https://doi.org/10.3176/phys.math.1991.4.11

Proc. Estonian Acad. Sci. Phys. Math., 1991, **40**, N 4, 334—336

УДК 519.816, 621.391.176

Ivar PETERSEN *

ON MASS CONCENTRATIONS ON THE BOUNDARIES BETWEEN OPTIMAL MULTIDECISION REGIONS

Ivar PETERSEN. MASSIDE KONTSENTRATSIOONIST OPTIMAALSETE MULTIOTSUSTUSPIIRKONDADE RAJAPINDADEL

Ивар ПЕТЕРСЕН. О КОНЦЕНТРАЦИИИ МАСС НА ГРАНИЦАХ МЕЖДУ ОПТИМАЛЬНЫМИ ОБЛАСТЯМИ МУЛЬТИРЕШЕНИЯ

(*Presented by N. Alumäe*)

In Petersen [1] we formulated the optimal *multidecision problem* as the problem: *Given* $X \subset R^n$ (*the set of admissible decisions*), $f: X \times R^m \to R^+$ (*the loss-function*), μ — *a nonnegative finite measure on the Borel* σ-*field* $\sum$ *of subsets of* R^m (*the space of individuals*), *and a natural number* N. *Find such an admissible multidecision* $\mathbf{x}^* = (x_1^* \ldots, x_N^*)$, $x_1^*, \ldots, x_N^* \in X$ *and such a partition* $\mathbf{Y}^* = (Y_1^*, \ldots, Y_N^*)$ *of the space of individuals into* N *decision regions* $Y_1^*, \ldots Y_N^* \in \sum$ *that minimize the summary losses*

$$F(\mathbf{x}, \mathbf{Y}) = \sum_{i=1}^{N} \int_{Y_i} f(x_i, y)\, \mu(dy) \to \min \tag{1}$$

subject to $x_i \in X$, $Y_i \in \sum$, $i = 1, \ldots, N$, $R^m \subset \bigcup_{i=1}^{N} Y_i$.

A pair $\mathbf{x}^*$, $\mathbf{Y}^*$ is a *local minimum* of the multidecision problem (1) if for some $\delta > 0$

$$F(\mathbf{x}^*, \mathbf{Y}^*) < F(\mathbf{x}, \mathbf{Y}) \tag{2}$$

for every $\mathbf{x}$, $0 < \|\mathbf{x} - \mathbf{x}^*\| < \delta$, and for every N-partition $\mathbf{Y}$ of R^m.

* Eesti Teaduste Akadeemia Küberneetika Instituut (Institute of Cybernetics, Estonian Academy of Sciences). 200108 Tallinn, Akadeemia tee 21. Estonia.

In connection with the optimal quantization problem of continuous signals Lloyd [2] showed by an intuitive argumentation that if $n=m$, $f(x, y)=\|y-x\|^2$, $X=R^n$ (in fact he considered the case $n=1$) then the boundaries of the optimal decision regions cannot go through atoms of the measure μ. This result has been later used (e. g. Gray, Kieffer and Linde [3], Petersen [4]) by characterizing the minimum point of (1) by means of calculus. Applied to geographic theory (e. g. Gusein-Zade [5]) one can interprete the result as a proof that optimal state borders cannot go through cities.

The aim of this note is to generalize the Lloyd's result to the case of the more general problem (1).

Theorem. *Let* $\mathbf{x}^*$, $\mathbf{Y}^*$ *be a local minimum of the multidecision problem* (1) *and let* B *be a Borel subset of the common boundary between* $Y^*_{i_1}, \ldots$ $\ldots, Y^*_{i_s}$ $(s\geqslant 2)$. *If at least for one* $i=i_1, \ldots, i_s$ *there exists a sequence of points* $\{z_k\}$, $z_k\in X, z_k\neq x^*_i$ $z_k\to x^*_i$ *when* $k\to\infty$, *satisfying for almost all* $y\in B$ *and for all* k

$$[f(z_k, y)-f(x^*_i, y)]/\|z_k-x^*_i\|\geqslant\delta>0, \tag{3}$$

$$\int\limits_{Y^*_i}[f(z_k, y)-f(x^*_i, y)]\mu(dy)/\|z_k-x^*_i\|\to 0, \tag{4}$$

then $\mu(B)=0$.

