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SOME QUADRATURE FORMULAE WITH SIMPLIFIED ERROR ESTIMATE FOR *n*-CONVEX FUNCTIONS

(Presented by G. Vainikko)

1. Introduction and preliminaries

The error of a quadrature formula is often treated by introducing high-order derivatives. We shall demonstrate that in the error estimate the order of derivatives can be lowered if the integrand has some convexity properties. This fact is known for the trapezoidal rule [1, 2] and for some Gauss's and Lobatto's rules [3]. However, we have the error estimates via L_1-norm and thus our rules are applicable in the case if the integrand is a product of two functions, e. g. the finite Fourier transform. Some general results on quadrature formulae for convex functions are given by Brass [4], Heindl [5] and Bourdeau, Dubuc [6].

Let us recall some definitions and properties concerning convex and n-convex functions.

Definition 1. *Any function f for which*

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)$$

for all $x, y\in[a, b]$, $0\leqslant t\leqslant 1$ is said to be convex (up-down) on the interval $[a, b]$.

It is known [7] that the convex function is continuous on (a, b) and has there increasing left- and right-hand derivatives. Moreover, $f'_-\leqslant f'_+$ and for all $x, y\in(a, b)$, $x<y$

$$f'_+(x)\leqslant\frac{f(y)-f(x)}{y-x}\leqslant f'_-(y). \tag{1.1}$$

In addition,

$$\int_a^b f'_-=\int_a^b f'_+=f(b)-f(a).$$

Definition 2. *A function f defined on $[a, b]$ is said to be n-convex (concave) iff for all choices of $(n+1)$ distinct points, $x_0, \ldots, x_n$, on $[a, b]$ the n-th divided difference of f at these $(n+1)$ points $[x_0, \ldots, x_n]f\geqslant 0$ $(\leqslant 0)$.*

A subclass of n-convex functions is the class for which $f^{(n)}\geqslant 0$. Obviously, a 2-convex function is convex and vice versa. Let L_n denote the Lagrange interpolating polynomial of the degree n of the function f in some $(n+1)$ distinct nodes in $[a, b]$.

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Theorem A. ([8], Th. 5). *Let* $P_k=(x_k, y_k)$, $1\leqslant k\leqslant n$, $n\geqslant 2$, $a\leqslant x_1<\ldots<x_n\leqslant b$, *be any n distinct points on the graph of the function f. Then f is n-convex iff for all such sets of n distinct points, the graph lies alternately above and below the curve* $y=L_{n-1}(x)$, *lying below if* $x_{n-1}\leqslant x\leqslant x_n$. *Further,* $L_{n-1}\leqslant f$ *in* $[x_n, b]$; *and* $L_{n-1}\leqslant f$ $(\geqslant f)$ *in* $[a, x_1]$, *n being even (odd).*

Theorem B. ([8], Th. 7; Cor. 15) *If f is n-convex in* $[a, b]$, *then*
(*i*) $f^{(r)}$ *do exist and are continuous in* $[a, b]$, $1\leqslant r\leqslant n-2$;
(*ii*) *both* $f_-^{(n-1)}$, $f_+^{(n-1)}$ *exist in* (a, b), *are monotonic increasing and if* $a\leqslant x_1<\ldots<x_n\leqslant x\leqslant y_1<\ldots<y_n\leqslant b$ *then* $(n-1)![x_1, \ldots, x_n]f\leqslant \leqslant f_-^{(n-1)}(x)\leqslant f_+^{(n-1)}(x)\leqslant(n-1)![y_1, \ldots, y_n]f$;
(*iii*) $f^{(n-2)}$ *is convex.*

2. The main result for the even n

Let us subdivide $[a, b]$ into N equal-length intervals $[x_k, x_{k+1}]$ with $x_k=a+kh$ $(k=0, \ldots, N)$, $N\equiv 0 \pmod{2n}$, $h=(b-a)/N$, $y_k=f(x_k)$. Let f defined on $[a, b]$ be n-convex and n be even. Then, also, in any subinterval $[\alpha, \beta]\subset[a, b]$ f is n-convex. Let us subdivide $[\alpha, \beta]$ into $2n$ equal-length intervals $[t_j, t_{j+1}]$. In the interval $[\alpha, \beta]$ we approximate f by its Lagrange interpolating polynomial L_{n-1} passing through the points $(t_{2j-1}, f(t_{2j-1}))$ $j=1, \ldots, n$. By the Theorem A we have

$$(-1)^{j+1}(f-L_{n-1})\geqslant 0 \quad \text{on} \quad [t_{2j-2}, t_{2j-1}],$$
$$(-1)^{j}(f-L_{n-1})\geqslant 0 \quad \text{on} \quad [t_{2j-1}, t_{2j}].$$

