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SEMI-SYMMETRIC ENVELOPES OF SOME SYMMETRIC CYLINDRICAL SUBMANIFOLDS

(Presented by G. Vainikko)

1. Introduction. Semi-symmetric submanifolds M^m in space forms $N^n(c)$, especially in Euclidean spaces E^n, generalize the locally symmetric submanifolds in the following sence.

The last ones are defined as admitting locally reflections with respect to the normal $(n-m)$-planes [1,2] and are analytically characterized by $\overline{\nabla}h=0$ (i.e. the second fundamental form h is parallel with respect to the van der Waerden-Bortolotti connection $\overline{\nabla}=\nabla\oplus\nabla^{\perp}$; therefore they are often also called «parallel submanifolds» [3]).

The semi-symmetric ($\equiv$semi-parallel) submanifolds M^m in $N^n(c)$ are defined (see [4,5] and references in [6]) as satisfying the integrability condition $\overline{\nabla}_{[X}\overline{\nabla}_{Y]}h=0$ (or, equivalently, $\overline{R}(X,Y)h=0$) of the system $\overline{\nabla}h=0$, where X and Y are two arbitrary vector fields on M^m. The formulation given in the last brackets shows that this condition is, due to the Gauss equation, an algebraic relation on the components of h with respect to the frames of the adapted orthonormal frame bundle. Hence the semi-symmetricity is a pointwise property of a submanifold. Geometrically this is expressed by the next result.

Theorem A (see [7]). *A submanifold M^m in $N^n(c)$ is semi-symmetric iff M^m in its every point x has the 2nd-order tangency with a locally symmetric m-dimensional submanifold $\overline{M}^m(x)$, i.e. is a 2nd-order envelope of the latter.*

Here the 2nd-order tangency means that for every path λ in M^m through x there exists a path $\tilde{\lambda}$ in M^m which has the 2nd-order tangency with λ at x.

In some cases such envelopes are trivial; e.g. the 2nd-order envelope of m-dimensional spheres in $N^n(c)$ is a single sphere $S^m(r)$ if $m\geqslant 2$, because such envelope consists of umbilic points. More general result gives

Theorem B (see [8,9]). *The 2nd-order envelope of the products $S^{m_1}(R_1)\times\ldots\times S^{m_k}(R_k)$ in E^n, where $m_1\geqslant 2,\ldots,m_k\geqslant 2$, $n\geqslant m_1+\ldots+m_k+k$, is a single such product.*

As every line M^1 in $N(c)$ is the 2nd-order envelope of its curvature circles, the conditions $m_1\geqslant 2,\ldots,m_k\geqslant 2$ are essential here. More general is the case of the 2nd-order envelope of the products $S^{m_1}(R_1)\times\ldots\times S^{m_k}(R_k)\times E^{m_0}$ in E^n, $n\geqslant n_0=m_1+\ldots+m_k+m_0+k$, without any restriction on the dimensions $m_0, m_1,\ldots,m_k$. If $m_0\geqslant 1$, this product is called a cylinder on $S^{m_1}(R_1)\times\ldots\times S^{m_k}(R_k)$, and due to [10] it is the most general case of normally flat ($\equiv\nabla^{\perp}$ is flat) symmetric submanifold; such cylinder lies in E^{n_0}. It follows that the 2nd-order envelope of these cylinders is the most general case of a normally flat semi-symmetric submanifold in E^n considered in [9,11].

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The first aim of this paper is to give a geometric description of the latter by means of the concept of warped product. Our Theorem 1 finishes the investigations of normally flat semi-symmetric submanifolds M^m in E^n started in [12] and developed in [9, 11] (see also [6]).

The second aim of this paper is to start the investigations of 2nd-order envelopes of the so-called Veronese cylinders.

Veronese submanifold $V^m(R)$ in $E^{1/2m(m+3)}$ is some kind of opposite for a sphere $S^m(R)$ in E^{m+1}: if $S^m(R)$ is the irreducible ($\equiv$nonproduct) symmetric submanifold M^m with the first normal space of minimal nonzero dimension 1 at every point $x \in M^m$ then $V^m(R)$ is the irreducible symmetric submanifold M^m with the first normal space of maximal dimension $\frac{1}{2}m(m+1)$ at every point $x \in M^m$ (see [13] and references in it; also [14]). Both of them are the standard immersions of the m-dimensional compact simply connected Riemannian manifold of constant curvature R^{-2}, respectively, of the 1st and 2nd order.