Proof. Assume, on the contrary, $\mu(B)=p>0$. Let $j\neq i$ and $j\in\{i_1, \ldots, i_s\}$. The necessary conditions for local optimality (e. g. Petersen [1]) imply $f(x^*_i, y)=f(x^*_j, y)$ for almost all $y\in B$ and therefore

$$\int\limits_B f(x^*_i, y)\mu(dy)=\int\limits_B f(x^*_j, y)\mu(dy), \tag{5}$$

and that we can take $B\subset Y^*_i$. From (3) we have

$$\int\limits_B[f(z_k, y)-f(x^*_i, y)]\mu(dy)\geqslant p\delta\|z_k-x^*_i\|, \quad k=1, 2, \ldots \tag{6}$$

Condition (4) implies the existence of a sequence of numbers $\{\varepsilon_k\}$, $\varepsilon_k>0$, $\varepsilon_k\to 0$ such that

$$\int\limits_{Y^*_i}[f(z_k, y)-f(x^*_i, y)]\mu(dy)=\varepsilon_k\|z^k-x^*_i\|. \tag{7}$$

From (6) and (7) we have

$$\int\limits_{Y^*_i\setminus B} f(z_k, y)\mu(dy)\leqslant\int\limits_{Y^*_i\setminus B} f(x^*_i, y)\mu(dy)+(\varepsilon_k-p\delta)\|z_k-x^*_i\|. \tag{8}$$

Let us now modify the partition $\mathbf{Y}^*$, carrying from Y^*_i the subset B over to Y^*_j. In the so obtained new partition $\mathbf{Y}'$ we have $Y'_i=Y^*_i\setminus B$ and $Y'_j=Y_j\cup B$, while $Y'_t=Y^*_t$ for $t\neq i, j$. From (5) it follows $F(\mathbf{x}^*, \mathbf{Y}^*)=F(\mathbf{x}^*, \mathbf{Y}')$. Let us substitute in $\mathbf{x}^*$ instead of x^*_i the point z_k and denote the obtained new multidecision by $\mathbf{z}^*_k$. The values $F(\mathbf{x}^*, \mathbf{Y}^*)$ and $F(\mathbf{z}^*_k\ \mathbf{Y}')$ may differ only in terms with indices i and j. Inequality (8) implies

$$\int_{Y'_i} f(z_k, y)\,\mu(dy) + \int_{Y'_j} f(x^*_j, y)\,\mu(dy) =$$

$$= \int_{Y^*_i \setminus B} f(z_k, y)\,\mu(dy) + \int_{Y^*_j \setminus B} f(x^*_j, y)\,\mu(dy) + \int_B f(x^*_j, y)\,\mu(dy) \leqslant$$

$$\leqslant \int_{Y^*_i \setminus B} f(x^*_i, y)\,\mu(dy) + (\varepsilon_k - p\delta)\,\|z_k - x^*_i\| + \int_{Y^*_j} f(x^*_j, y)\,\mu(dy) +$$

$$+ \int_B f(x^*_i, y)\,\mu(dy) = \int_{Y^*_i} f(x^*_i, y)\,\mu(dy) +$$

$$+ \int_{Y^*_j} f(x^*_j, y)\,\mu(dy) + (\varepsilon_k - p\delta)\,\|z_k - x^*_i\|.$$

Here $\varepsilon_k - p\delta < 0$ for all sufficiently large k. Therefore $F(\mathbf{z}^*_k, \mathbf{Y}') < F(\mathbf{x}^*, \mathbf{Y}^*)$, where $\mathbf{z}^*_k$ can be taken arbitrarily close to $\mathbf{x}^*$. This contradicts the assumption that $\mathbf{x}^*$, $\mathbf{Y}^*$ is a local minimum of $F(\mathbf{x}, \mathbf{Y})$.

REFERENCES

1. *Petersen, I.* Proc. Estonian Acad. Sci. Phys. Math., 1990, **39**, 3, 205—212.
2. *Lloyd, S. P.* Unpublished Bell Laboratories Memorandum, 1957, Published in IEEE Trans. Inform. Theory, 1982, **IT-28**, 129—136.
3. *Gray, R. M., Kieffer, J. C., Linde, Y.* Inform. Contr., 1980, **45**, 178—198.
4. *Petersen, I.* Math. Soc. Sci., 1991, **21**, 81—93.
5. Гусейн-Заде С. М. Модели размещения населения и населенных пунктов. Москва, Изд-во МГУ, 1988.

Received
Feb. 1, 1991