Therefore, the number of positivity and negativity intervals of $f-L_{n-1}$ are equal and so the approximation error in the sense of L_1-norm could be small. This construction is done in any interval $[x_{2nk}, x_{2n(k+1)}]$ and it gives us the main idea of the proof of the next theorem. Note that in all the following error estimates the bound expressed through left- and right-hand derivatives holds if these one-sided derivatives exist (see Theorem B, (*ii*)).

Theorem 1. *Suppose f is n-convex (concave) where n is even and* $N\equiv 0 \pmod{2n}$. *Then there exists a piecewise polynomial (of the degree* $\leqslant n-1$) *approximation* $f_{n,N}$ *of f such that*

$$(c_n:=2^{n-1}(n/2)!1\cdot 5\cdot\ldots(2n-3)/n!<2^{n-1}(\pi n)^{-1/4})$$

$$\int_a^b |f-f_{n,N}|\leqslant c_n h^{n-1}|f^{(n-2)}(b)+f^{(n-2)}(a)-f^{(n-2)}(a+h)-f^{(n-2)}(b-h)|$$
$$\leqslant c_n h^n|f_-^{(n-1)}(b)-f_+^{(n-1)}(a)|, \tag{2.1}$$

$$\int_a^b f_{n,N}=nh\sum_{j=1}^{n/2}(-1)^{j+1}S(n, j)\sum_{k=1}^{N/(2n)}(y_{2n(k-1)+2j-1}+y_{2nk-2j+1}), \tag{2.2}$$

$$S(n, j):=\frac{(n/2)^{n-2}}{(j-1)!(n-j)!}\int_0^1\left[u^2-\left(\frac{1}{n}\right)^2\right]\left[u^2-\left(\frac{3}{n}\right)^2\right]\ldots$$
$$\ldots[n-2j+1]\ldots\left[u^2-\left(\frac{n-1}{n}\right)^2\right]du \tag{2.3}$$

(the factor $u^2-((n-2j+1)/n)^2$ must be replaced by $n-2j+1$).

The most important cases are $n=2$ and $n=4$. The case $n=2$ is developed by the author in [9] (but the constant 2 in the error estimate there is superfluous) and the constant $c_2=1$ cannot be improved as we see, if $f(x)=|x|\leqslant 1$ and $N=4$. For the case $n=2$ we give an application to finite Fourier (-cosine) transform [9].

Corollary 1. *Suppose f is convex (concave) and $N \equiv 0 \pmod 4$. Then*

$$\Big|\int_a^b f(x)\cos(xt)\,dx - \Big\{$$

$$-\frac{1}{t}\Big\{y_{N-1}\sin bt - y_1 \sin at - \sum_{k=4,8,\ldots}^{N-4} w_k + \Big(\cos 2ht - \frac{\sin 2ht}{ht}\Big)\sum_{k=2,6,\ldots}^{N-2} w_k\Big\}\Big| \leqslant$$

$$\leqslant h|f(b)+f(a)-f(a+h)-f(b-h)| \leqslant h^2|f'_-(b)-f'_+(a)|,$$

where $w_k := (y_{k+1}-y_{k-1})\sin x_k t$.