Theorem C (see [13, 15, 16]). *The 2nd-order envelope M^m of $V^m(R)$ in $E^{\frac{1}{2}m(m+3)}$, $m \geqslant 2$, is trivial, i. e. it is a single $V^m(R)$. In $E^{\frac{1}{2}m(m+3)+1}$ there exist nontrivial 2nd-order envelopes M^m of $V^m(R)$.*

Below, these results will be proved anew in the context of the present paper (see Section 4).

The point is that such nontrivial 2nd-order envelopes of $V^m(R)$ are the most general cases in classifications of semi-symmetric submanifolds (for $m=2$ see [4], for $m=3$ see [17]).

The product $V^m(R) \times E^q$ in $E^{\frac{1}{2}m(m+3)+q}$ is called the Veronese cylinder. It is a symmetric submanifold with $\frac{1}{2}m(m+1)$-dimensional first normal space at every point, hence the 2nd-order envelope M^{m+q} of $V^m(R) \times E^q$ in E^n, $n \geqslant \frac{1}{2}m(m+3)+q$ is a semi-symmetric submanifold.

From Theorem 1 below it follows that the 2nd-order envelope of spherical cylinders $S^m(R) \times E^q$ in E^n, $n \geqslant m+q+1$, $m \geqslant 2$, is either a single cylinder $S^m(R) \times E^q$ in E^{m+q+1} or a product $M^{m+1} \times E^{q-1}$, where M^{m+1} is a spherical (or round) cone with point-vertex and 1-dimensional generators (see also [9]). The problem is which is the geometric construction of the 2nd-order envelope M^{m+q} of $V^m(R) \times E^q$ in E^n, $n \geqslant \frac{1}{2}m(m+3)+q$. Theorem 2 below states that if $m \geqslant 3$, then such envelope is a product $M^m \times E^q$ where M^m is the 2nd-order envelope of $V^m(R)$ in E^{n-q} (see Theorem C), but if $m=2$, there is another possibility: M^{2+q} can be a product $M^3 \times E^{q-1}$ where M^3 is a cone with point-vertex in E^{n-q+1}, which is a 2nd-order envelope of $V^2(R) \times E^1$.

The next problem arises — can such a cone M^3 in E^{n-q+1} be realized by all the values of $n-q+1 \geqslant 6$? The first results in this direction are formulated in the Proposition (Section 5). If $n-q+1=6$, then such cone M^3 is impossible; more exactly, the only 2nd-order envelope of $V^2(R) \times E^1$ in E^6 is a single $V^2(R) \times E^1$. For the case $n-q+1=7$ we can say that in E^7 there are no cones M^3 2nd-order enveloping the Veronese cylinders $V^2(R) \times E^1$, so that R at $x \in M^3$ depends only on the distance between x and the vertex of the cone M^3. If this is not the case or if $n-q+1>7$, the problem of the existence of such cones M^3 remains open. The conjucture is formulated that they do not exist.

2. 2nd-order envelope of normally flat symmetric cylinders. Such a cylinder in E^n is, according to [10], a product $S^{m_1}(R_1)\times\ldots\times S^{m_k}(R_k)\times E^{m_0}$, $n\geqslant n_0=m_0+\ldots+m_k+k$, $m_0\geqslant 1, \ldots, m_k\geqslant 1$. Let $m_1>1, \ldots, m_p>1$ and $m_{p+1}=\ldots=m_{p+q}=1$, $p+q=k$. The moving orthonormal frame in E^n can be adapted to the point x of the envelope M^m so that

