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ON STATISTICAL ESTIMATION OF A PROBABILITY FUNCTION

(Presented by J. Engelbrecht)

1. In stochastic programming problems defined in n-dimensional Euclidean space R^n one often meets probability functions $v(x) = P\{f(x,\xi)<0\}$ or $v(x)=P\{f(x,\xi)\leqslant 0\}$, where ξ is an s-dimensional random parameter having density $p(y)$, f: $R^n \times R^s \to R^1$ and P denotes probability. For example, in problems with probability constraints it is required that every constraint $f(x,\xi_i)\leqslant 0$, depending on a random vector ξ_i, is satisfied with certain probability α_i, i. e. $P\{f_i(x,\xi_i)\leqslant 0\}\geqslant \alpha_i$. In another problem it may happen that some probability function is desired to be maximized or minimized. It is well known that it is rather complicated to solve problems containing probability functions, because calculating values of the multiple integral $v(x)$ and its derivatives is, as a rule, very laborconsuming. If the distribution of ξ is not known then the evaluation of these integrals is impossible altogether.

One way to construct methods for solving such problems is as follows: probability functions are replaced everywhere in the optimization problem through their statistical estimates and then the problem obtained is solved.

In this paper, an asymptotically unbiased estimate for the probability function $v(x)=P\{f(x,\xi)<0\}$ is constructed, and it is shown how this estimate can be used to find an approximate local minimum point of $v(x)$.

2. Let us consider the probability function $v(x)=P\{f(x,\xi)<0\}$. Assume that for every x the function $f(x,y)$ is continuous in y. In this case for every x $f(x,\xi)$ is a random variable and it denotes by $w(x,t) = P\{f(x,\xi)<t\}$ its distribution function. Then $v(x)=P\{f(x,\xi)<0\} = w(x,0)$. To find some statistical estimate for $v(x)$ one can use some estimate constructed for a distribution function $F(t)$. Let η be a random variable and $F(t)$ its distribution function. As an estimate of $F(t)$ at a given point t, it is natural to take the sample distribution function

$$F^k(t) = \frac{1}{k}\sum_{i=1}^{k} H(t-\eta_i), \tag{1}$$

where $H(z)=\begin{cases}1, & z\geqslant 0\\ 0, & z<0\end{cases}$, and η_i, $i=1, 2, \ldots, k$, are independent realizations on $\eta \cdot F^k(t)$ is a binomially distributed random variable, $EF^k(t) = F(t)$ and $\operatorname{var} F^k(t) = \frac{1}{k}F(t)(1-F(t))$. Therefore, as an estimate for $v(x)=P\{f(x,\xi)<0\}$, we could choose

$$w_k(x) = F^k(0) = \frac{1}{k}\sum_{i=1}^{k} H(-f(x,\xi_i)).$$

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However, $H(z)$ is not differentiable, and neither is $w_k(x)$ differentiable in x. Differentiability of the estimate is needed in order to establish conditions guaranteeing 1) the existence of a local solution x_k^* of the problem containing estimates, and 2) that the distance between x_k^* and a (local) solution x^* of the initial problem is sufficiently small. To obtain a differentiable estimate for a distribution function $F(t)$, we can rely on the paper by R.-D. Reiss [1]. Let $M(z)$ be a continuously differentiable distribution function. As an estimate for $F(t)$ let us take the sum

$$G^k(t)=\frac{1}{k}\sum_{i=1}^{k} M\left(\frac{t-\eta_i}{h_k}\right),$$

where h_k is a smoothing constant, $\lim\limits_{k\to\infty} h_k=0$. As a particular case of Lemma 2.1 [1] we have

Lemma 1. *If*

A1. $\int\limits_{-\infty}^{+\infty} zM'(z)\,dz<\infty, \qquad \int\limits_{-\infty}^{+\infty} z^2M'(z)\,dz<\infty,$

A2. *The distribution function $F(t)$ is twice differentiable at t and $\sup\limits_{t\in R^1}|F''(t)|<\infty$, then*

$$\lim_{k\to\infty} E\,G^k(t)=F(t),$$

and

$$\lim_{k\to\infty} \operatorname{var}\sqrt{k}\,G^k(t)=F(t)\,(1-F(t)).$$

Corollary. *If*

A3. *The distribution function of the random variable $\eta=f(x,\xi)$ satisfies the assumption* A2, *then*

$$v_k(x)=\frac{1}{k}\sum_{i=1}^{k} M\left(-\frac{f(x,\xi_i)}{h_k}\right) \tag{2}$$

is an asymptotically unbiased estimate of the probability function $v(x)=P\{f(x,\xi)<0\}$ at x and

$$\lim_{k\to\infty} \operatorname{var}\sqrt{k}\,v_k(x)=v(x)\,(1-v(x)).$$

3. Consider now the unconstrained minimization problems

$$\min_{x\in R^n} v(x) \tag{3}$$

and

$$\min_{x\in R^n} v_k(x). \tag{4}$$

As an approximate solution of (3), consider a solution of (4). In connection with such approach two questions arise: 1) does the problem (4) have a solution x_k^* and 2) if it has then in what sense x_k^* approximates a solution x^* of (3)?