Corollary 2. *Suppose f is 4-convex (concave) and $N \equiv 0 \pmod 8$. Then*

$$\int_a^b f_{4,N} = \frac{h}{6}\Big\{13\sum_{k=1}^{N/8}(y_{8k-7}+y_{8k-1}) + 11\sum_{k=1}^{N/8}(y_{8k-5}+y_{8k-3})\Big\},$$

$$\int_a^b |f-f_{4,N}| \leqslant (10/3)h^3|f''(b)+f''(a)-f''(a+h)-f''(b-h)|$$

$$\leqslant (10/3)h^4|f'''_-(b)-f'''_+(a)|.$$

3. A result for the odd n

In this section we use notations from Section 2 with the exception $N \equiv 0 \pmod{n+1}$. This difference from the even n case is caused by alternating property of n-convex function with its Lagrange interpolating polynomial (see Theorem A). Indeed, now it is natural to subdivide $[a,b]$ into subintervals $[x_{(n+1)(k-1)}, x_{(n+1)k}]$ $(k=1,\ldots,N/(n+1))$, because in this case the polynomial L_{n-1} passing through the points $P_{(n+1)(k-1)+j}$ $(j=1,\ldots,n)$ induces for the difference $f-L_{n-1}$ the equal number of subintervals, where this difference is alternately positive and negative.

Theorem 2. *Suppose f is n-convex (concave), where n is odd and $N \equiv 0 \pmod{n+1}$. Then there exists a piecewise polynomial (of the degree $\leqslant n-1$) approximation $f_{n,N}$ of f such that the* (2.1) *holds with $c_n=1$ and*

$$\int_a^b f_{n,N} = h\sum_{j=0}^{(n-1)/2}{}' (-1)^j T(n,j)\sum_{k=1}^{N/(n+1)}(y_{(n+1)k+j-n}+y_{(n+1)k-j-1}), \qquad (3.1)$$

$$T(n,j) = \frac{2((n+1)/2)^n}{j!(n-j-1)!}\int_0^1\Big[u^2\Big(u^2-\Big(\frac{2}{n+1}\Big)^2\Big)\ldots$$

$$\ldots\Big(u^2-\Big(\frac{n-1}{n+1}\Big)^2\Big)\Big/\Big(u^2-\Big(\frac{n-2j-1}{n+1}\Big)^2\Big)\Big]du. \qquad (3.2)$$

$\sum'$ means that in the sum the final term must be halved.

As the most important cases we consider $n=1, 3, 5$.

Corollary 3. *If f is monotone (but not necessarily continuous) and N is even, then*

$$\int_a^b f_{1,N} = 2h\sum_{k=1}^{N/2} y_{2k-1}, \qquad \int_a^b |f-f_{1,N}| \leqslant h|f(b)-f(a)|.$$

Corollary 4. *Suppose f is 3-convex (concave) and* $N\equiv 0 \pmod 4$. *Then*

$$\int_a^b f_{3,N}=\frac{4}{3}h\left\{2\sum_{k=1}^{N/2} y_{2k-1}-\sum_{k=1}^{N/4} y_{4k-2}\right\}.$$

Corollary 5. *If f is 5-convex (concave) and* $N\equiv 0 \pmod 6$, *then*

$$\int_a^b f_{5,N}=\frac{3}{10}h\left\{11\sum_{k=1}^{N/6}(y_{6k-5}+y_{6k-1})-14\sum_{k=1}^{N/6}(y_{6k-4}+y_{6k-2})+26\sum_{k=1}^{N/6} y_{6k-3}\right\}.$$

4. The proofs of the main results

Proof of Theorem 1. Let f be n-convex and n even. We split $[a,b]$ into subintervals, i. e.

$$\int_a^b=\sum_{k=1}^{N/(2n)}\int_{x_{2n(k-1)}}^{x_{2nk}}. \tag{4.1}$$

In order to shorten the following formulae we consider instead of $[x_{2n(k-1)}, x_{2nk}]$ the interval $[-1,1]$. We subdivide the latter into $2n$ equal-length intervals with nodes $t_j=-1+j/n$, $j=1,\ldots,2n-1$ and approximate f by its Lagrange interpolating polynomial L_{n-1} equal to f in nodes t_{2j-1}, $j=1,\ldots,n$. Explicitly,

$$L_{n-1}(t)=\left(\frac{n}{2}\right)^{n-1}\sum_{j=1}^{n}\frac{(-1)^j}{(j-1)!\,(n-j)!}f(t_{2j-1})\prod_{j\neq l=1}^{n}(t-t_{2l-1}).$$

