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ON THE DETERMINITY OF STANDARD WREATH PRODUCT BY ITS SEMIGROUP OF ENDOMORPHISMS

(Presented by R.-K. Loide)

Let G and H be groups. If the isomorphism of semigroups $\operatorname{End} G$ and $\operatorname{End} H$ implies the isomorphism of groups G and H, then we can say that the group G is determined by its semigroup of endomorphisms. In general, the necessary and sufficient conditions for determinity of an arbitrary group by its semigroup of endomorphisms are unknown. In this paper we study the determinity of the standard wreath product $G=A \operatorname{Wr} B$ of groups A and B by its semigroup of endomorphisms.

The following theorems are proved.

Theorem 2.1. *Let p be a prime, A a cyclic group of order p^n and B a finite group. Assume that*
1° *B is a direct product of its Sylow p-subgroup and Hall p-subgroup;*
2° *B is determined by its semigroup of endomorphisms;*
3° *the group of units of the group ring of B over integers modulo p^n is solvable.*
Then the group $A \operatorname{Wr} B$ is determined by its semigroup of endomorphisms.

Theorem 3.1. *The standard wreath product $A \operatorname{Wr} B$ of finite Abelian groups A and B is determined by its semigroup of endomorphisms.*

Theorem 3.2. *Let $A_0, A_1, \ldots, A_n$ be finite Abelian p-groups $(n \geqslant 1)$, where p is a prime and let $B_1 = A_1 \operatorname{Wr} A_0$, $B_k = A_k \operatorname{Wr} B_{k-1}$ for all $k \in \{2, \ldots, n\}$. Then the group B_n is determined by its semigroup of endomorphisms.*

1. Introduction

If A and B are groups, then the standard wreath product of A and B, denoted $A \operatorname{Wr} B$, is the semidirect product $A^B \rtimes B$ of A^B by B, where A^B is the set of all functions $f: B \to A$ and

$$(fg)(b) = f(b) \cdot g(b),$$

$$c^{-1}fc = f^c, \qquad f^c(b) = f(bc^{-1})$$

for all $b, c \in B$ and $f, g \in A^B$. The general properties of wreath products are presented in [1].

Let G be a fixed group. If for a suitable group H from the isomorphism of semigroups of all endomorphisms of groups G and H follows the isomorphism of groups G and H, then we can say that the group G is determined by its semigroup of endomorphisms in the class of all groups. In this paper we study a problem when the wreath product $A \operatorname{Wr} B$ is determined by its semigroup of endomorphisms in the class of all groups. The principal results are formulated in Theorems 2.1, 3.1 and 3.2.

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We shall use the following notations: $\operatorname{End} G$ — a semigroup of all endomorphisms of a group G; $\operatorname{Aut} G$ — a group of all automorphisms of a group G; $Z(G)$ — a centre of a group G; $C(k)$ — a cyclic group of order k; Z_k — a ring of residual classes modulo k; $\hat{g}$ — an inner automorphism, generated by an element g; $\hat{C}=\{\hat{g} \mid g \in C\}$; $\langle a, b, \ldots\rangle$ — a subgroup, generated by elements $a, b, \ldots$; $\langle A, B, \ldots\rangle$ — a subgroup, generated by subsets $A, B, \ldots$; $K_G(x)=\{y \in \operatorname{End} G \mid yx=xy=y\}$; $D_G(x)=\{y \in \operatorname{Aut} G \mid yx=xy=x\}$.

2. Main theorem

The main result of this paper is

Theorem 2.1. *Let p be a prime, A be a cyclic group of order p^n and B be a finite group such that:*
1° *the group B is a direct product of its Sylow p-subgroup and Hall p'-subgroup;*
2° *the group B is generated by its semigroup of endomorphisms in the class of all groups;*
3° *the group of all invertible elements of the group ring of B over the ring of residual classes modulo p^n is solvable.*
Then the wreath product $A \operatorname{Wr} B$ is generated by its semigroup of endomorphisms in the class of all groups.