1) $e_{i\rho}\in T_xS^{m_\rho}(R_\rho)\subset T_xM^m$; $\quad m=n_0-k$, $\quad \varrho\in\{1, \ldots, p\}$,
2) $e_a\in T_xS^1(R_a)\subset T_xM^m$; $\quad a\in\{m^*+1, \ldots, m^*+q\}$, $\quad m^*=\sum_\rho m_\rho$,
3) $e_\alpha\in T_xE^{m_0}\subset T_xM^m$; $\quad \alpha\in\{m^*+q+1, \ldots, m\}$,
4) $e_{m+\rho}\|\overrightarrow{xz_\rho}$, $\quad e_{m_*+a}\|\overrightarrow{xz_a}$, where $m_*=m+p-m^*$, but z_ρ and z_a are the centres of $S^{m_\rho}(R_\rho)$ and $S^1(R_a)$ at x, respectively. Then for the formulae

$$dx=e_I\omega^I, \quad de_I=e_J\omega^J_I, \quad \omega^J_I+\omega^I_J=0,$$
$$d\omega^I=\omega^J\wedge\omega^I_J, \quad d\omega^J_I=\omega^K_I\wedge\omega^J_K$$

the equations are valid:

$$\omega^{m+\rho}=\omega^{m_*+a}=\omega^\xi=0; \quad \xi\in\{n_0+1, \ldots, n\},$$
$$\omega^{m+\sigma}_{i_\rho}=\varkappa_\rho\delta^\sigma_\rho\omega^{i_\rho}, \quad \omega^{m+\sigma}_u=0, \quad \varkappa_\rho=R^{-1}_\rho,$$
$$\omega^{m_*+a}_{i_\rho}=0, \quad \omega^{m_*+a}_u=\varkappa_a\delta^a_u\omega^a, \quad \varkappa_a=R^{-1}_a,$$
$$\omega^\xi_i=0, \quad \omega^\xi_u=0$$

(see [11, 6]); here $u\in\{m^*+1, \ldots, m\}$, i.e. u run the sum of sets for a and α.

After the exterior differentiation and application of the Cartan lemma, these equations lead to the next ones (cf. [11, 6]):

$$\omega^{j_\sigma}_{i_\rho}=0 \ (\varrho\neq\sigma), \quad \omega^u_{j_\rho}=\lambda_{\rho u}\omega^i,$$
$$\omega^b_a=\varkappa_a c^b_a\omega^a-\varkappa_b c^a_b\omega^b, \quad \omega^\alpha_a=\lambda_{a\alpha}\omega^a,$$
$$\omega^{m+\sigma}_{m+\rho}=0 \ (\varrho\neq\sigma), \quad \omega^{m_*+a}_{m+\rho}=-\varkappa_a\varkappa^{-1}_\rho\lambda_{\rho a}\omega^a, \quad \omega^\xi_{m+\rho}=0, \tag{2.1}$$
$$\omega^{m_*+b}_{m_*+a}=\varkappa_a c^a_b\omega^a-\varkappa_b c^b_a\omega^b, \quad \omega^\xi_{m_*+a}=\varphi^\xi_a\omega^a,$$
$$d\varkappa_a=\gamma_a\omega^a+\varkappa^2_a\sum_u c^u_a\omega^u, \quad c^\alpha_a=\varkappa^{-1}_a\lambda_{a\alpha},$$
$$d\varkappa_\rho=\varkappa_\rho\lambda_{\rho u}\omega_u.$$

From these new equations of the first row the same procedure gives that

$$\sum_u\lambda_{\rho u}\lambda_{\sigma u}=0 \quad (\varrho\neq\sigma), \tag{2.2}$$

$$d\lambda_{\rho u}=\lambda_{\rho v}\omega^v_u+\lambda_{\rho u}\lambda_{\rho v}\omega^v. \tag{2.3}$$