Further,

$$\int_{-1}^{1}\prod_{j\neq l=1}^{n}(t-t_{2l-1})\,dt=-\frac{2}{n}\int_0^1\left(t^2-\left(\frac{1}{n}\right)^2\right)\left(t^2-\left(\frac{3}{n}\right)^2\right)\ldots$$

$$\ldots(n-2j+1)\ldots\left(t^2-\left(\frac{n-1}{n}\right)^2\right)dt,$$

where in the last integral the factor $t^2-((n-2j+1)/n)^2$ must be replaced by $n-2j+1$. Therefore, using definition (2.3) we have

$$\int_{-1}^{1}L_{n-1}=\sum_{j=1}^{n}(-1)^{j+1}S(n,j)f(t_{2j-1}).$$

Obviously, $S(n,j)=-S(n,n-j+1)$, which implies

$$\int_{-1}^{1}L_{n-1}=\sum_{j=1}^{n/2}(-1)^{j+1}S(n,j)\,(f(t_{2j-1})+f(t_{2n-2j+1})). \tag{4.2}$$

The interval $[x_{2n(k-1)}, x_{2nk}]$ may be mapped into $[-1,1]$ by the transformation $x=a+(2k-1+t)nh$. Thus, by (4.1), (4.2) we get (2.2). Our aim is now to prove (2.1). Let us denote $g_n:=f-L_{n-1}$, where as above, L_{n-1} is determined by nodes t_{2j-1} $(j=1,\ldots,n)$. To estimate the error in the L_1-norm we consider

$$E := \int_{-1}^{1} |g_n| = \sum_{j=0}^{n-1} \left(\int_{t_{2j}}^{t_{2j+1}} + \int_{t_{2j+1}}^{t_{2j+2}} \right) |g_n|.$$

If we take two sequences of constants $a_n(j) \geqslant 1$, $b_n(j) \geqslant 1$, then by Theorem A we have

$$E \leqslant \sum_{j=0}^{n-1} \left((-1)^j a_n(j) \int_{t_{2j}}^{t_{2j+1}} g_n + (-1)^{j+1} b_n(j) \int_{t_{2j+1}}^{t_{2j+2}} g_n \right).$$

After the change of variable in the last integrals and summing in a suitable way we have $(h := 1/n)$

$$E \leqslant \int_{-1}^{t_1} \sum_{j=0}^{n-1} \left((-1)^j a_n(j) g_n(t+2jh) + (-1)^{j+1} b_n(j) g_n(t+(2j+1)h) \right) dt =$$

$$= \int_{-1}^{t_1} \left\{ \sum_{j=0}^{n/2-1} [b_n(2j+1) g_n(t+(4j+3)h) - a_n(2j+1) g_n(t+(4j+2)h) - \right.$$

$$\left. - b_n(2j) g_n(t+(4j+1)h) + a_n(2j) g_n(t+4jh)] \right\} dt. \tag{4.3}$$

This kind of summation is needed for minimizing the constant c_n in error estimate. Let us evaluate now the $(n-1)$th divided difference of $g := g_n$ at the points $\{t+(4j+2)h, t+(4j+3)h\}$, $\{t+4jh, t+(4j+1)h\}$ with $j=0, \ldots, n/2-1$, $-1 \leqslant t \leqslant t_1$, respectively. Thus

$$[t+2h, t+3h, \ldots, t+(2n-1)h] g =$$

$$= h^{1-n} \sum_{j=0}^{n/2-1} \left(\frac{g(t+(4j+3)h)}{c_n(2j+1)} - \frac{g(t+(4j+2)h)}{c_n(2j)} \right), \tag{4.4}$$

$$[t, t+h, \ldots, t+(2n-3)h] g = h^{1-n} \sum_{j=0}^{n/2-1} \left(\frac{g(t+(4j+1)h)}{c_n(2j+1)} - \frac{g(t+4jh)}{c_n(2j)} \right)$$

where

$$c_n(2j) = (-1)^{n/2-j} 2^{n-2} j! (n/2-j-1)! (4j-1)(4j-5) \ldots (4j-(2n-3)),$$

$$c_n(2j+1) = (-1)^{n/2-j-1} 2^{n-2} j! (n/2-j-1)! (4j+1)(4j-3) \ldots$$

$$\ldots (4j-(2n-5)).$$

It can be shown that $c_n(j) > 0$ and

$$\max_{0 \leqslant j \leqslant n-1} c_n(j) = c_n(0) = 2^{n-2}(n/2-1)! 1 \cdot 5 \cdot 9 \cdot \ldots \cdot (2n-3). \tag{4.5}$$