Proof. Suppose that A and B are such groups that the assumptions of theorem are satisfied. Denote $G=A \operatorname{Wr} B$. Let G^* be another group such that the semigroups of all endomorphisms of G and G^* are isomorphic:

$$\operatorname{End} G \cong \operatorname{End} G^*. \tag{2.1}$$

We shall show that G and G^* are also isomorphic. The image of an element z of $\operatorname{End} G$ under the isomorphism (2.1) we denote always as z^*.

Assume below that x is the projection of $G=A \operatorname{Wr} B=A^B \leftthreetimes B$ on his subgroup B. By [2], Corollary 4.1 and Assumption 3°, the group $D_G(x)$ is solvable. Therefore

$$G^*=\operatorname{Ker} x^* \leftthreetimes \operatorname{Im} x^*,$$
$$\operatorname{Ker} x^* \cong A^B \cong C(p^n) \times \cdots \times C(p^n) \quad (|B| \text{ factors}),$$
$$\operatorname{End}(\operatorname{Im} x^*) \cong \operatorname{End} B$$

([3], theorem). Due to Assumption 2° the groups $\operatorname{Im} x^*$ and B are isomorphic. The groups B and $\operatorname{Im} x^*$ will be identified later. Let $T=\operatorname{Ker} x^*$. Consequently,

$$G^*=T \leftthreetimes B, \tag{2.2}$$

$$T \cong C(p^n) \times \ldots \times C(p^n) \quad (|B| \text{ factors}). \tag{2.3}$$

By Assumption 1°, $B=B_p \times B_{p'}$, where B_p and $B_{p'}$ are the Sylow p-subgroup and Hall p'-subgroup of B, respectively.

The group $G=A \operatorname{Wr} B$ splits into the following semidirect product:

$$G=([B, A^B] \leftthreetimes B) \leftthreetimes A_1,$$

where

$$A_1=\{f \in A^B \mid f(b)=1 \text{ for } b \neq 1\},$$

and $[B, A^B]$ is a commutator-group between B and A^B. Clearly,

$$A_1 \cong A \cong C(p^n).$$

Denote now by y the projection of G onto its subgroup A_1. By [4], Lemma 1.6 the semigroups $\operatorname{End}(\operatorname{Im} y) = \operatorname{End} A_1$ and $\operatorname{End}(\operatorname{Im} y^*)$ are isomorphic. As every finite group is generated by its semigroup of endomorphisms in the class of all groups ([4], Theorem 4.2), then $\operatorname{Im} y^*$ and $\operatorname{Im} y = A_1$ are isomorphic:

$$\operatorname{Im} y^* \cong A_1 \cong C(p^n).$$

The further proof of Theorem 2.1 is developed in the following three lemmas.

Lemma 2.2. ([3], Proposition 6). *If $z \in \operatorname{End} G$ and $yz = 0$, then $z = xz$.*

Lemma 2.3. *The group $\dot{T}$ splits into the direct product*

$$T = \Pi_{b \in B} \langle a\hat{b} \rangle,$$

where a is the generator of $\operatorname{Im} y^* \cong C(p^n)$.

Proof. By construction $xy = yx = 0$. Hence, $x^*y^* = y^*x^* = 0^*$, $a \in \operatorname{Im} y^* \subset \operatorname{Ker} x^* = T$ and the group

$$H = \langle a\hat{b} \mid b \in B \rangle$$

is a subgroup of T. Thus, H is $\hat{B}$-invariant. Let us show that $H = T$.