In order to find answers to these questions the Frechet' differentiability of $v(x)$ is needed. Under certain conditions laid on $f(x,y)$ the function $w(x,t)=P\{f(x,\xi)<t\}$ is differentiable as many times as needed [2], and its first derivatives can be expressed through surface integrals as follows:

$$w'_t(x,t)=\int_{S_t}\frac{p(y)}{\|f'_y(x,y)\|}dS_t,$$

$$w'_x(x,t)=-\int_{S_t}\frac{f'_x(x,y)}{\|f'_y(x,y)\|}p(y)\,dS_t,$$

$$v'(x)=w'_x(x,0)=-\int_{S}\frac{f'_x(x,y)}{\|f'_y(x,y)\|}p(y)\,dS,$$

where $S_t=\{y|f(x,y)=t\}$ and $S=\{y|f(x,y)=0\}$ [3]. The results in [2] enable us to find higher-order derivatives of $v(x)$ and $w(x,t)$ as well.

Lemma 2. *If*

A4. *$v(x)$ is twice differentiable in t, $w(x,t)$ twice differentiable in x,*

A5. *$w'_x(x,t)$ and $w''_{xx}(x,t)$ are twice differentiable in t, $\|w^{(3)}_{xtt}(x,t)\|$ and $\|w^{(4)}_{xxtt}(x,t)\|$ are bounded, then*

$$\lim_{k\to\infty}Ev'_k(x)=v'(x)$$

and

$$\lim_{k\to\infty}Ev''_k(x)=v''(x).$$

Proof. Due to (2) and Theorem 108 [4], we have

$$Ev_k(x)=\int_{R^s}M\left(-\frac{f(x,y)}{h_k}\right)p(y)\,dy=$$

$$=\int_{-\infty}^{+\infty}M\left(-\frac{\tau}{h_k}\right)d\tau\int_{S_\tau}\frac{p(y)}{\|f'_y(x,y)\|}dS_\tau=\int_{-\infty}^{+\infty}M\left(-\frac{\tau}{h_k}\right)w'_t(x,\tau)\,d\tau.$$

Therefore,

$$Ev'_k(x)=\int_{-\infty}^{+\infty}M\left(-\frac{\tau}{h_k}\right)w''_{tx}(x,\tau)\,d\tau$$

and

$$Ev''_k(x)=\int_{-\infty}^{+\infty}M\left(-\frac{\tau}{h_k}\right)w^{(3)}_{txx}(x,\tau)\,d\tau.$$

Integrating by parts we obtain

$$Ev'_k(x)=\int_{-\infty}^{+\infty}M\left(-\frac{\tau}{h_k}\right)w''_{tx}(x,\tau)\,d\tau=M\left(-\frac{\tau}{h_k}\right)w'_x(x,\tau)\Big|_{-\infty}^{+\infty}-$$

$$-\frac{1}{h_k}\int_{-\infty}^{+\infty}w'_x(x,\tau)M'\left(-\frac{\tau}{h_k}\right)d\tau=\int_{-\infty}^{+\infty}w'_x(x,-h_kz)M'(z)\,dz=$$

$$=\int_{-\infty}^{+\infty}M'(z)\left[w'_x(x,0)-w''_{xt}(x,0)h_kz+\frac{1}{2}w^{(3)}_{xtt}(x,-\theta(z)h_kz)(h_kz)^2\right]dz=$$

$$= w'_x(x,0) - w''_{xt}(x,0)\, h_k \int_{-\infty}^{+\infty} M'(z)\, z\, dz +$$

$$+ \frac{h_k^2}{2} \int_{-\infty}^{+\infty} w^{(3)}_{xtt}(x, -\theta(z) h_k z)\, M'(z)\, z^2\, dz, \quad \text{where } 0 \leqslant \theta(z) \leqslant 1.$$

Under the assumptions A4, A5, and the conditions laid upon $M(z)$, we have $\lim_{k\to\infty} Ev'_k(x) = v'(x)$. Similarly, we obtain that $\lim_{k\to\infty} Ev''_k(x) = v''(x)$. Lemma is proved.