In (4.3) let us take $a_n(2j) = a_n(2j+1) = c_n(0)/c_n(2j)$ and $b_n(2j) = b_n(2j+1) = c_n(0)/c_n(2j+1)$. Hence, $a_n(j), b_n(j) \geqslant 1$, and from (4.3) and (4.4) one obtains

$$E \leqslant c_n(0) h^{n-1} \int_{-1}^{t_1} \{[t+2h, t+3h, \ldots, t+(2n-1)h](f-L_{n-1}) -$$

$$- [t, t+h, \ldots, t+(2n-3)h](f-L_{n-1})\} dt =$$

$$= c_n(0) h^{n-1} \int_{-1}^{t_1} \{[t+2h, t+3h, \ldots, t+(2n-1)h] f -$$

$$- [t, t+h, \ldots, t+(2n-3)h] f\} dt.$$

The last part follows from the fact that the divided difference [n distinct points] L_{n-1}=coefficient of t^{n-1} of L_{n-1}. Using Theorem B, (*ii*) and denoting $g=f^{(n-2)}$ we get

$$E\leqslant\frac{c_n(0)}{(n-1)!}h^{n-1}\int_{-1}^{t_1}(g'_-(t+(2n-1)h)-g'_+(t))\,dt. \tag{4.6}$$

By (4.5) the constant in Theorem 1 is $c_n=c_n(0)/(n-1)!$. Let $d_n:=$ $:=2^{1-n}c_n$, then (n even) $d_n=1\cdot5\cdot9\cdot\ldots\cdot(2n-3)/(2\cdot6\cdot10\cdot\ldots\cdot(2n-2))$. From the Wallis's formula for π we have the following inequality

$$1\cdot3\cdot5\cdot\ldots\cdot(2n-1)/(2\cdot4\cdot6\cdot\ldots\cdot2n)\leqslant(\pi n)^{-1/2},$$

which implies (n even)

$$d_n^2=\frac{1\cdot1\cdot5\cdot5\cdot\ldots\cdot(2n-3)^2}{2\cdot2\cdot6\cdot6\cdot\ldots(2n-2)^2}<\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot2n}\leqslant(\pi n)^{-1/2}$$

and this completes the proof of the inequality for c_n. Further, as $g=f^{(n-2)}$ is convex (Theorem B, (*iii*)), then by (1.2) and (4.6)

$$E\leqslant c_nh^{n-1}\{[g(1)-g(1-h)]-[g(-1+h)-g(-1)]\},$$

where $h=1/n$. This inequality gives the error estimate in the sense of L_1-norm on the interval $[-1,1]$. Consequently, for the error on $[x_{2n(k-1)},x_{2nk}]$ the inequality

$$E_k:=\int_{x_{2n(k-1)}}^{x_{2nk}}|g_n|\leqslant$$

$$\leqslant c_nh^{n-1}\{[g(x_{2nk})-g(x_{2nk-1})]-[g(x_{2n(k-1)+1})-g(x_{2n(k-1)})]\},$$

where $h=(b-a)/N$, holds. Using (4.1) and due to convexity of $g=f^{(n-2)}$ we get (2.1). Theorem 1 is proved.

P r o o f o f T h e o r e m 2. In the outline it is similar to the case when n is even. In the present case $N\equiv0(\mathrm{mod}\,n+1)$ and

$$\int_a^b=\sum_{k=1}^{N/(n+1)}\int_{x_{(n+1)(k-1)}}^{x_{(n+1)k}}. \tag{4.7}$$

As above, we confine ourselves to the case $[-1,1]$. Now we subdivide $[-1,1]$ into $n+1$ equal-length intervals with nodes $t_j=-1+jh$, $h=$ $=2/(n+1)$. The Lagrange polynomial L_{n-1} determined by the same nodes t_j, $j=1,\ldots,n$ (n odd) is

$$L_{n-1}(t)=h^{1-n}\sum_{j=1}^{n}\frac{(-1)^{j-1}}{(j-1)!(n-j)!}f(t_j)\prod_{j\neq l=1}^{n}(t-t_l).$$

Futher,

$$\int_{-1}^{1}\prod_{j\neq l=1}^{n}(t-t_l)\,dt=2\int_0^1\frac{t^2(t^2-(1h)^2)\ldots(t^2-((n-1)h/2)^2)}{t^2-((n-2j+1)h/2)^2}\,dt,$$

and by the definition (3.2) and by property $T(n,j)=T(n,n-j-1)$ we have

$$\int_{-1}^{1}L_{n-1}=\frac{2}{n+1}\sum_{j=0}^{(n-1)/2}{}'(-1)^jT(n,j)\,(f(t_{j+1})+f(t_{n-j})),$$

where $\sum'$ means that the final term in the sum must be halved. By transformation $x=a+(n+1)(2k-1+t)h/2$ in (4.7) one obtains (3.1).