Let $\overline{G^*} = G^*/T^p$, where $T^p = \{g^p \mid g \in T\}$. For every $g \in G^*$ and $K \subset G^*$ denote $\bar{g} = gT^p$, $\overline{K} = \{\bar{g} \mid g \in K\}$. Then, by (2.2) and (2.3),

$$\overline{G^*} = \overline{T} \rightthreetimes \overline{B}; \quad \overline{B} \cong B;$$

$$\overline{T} \cong C(p) \times \ldots \times C(p) \quad (|B| \text{ factors}).$$

The subgroup $\overline{H}$ of $\overline{G^*}$ is $\hat{\overline{B}}$-invariant,

By contradiction assume that $\overline{H} \neq \overline{T}$. There exists a maximal $\hat{\overline{B}}$-invariant proper subgroup $\overline{D}$ of $\overline{T}$ such that $\overline{H} \subset \overline{D}$ $(D = \{g \in T \mid \bar{g} \in \overline{D}\})$. Suppose $\mathbf{G} = G^*/D$. Then

$$\mathbf{G} = \mathbf{T} \rightthreetimes \mathbf{B},$$

where

$$\mathbf{T} = \{gD \mid g \in T\} = T/D \cong \overline{T}/\overline{D},$$

$$\mathbf{B} = \{bD \mid b \in B\} = BD/D \cong B.$$

The group $\mathbf{T}$ is a non-trivial elementary Abelian p-subgroup of $\mathbf{G}$. By construction, $\mathbf{T}$ is $\hat{\mathbf{B}}$-invariant and has not non-trivial $\hat{\mathbf{B}}$-invariant subgroups. Since $\mathbf{B} = \mathbf{B}_p \times \mathbf{B}_{p'}$, where $\mathbf{B}_p$ is a Sylow p-subgroup and $\mathbf{B}_{p'}$ is a Hall p'-subgroup of $\mathbf{B}$, we have

$$\mathbf{G} = \mathbf{T} \rightthreetimes (\mathbf{B}_p \times \mathbf{B}_{p'}) = (\mathbf{T} \rightthreetimes \mathbf{B}_p) \rightthreetimes \mathbf{B}_{p'}. \tag{2.4}$$

Let

$$\mathbf{F} = \mathbf{T} \cap Z(\mathbf{T} \rightthreetimes \mathbf{B}_p)$$

and

$$F = \{g \in T \mid gD \in \mathbf{F}\}.$$

The group $\mathbf{F}$ is non-trivial ([5], Theorem 2.6.4) and thus

$$\overline{D} \subset \overline{F} \subset \overline{T}, \quad \overline{F} \neq \overline{D}. \tag{2.5}$$

It is clear that $\mathbf{F}$ is $\hat{\mathbf{B}}$-invariant. Therefore, $\overline{F}$ is $\hat{\overline{B}}$-invariant. By (2.5) and the definition of $\overline{D}$ we have $\overline{F}=\overline{T}$. Consequently, $\mathbf{F}=\mathbf{T}$, $\mathbf{T} \leftthreetimes \mathbf{B}_p = \mathbf{T} \leftthreetimes \mathbf{B}_p$, and by (2.4)

$$\mathbf{G} = (\mathbf{T} \leftthreetimes \mathbf{B}_p) \leftthreetimes \mathbf{B}_{p'} = (\mathbf{T} \leftthreetimes \mathbf{B}_{p'}) \leftthreetimes \mathbf{B}_p. \tag{2.6}$$

Since $\overline{D}$ is $(\widehat{\overline{B_{p'}}})$-invariant, then due to [5] Theorem 3.3.2 there exists a $(\widehat{\overline{B_{p'}}})$-invariant subgroup $\overline{S}$ of $\overline{T}$, such that $\overline{T} = \overline{S} \times \overline{D}$ $(S = \{g \in T \mid \overline{g} \in \overline{S}\})$. Then $\langle \overline{S}, \overline{B_{p'}} \rangle = \overline{S} \leftthreetimes \overline{B_{p'}}$ and

$$\mathbf{T} \leftthreetimes \mathbf{B}_{p'} \cong \overline{S} \leftthreetimes \overline{B_{p'}}. \tag{2.7}$$