Let us fix the value ϱ and consider the distribution of $T_xS^{m_\rho}(R_\rho)$ on M^m. This distribution is the annulet of the system $\omega^{i_\sigma}=0$ $(\sigma\neq\varrho)$, $\omega^u=0$. Due to $d\omega^{i_\sigma}=\omega^{j_\sigma}\wedge(\omega^{i_\sigma}_{j_\sigma}+\delta^{i_\rho}_{j_\sigma}d\ln\varkappa_\sigma)$, $d\omega^u=\omega^v\wedge\omega^u_v$ this distribution is a foliation. For every of its integral submanifold the formulae

$$dx=e_{i_\rho}\omega^{i_\rho}\ (\varrho \text{ fixed}),\quad de_{i_\rho}=e_{j_\rho}\omega^{j_\rho}_{i_\rho}+\omega^{i_\rho}\Big(\sum_u \lambda_{\rho u}e_u+\varkappa_\rho e_{m+\rho}\Big) \tag{2.4}$$

are valid, hence this submanifold is a sphere $S^{m_\rho}(r_\rho)$ with the radius

$$r_\rho=\Big(\sum_u \lambda^2_{\rho u}+\varkappa^2_\rho\Big)^{-\frac{1}{2}}$$

and the normal unit vector

$$n_\rho=r_\rho\Big(\sum_u \lambda_{\rho u}e_u+\varkappa_\rho e_{m+\rho}\Big).$$

From (2.2) it follows that $\langle n_\rho, n_\sigma\rangle=\delta_{\rho\sigma}$.

All such spheres $S^{m_1}(r_1), \ldots, S^{m_p}(r_p)$, going through $x\in M^m$, lie on the sphere $S^{m^*+p-1}(r)$, whose radius is $r=\Big(\sum_\rho r^2_\rho\Big)^{\frac{1}{2}}$ and whose centre has the radius vector $y=x+\sum_\rho r_\rho n_\rho$.

To prove it we deduce that

$$dr_\rho=r_{\rho u}\omega^u,\quad r_{\rho u}=-r_\rho\lambda_{\rho u},\quad dn_\rho=-r^{-1}_\rho e_{i_\rho}\omega^{i_\rho}. \tag{2.5}$$

Thus $\varkappa_\rho r_\rho=c_\rho=\text{const}$ and

$$dy=f_u\omega^u,\quad f_u=e_u+\sum_\rho r_{\rho u}n_\rho,\quad dr=r^{-1}\sum_\rho r_\rho r_{\rho u}\omega^u. \tag{2.6}$$

Along every integral submanifold of the foliation given by $\omega^u=0$, we have $dy=0$, $dr=0$ and, therefore, this submanifold lies on the sphere mentioned above.

Let us consider the submanifold $Q^{m'}$, $m'=q+m_0$ of the centres of these spheres $S^{m^*+p-1}(r)$. The straightforward computation shows that $\langle f_u, n_\sigma\rangle=0$ and

$$df_u=f_v\omega^v_u+\delta^a_u\varkappa_a\omega^a e_{m_*+a}. \tag{2.7}$$

From (2.4) and (2.5) it follows that for every fixed ϱ the $(m_\rho+1)$-planes of the spheres $S^{m_\rho}(r_\rho)$, generating M^m, are parallel in E^n. Moreover, for every two distinct values ϱ and σ corresponding $(m_\rho+1)$-plane and $(m_\sigma+1)$-plane are totally orthogonal to each other, because every of e_{i_ρ}, n_ρ is orthogonal to every of e_{j_σ}, n_σ, if $\varrho\neq\sigma$. All of them are orthogonal to the $(n-m^*-p)$-plane, in which the submanifold $Q^{m'}$ is immersed. The last assertion follows from (2.4) and (2.5) if to start with obvious identities $\langle dy, e_{i_\rho}\rangle=0$, $\langle dy, n_\rho\rangle=0$ and deduce successively $\langle d^2y, e_{i_\rho}\rangle=0$, $\langle d^2y, n_\rho\rangle=0$; $\langle d^3y, e_{i_\rho}\rangle=0$, $\langle d^3y, n_\rho\rangle=0$, etc.

The last properties of M^m lead us to the next general concept.