As defined before, let $g_n = f - L_{n-1}$ in $[x_{(n+1)k}, x_{(n+1)(k+1)}]$. As the binomial coefficients $C_n^j \geqslant 1$, then, by Theorem A,

$$E_k := \int_{x_{(n+1)k}}^{x_{(n+1)(k+1)}} |g_n| \leqslant \sum_{j=0}^{n} (-1)^{j+1} C_n^j \int_{x_{(n+1)k+j}}^{x_{(n+1)k+j+1}} g_n.$$

Hence, making a change of variables, we obtain

$$E_k \leqslant \int_{x_{(n+1)k}}^{x_{(n+1)k+1}} \sum_{j=1}^{n} (-1)^{j+1} C_n^j g_n (x+jh)\, dx =$$

$$= n!\, h^n \int_{x_{(n+1)k}}^{x_{(n+1)k+1}} [x, x+h, \ldots, x+nh] g_n \, dx.$$

L_{n-1} is a polynomial of the degree $\leqslant n-1$, thus $[x, \ldots, x+nh] L_{n-1} = 0$. By the recursion formula for the divided differences we have

$$nh[x, \ldots, x+nh] f = [x+h, \ldots, x+nh] f - [x, \ldots, x+(n-1)h] f.$$

Thus by Theorem B, (*ii*) we get

$$E_k \leqslant (n-1)!\, h^{n-1} \int_{x_{(n+1)k}}^{x_{(n+1)k+1}} ([x+h, \ldots, x+nh] f - [x, \ldots, x+(n-1)h] f)\, dx \leqslant$$

$$\leqslant h^{n-1} \int_{x_{(n+1)k}}^{x_{(n+1)k+1}} \{f_-^{(n-1)}(x+nh) - f_+^{(n-1)}(x)\}\, dx.$$

The rest follows using a similar argument as in the case of even n.

Remark. By A. Markoff's Theorem ([10], No 51), the best L_1-approximation will be attained if we take the nodes $t_j = \cos(j\pi/(n+1))$ on $[-1, 1]$. In cases $n=1$, $n=2$ this is done, but for $n>2$ due to nonequidistant nodes there exist some well-known technical complications.

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MÕNED KVADRATUURVALEMID LIHTSUSTATUD VEAHINNANGUGA n-KUMERATE FUNKTSIOONIDE KORRAL

Kvadratuurvalemite enamlevinud veahinnangud nõuavad funktsioonidelt kõrget järku tuletiste olemasolu. Käesolevas töös on näidatud, et tuletiste järku võib alandada, kui integreeritaval funktsioonil on teatud kumeruse omadused. Erinevalt varasematest töödest selles suunas [1–6] on siin veahinnang antud L_1-normi mõttes, mis on kasulik korrutise integreerimisel. Näitena on toodud lõpliku Fourier' teisenduse ligikaudse arvutamise valem koos veahinnanguga.

Анди КИВИНУКК

НЕКОТОРЫЕ КВАДРАТУРНЫЕ ФОРМУЛЫ С УПРОЩЕННОЙ ОЦЕНКОЙ ТОЧНОСТИ ДЛЯ n-ВЫПУКЛЫХ ФУНКЦИЙ

Наиболее распространенные оценки точности квадратурных формул требуют существования производных высокого порядка. В работе показано, что порядок производных можно понизить, если интегрируемая функция в некотором смысле выпукла. От более ранних работ в этом направлении [1–6] настоящая отличается тем, что точность оценивается по норме в L_1. Это обстоятельство полезно для интегрирования произведения. В качестве примера приведена одна формула с оценкой точности для вычисления конечного преобразования Фурье.