Define now a map

$$v \colon \overline{S} \leftthreetimes \overline{B_{p'}} \to T^{p^{n-1}} \leftthreetimes B$$

by the equation

$$(\overline{bh})\, v = b h^{p^{n-1}},$$

where $b \in B_{p'}$ and $h \in S$. The map v is defined correctly. Indeed, if $\overline{b}\overline{h} = \overline{b}_1 \overline{h}_1$, where $b, b_1 \in B_{p'}$ and $h, h_1 \in S$, then

$$b = b_1, \quad \overline{h} = \overline{h}_1, \quad h^{-1} h_1 \in T^p,$$
$$(h^{-1} h_1)^{p^{n-1}} = 1, \quad h^{p^{n-1}} = h_1^{p^{n-1}},$$

and $(\overline{b}\overline{h})v = (\overline{b}_1 \overline{h}_1)v$. The direct calculations show that v is a homomorphism.

Let z^* be the product of a natural homomorphism $G^* \to \mathbf{G} = G^*/D$, a projection $\mathbf{G} \to \mathbf{T} \leftthreetimes \mathbf{B}_{p'}$ (see (2.6)), an isomorphism (2.7) and v. Then z^* is an endomorphism of G^* and, by construction of z^*,

$$\operatorname{Ker} x^* \not\subset \operatorname{Ker} z^* \tag{2.8}$$

and $\operatorname{Im} y^* \subset \operatorname{Ker} z^*$. Therefore, $y^* z^* = 0^*$. According to Lemma 2.2 and (2.1) $z^* = x^* z^*$. Hence, $\operatorname{Ker} x^* \subset \operatorname{Ker} z^*$. This contradicts (2.8). The contradiction obtained shows that $\overline{T} = \overline{H}$.

The subgroup H of T has $|B|$ generators. From equation $\overline{H} = \overline{T} \cong C(p) \times \ldots \times C(p)$ ($|B|$ factors) and (2.3) it follows that $T = H$. The last equation is equivalent to the statement of lemma. The lemma is proved.

L e m m a 2.4. *The groups G^* and $A^* \operatorname{Wr} B$ are isomorphic.*

P r o o f. Any element g of G^* is by (2.2) and Lemma 2.3 uniquely expressed as

$$g = c \cdot \Pi_{b \in B} \, (\hat{ab})^{i(b)},$$

where $c \in B$ and $i(b) \in Z_{p^n}$. Define a map $\tau \colon G^* \to A^* \operatorname{Wr} B$ by equation $g\tau = cf$, where $f \in (A^*)^B$ and $f(b) = a^{i(b)}$. It is clear that τ is injective. The direct calculations show that τ is a homomorphism. Consequently, τ is an isomorphism. The lemma is proved and so is Theorem 2.1.

3. Corollaries from the main theorem

By Theorem 2.1 and the results of [6] and [7], it is possible to prove two following theorems.

Theorem 3.1. *If A and B are finite Abelian groups then the wreath product $A \operatorname{Wr} B$ is generated by its semigroup of endomorphisms in the class of all groups.*

Proof. Suppose that A and B are finite Abelian groups. Denote $G = A \operatorname{Wr} B$. The group A splits into the direct product $A = A_1 \times \ldots \times A_n$, where the subgroups $A_1, \ldots, A_n$ are primary and cyclic. Then the subgroup A^B of G splits into the direct product $A^B = A_1^B \times \ldots \times A_n^B$ and

$$G = A^B \leftthreetimes B = (A_1^B \times \ldots \times A_n^B) \leftthreetimes B. \tag{3.1}$$

