3\sqrt{7}/4$, $b=\sqrt{7}/4$.

We conclude this section with the remark on the hierarchy of generalized Christoffel symbols given in [6]. This particular method works well in the $\lambda=2$ and $\lambda=3$ cases, but in the $\lambda\geqslant 4$ case it needs some improvement, because the equations of motion derived via the method of generalized Christoffel symbols contain superfluous representations. Moreover, they do not add to the Lagrangian theory. Also, the redefinition of field equation proposed in [6]

$$W^{\mu_1\mu_2\mu_3\mu_4}-\frac{1}{2}\sum\eta^{\mu_1\mu_2}W^{\mu_3\mu_4\rho}{}_{\rho}=0 \qquad (7.7)$$

is not in accordance with (7.6). In the $\varkappa=-3$ case, (7.6) gives the transformation between the equations corresponding to $a=b=-3\sqrt{7}/4$ and $a=-3\sqrt{7}/4$, $b=\sqrt{7}/4$.

8. Conclusions

In this paper a general form of arbitrary-helicity boson gauge-invariant wave equations is given. The proposed general form follows the program of M. Fierz and W. Pauli [1], demanding that all field equations and subsidiary conditions should be derived from an action principle. It appears that the higher-helicity ($\lambda \geqslant 4$) massless wave equations and Lagrangians should be modified to have the needed structure.

The proposed form of equations becomes important in the interacting field case, since the construction of interaction Lagrangians needs the knowledge of bilinear form consistent with a given equation.

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TÄISARVULISE SPIRAALSUSEGA MASSITUD KALIBRATSIOONIVÄLJAD

Kasutades spinniprojektorite formalismi on analüüsitud Pauli-Fierzi programmile vastavat täisarvulise spiraalsusega massitute kalibratsiooniväljade üldist teooriat. On vaadatud spiraalsuste 2, 3 ja 4 üldist realisatsiooni sümmeetriliste tensorväljadega.

Рейн-Карл ЛОЙДЕ, Ильмар ОТС, Рейн СААР

БЕЗМАССОВЫЕ КАЛИБРОВОЧНЫЕ ПОЛЯ ЦЕЛОЧИСЛЕННОЙ СПИРАЛЬНОСТИ

С использованием формализма спинпроекторов проведен анализ общей теории безмассовых калибровочных полей целочисленной спиральности, соответствующей программе Паули—Фирца. Рассмотрена общая реализация спиральностей 2, 3 и 4 для симметричных тензорных полей.