Let us introduce now the following assumptions.

A6. The problem (3) has a local solution x^*, i.e. $v(x^*) \leqslant v(x)$ in some neighbourhood $U(x^*, \delta) = \{x \mid \|x - x^*\| < \delta\}$ of x^*.

A7. The function $v(x)$ is twice differentiable at x^* and $u^T v''(x^*) u \geqslant m\|u\|^2$ for every $u \in R^n$ and for some constant $m > 0$.

A8. $$\left\| M''_{xx}\left(-\frac{f(x^1,\xi)}{h_k}\right) - M''_{xx}\left(-\frac{f(x^2,\xi)}{h_k}\right) \right\| \leqslant \frac{c(\xi)}{h_k^r} \|x^1 - x^2\|$$

for every $x^1, x^2 \in U(x^*, \delta)$, where r is some integer and $\operatorname{var} c(\xi) < \infty$.

A9. For every $x \in U(x^*, \delta)$ and for sufficiently large k

$$\operatorname{var} M'_x f(x,\xi) \leqslant \frac{\sigma_1^2}{h_k^r} \quad \text{and} \quad \operatorname{var} M''_{xx} f(x,\xi) \leqslant \frac{\sigma_2^2}{h_k^r}.$$

T h e o r e m 1. *Let the assumptions* A1, A3—A9, *and*

A10. $\lim_{k\to\infty} k h_k^r = \infty$

hold.

Then, for sufficiently large k

1) *the problem*

$$\min_{x\in R^n} Ev_k(x) \tag{5}$$

has a local solution $\bar{x}_k$,

2) *the problem* (4) *has a local solution* x_k^* *with positive probability not less than*

$$P_k = 1 - \frac{1}{kh_k^r}\left(\frac{\sigma_2^2}{\delta_2^2} + \frac{\operatorname{var} c(\xi)}{\delta_3^2} + \frac{16\sigma_1^2\,[Ec(\xi) + \delta_3]^2}{(m - \delta_1 - \delta_2)^4}\right),$$

where $\delta_1, \delta_2, \delta_3$ *are arbitrary fixed constants,* $0 < \delta_1 < m$, $0 < \delta_1 + \delta_2 < m$, $\delta_3 > 0$ *and*

3) $P\{\|x_k^* - x^*\| \leqslant \varepsilon\} \geqslant$

$$\geqslant 1 - \frac{1}{kh_k^r}\left\{\frac{\sigma_2^2}{\delta_2^2} + \frac{\operatorname{var} c(\xi)}{\delta_3^2} + \frac{16\sigma_1^2}{\varepsilon^2[2(m - \delta_1 - \delta_2) - \varepsilon Ec(\xi)]^2}\right\}$$

for every $0 < \varepsilon < (m - \delta_1 - \delta_2)/2Ec(\xi)$.

The proof of this theorem is reduced to direct application of the Theorem in [5].

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TÕENÄOSUSFUNKTSIOONI STATISTILISEST HINDAMISEST

On leitud tõenäosusfunktsioonile $v(x)=P\{f(x,\xi)<0\}$ diferentseeruv hinnang $v_k(x)=\frac{1}{k}\sum_{i=1}^{k} M\left(-\frac{f(x,\xi_i)}{h_k}\right)$, kus ξ_i, $i=1,\ldots,k$, on juhusliku vektori ξ sõltumatud realisatsioonid ja $\lim\limits_{k\to\infty} h_k=0$. On näidatud, et $v_k(x)$ on funktsiooni $v(x)$ asümptootiliselt nihutamata hinnang ja ülesande $\min\limits_{x\in R^n} v_k(x)$ lahend koondub tõenäosuse järgi ülesande $\min\limits_{x\in R^n} v(x)$ lahendiks, kui $k\to\infty$.

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О СТАТИСТИЧЕСКОМ ОЦЕНИВАНИИ ФУНКЦИИ ВЕРОЯТНОСТИ

В настоящей работе для функции вероятности $v(x)=P\{f(x,\xi)<0\}$ найдена дифференцируемая оценка $v_k(x)=\frac{1}{k}\sum_{i=1}^{k} M\left(-\frac{f(x,\xi_i)}{h_k}\right)$, где ξ_i, $i=1,\ldots,k$, независимые реализации случайного вектора ξ и $\lim\limits_{k\to\infty} h_k=0$. Показано, что $v_k(x)$ является асимптотически несмещенной оценкой функции $v(x)$ и что при $k\to\infty$ решение задачи $\min\limits_{x\in R^n} v_k(x)$ сходится по вероятности к решению задачи $\min\limits_{x\in R^n} v(x)$.