In this connection

$$\langle B, A_i^B \rangle = A_i^B \leftthreetimes B = A_i \operatorname{Wr} B,$$

and

$$G = \left(\prod_{j=1,\, j\neq i}^{n} A_j^B \right) \leftthreetimes (A_i^B \leftthreetimes B)$$

for any $i \in \{1, 2, \ldots, n\}$. Let x_i and x are projections of G onto subgroups $A_i^B \leftthreetimes B$ and B, respectively. Then

$$A_i^B = \operatorname{Ker} x \cap \operatorname{Im} x_i, \qquad \operatorname{Im} x = B,$$

$$\operatorname{Im} x_i = A_i^B \leftthreetimes B = A_i \operatorname{Wr} B = (\operatorname{Ker} x \cap \operatorname{Im} x_i) \leftthreetimes \operatorname{Im} x \tag{3.2}$$

and

$$G = \left(\prod_{i=1}^{n} (\operatorname{Ker} x \cap \operatorname{Im} x_i) \right) \leftthreetimes \operatorname{Im} x. \tag{3.3}$$

Assume now that G^* is another group such that the semigroups $\operatorname{End} G$ and $\operatorname{End} G^*$ are isomorphic. Our purpose is to show that G and G^* are isomorphic.

Suppose that $x^*, x_1^*, \ldots, x_n^*$ are images of $x, x_1, \ldots, x_n$ by isomorphism $\operatorname{End} G \cong \operatorname{End} G^*$. From [6], Theorem 1 and Corollary 2 it follows that for the group G^* the equalities

$$G^* = \left(\prod_{i=1}^{n} (\operatorname{Ker} x^* \cap \operatorname{Im} x_i^*) \right) \leftthreetimes \operatorname{Im} x^*, \tag{3.4}$$

$$\operatorname{Im} x_i^* = (\operatorname{Ker} x^* \cap \operatorname{Im} x_i^*) \leftthreetimes \operatorname{Im} x^* \tag{3.5}$$

hold similarly to (3.2) and (3.3). Denote $B^* = \operatorname{Im} x^*$. Since $K_G(x) \cong K_{G^*}(x^*)$ and $K_G(x_i) \cong K_{G^*}(x_i^*)$, then due to [4], Lemma 1.6

$$\operatorname{End}(\operatorname{Im} x) \cong \operatorname{End}(\operatorname{Im} x^*) \tag{3.6}$$

and

$$\operatorname{End}(\operatorname{Im} x_i) \cong \operatorname{End}(\operatorname{Im} x_i^*) \tag{3.7}$$

for any i. From the commutativity of $B = \operatorname{Im} x$ it follows from (3.6) that $B \cong \operatorname{Im} x^* = B^*$ ([4], Theorem 4.2). By [2], Corollary 4.6 all the assumptions of Theorem 2.1 are true (taking $A = A_i$). Hence, from (3.7) follows the isomorphism $\operatorname{Im} x_i^* \cong \operatorname{Im} x_i$. From the construction of this isomorphism it follows that

$$\operatorname{Im} x_i^* = (\operatorname{Ker} x^* \cap \operatorname{Im} x_i^*) \rightthreetimes B^* \cong A_i^* \operatorname{Wr} B^* = (A_i^*)^{B^*} \rightthreetimes B^*, \tag{3.8}$$

where A_i^* is a some subgroup of $\operatorname{Ker} x^* \cap \operatorname{Im} x_i^*$ and $A_i^* \cong A_i$. The isomorphism (3.8) maps B^* identically and $\operatorname{Ker} x^* \cap \operatorname{Im} x_i^*$ onto $(A^*)^{B^*}$. From (3.4) and (3.5) it follows that G^* is isomorphic to

$$\left(\prod_{i=1}^{n} (A_i^*)^{B^*} \right) \rightthreetimes B^*, \tag{3.9}$$

where $\langle B^*, (A_i^*)^{B^*} \rangle = A_i^* \operatorname{Wr} B^*$. The group (3.9) is isomorphic to $(A_1^* \times \ldots \times A_n^*) \operatorname{Wr} B^*$. Since $A_i^* \cong A_i$ and $B^* \cong B$, then the groups G and G^* are by (3.1) isomorphic. The theorem is proved.

Theorem 3.2. *Let p be a prime, $A_0, A_1, \ldots, A_n$ are finite Abelian p-groups $(n \geqslant 1)$ and*

$$B_1 = A_1 \operatorname{Wr} A_0; \quad B_k = A_k \operatorname{Wr} B_{k-1}, \quad k = 2, 3, \ldots, n.$$

Then the group B_n is generated by its semigroup of endomorphisms in the class of all groups.

Proof. We prove the theorem by induction on n. If $n=1$ then the statement of theorem is true according to Theorem 3.1.

Assume now that $n>1$ and for $1 \leqslant k < n$ the group B_k is generated by its semigroup of endomorphisms in the class of all groups. We shall show that the group $B_n = A_n \operatorname{Wr} B_{n-1}$ is also generated by its semigroup of endomorphisms. We use Theorem 2.1 (for $B = B_{n-1}$, $A = A_n$). Since B_{n-1} is a p-group, the condition 1° of Theorem 2.1 is fulfilled. The condition 2° of Theorem 2.1 is fulfilled by the assumption of the induction. From [7], Theorem 1 the condition 3° of Theorem 2.1 is also fulfilled. Consequently, the group B_n is generated by its semigroup of endomorphisms in the class of all the groups. The theorem is proved.

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STANDARDSE PÕIMIKU MÄÄRATAVUSEST OMA ENDOMORFISMIPOOLRÜHMAGA

On tõestatud järgmised teoreemid.

Teoreem. 2.1. *Olgu p suvaline algarv, A tsükliline rühm järguga p^n ja B mingi lõplik rühm, kusjuures:*

1° *rühm B avaldub oma Sylow' p-alamrühma ja Halli p'-alamrühma otsekorrutisena;*

2° *rühm B on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis;*

3° *rühma B üle jäägiklassiringi mooduli p^n järgi võetud rühmaringi pööratavate elementide rühm on lahenduv.*

Siis põimik $A \operatorname{Wr} B$ on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.

Teoreem 3.1. *Lõplike Abeli rühmade A ja B põimik $A \operatorname{Wr} B$ on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.*

Teoreem 3.2. *Olgu $A_0, A_1, \ldots, A_n$ lõplikud Abeli p-rühmad ($n \geqslant 1$, p on algarv) ja $B_1 = A_1 \operatorname{Wr} A_0$, $B_k = A_k \operatorname{Wr} B_{k-1}$ iga $k \in \{2, \ldots, n\}$ korral. Siis rühm B_n on määratud oma endomorfismipoolrühmaga kõigi rühmade klassis.*

Пеэтер ПУУСЕМП

ОБ ОПРЕДЕЛЯЕМОСТИ СТАНДАРТНОГО СПЛЕТЕНИЯ ГРУПП ЕЕ ПОЛУГРУППОЙ ЭНДОМОРФИЗМОВ

Доказываются следующие теоремы.

Теорема 2.1. *Пусть p — произвольное простое число, A — циклическая группа порядка p^n и B — конечная группа, причем:*

1° *группа B является прямой суммой своих силовской p-подгруппы и холловской p'-подгруппы;*

2° *группа B определяется своей полугруппой всех эндоморфизмов в классе всех групп;*

3° *группа обратимых элементов группового кольца группы B над кольцом вычетов по модулю p^n разрешима.*

Тогда сплетение AWrB определяется его полугруппой эндоморфизмов в классе всех групп.

Теорема 3.1. *Сплетение $A \operatorname{Wr} B$ конечных абелевых групп A и B определяется его полугруппой эндоморфизмов в классе всех групп.*

Теорема 3.2. *Пусть $A_0, A_1, \ldots, A_n$ — конечные абелевы p-группы ($n \geqslant 1$, p — простое число) и $B_1 = A_1 \operatorname{Wr} A_0$, $B_k = A_k \operatorname{Wr} B_{k-1}$ для каждого $k \in \{2, \ldots, n\}$. Тогда группа B_n определяется ее полугруппой эндоморфизмов в классе всех групп.*