Let M^m be a smooth fibre bundle immersed in a Euclidean space E^n and having the next properties:

1) the base $Q^{m'}$ is a smooth submanifold in $E^{n'}\subset E^n$,

2) the fibre on arbitrary $y\in Q^{m'}$ is the product of spheres $S^{m_1}(r_1)\times\ldots\times S^{m_p}(r_p)$ in $(n-n')$-plane which is totally orthogonal to $E^{n'}$ and goes through y so that y is the centre of the sphere $S^{n-n'-1}(r)$ containing this product; $n-n'=\sum_\rho m_\rho+p$, $m-m'=\sum_\rho m_\rho$,

3) for every fixed value $\varrho \in \{1, \ldots, p\}$ the $(m_\rho+1)$-planes of $S^{m_\rho}(r_\rho)$ at two arbitrary different points of $Q^{m'}$ are parallel to each other.

Then M^m is said to be the warped product $Q^{m'} \times_{r_1} S^{m_1}(1) \times_{r_2} \ldots \ldots \times_{r_p} S^{m_p}(1)$ with warping functions $r_1, \ldots, r_p$.

A submanifold M^m in E^n is said to be irreducible if it is not a product of $M^{m_i} \subset E^{n_i}$, E^{n_i} and E^{n_j} $(i \neq j)$ being totally orthogonal, $m_i \geqslant 1$, and $E^n = E^{n_1} \times \ldots \times E^{n_k}$, $k \geqslant 2$.

Theorem 1. *Every irreducible 2nd-order envelope of symmetric products* $S^{m_1}(R_1) \times \ldots \times S^{m_k}(R_k) \times E^{m_0}$; $m_1 > 1, \ldots, m_p > 1$, $m_{p+1} = \ldots \ldots = m_k = 1$ *in* E^n *is the warped product* $Q^{m'} \times_{r_1} S^{m_1}(1) \times_{r_2} \ldots \times_{r_p} S^{m_p}(1)$; $p \leqslant m' = m_0 + k - p$, *satisfying the following conditions:*

(1) *intrinsically* $Q^{m'}$ *is locally euclidean,*

(2) *extrinsically* $Q^{m'}$ *has the rank* $k - p \leqslant m'$ *and by the bundle map the images of* $k - p$ *lines in* M^m, *2nd-order enveloping the circles* $S^1(R_{p+1}), \ldots, S^1(R_k)$ *are conjugate with each other in* $Q^{m'}$,

(3) *the warping functions* $r_1, \ldots, r_p$ *are nonconstant linear functions with respect to some local affine coordinates in* $Q^{m'}$ *and have mutually orthogonal gradients in* M^m.

Conversely, every warped product $M^m = Q^{m'} \times_{r_1} S^{m_1}(1) \times_{r_2} \ldots \ldots \times_{r_p} S^{m_p}(1)$ *in* E^n *with properties* (1), (2) *and* (3) *is an irreducible 2nd-order envelope of symmetric products* $S^{m_1}(R_1) \times \ldots \times S^{m_k}(r_k) \times E^{m_0}$ *(or, equivalently, an irreducible normally flat semi-symmetric submanifold).*

Proof. The first assertion concerning the warped product (except the inequality $p \leqslant m'$) is stated above.

To prove (1), we consider the curvature 2-forms

$$\Omega_u^v = d\omega_u^v + \omega_u^w \wedge \omega_w^v$$

of $Q^{m'}$. From (2.7) it follows that the 1st normal space of $Q^{m'}$ at its point y is spanned on e_{m_*+a} and the 2nd fundamental form of $Q^{m'}$ has, with respect to the frame $\{y, f_u, e_{m_*+a}, e_\xi\}$ in $E^{n'}$, zero components except $h_{aa}^{m_*+a} = \varkappa_a$. Thus

$$\Omega_u^v = \omega_u^{m_*+a} \wedge \omega_{m_*+a}^v = -\sum_a h_{uw}^{m_*+a} h_{vw'}^{m_*+a} \omega^w \wedge \omega^{w'} = 0.$$

To prove (2), we recall that rank $Q^{m'}$ is the rank of $m' \cdot q$ 1-forms $\omega_u^{m_*+a}$ where $q = k - p$ [18]. As $\omega_u^{m_*+a} = \delta_u^a \varkappa_a \omega^a$, we get that rank $Q^{m'} = k - p$. Recall also that the vectors $X = X^u e_u$ and $Y = Y^v e_v$ in $T_y Q^{m'}$ have conjugate directions if $h_{uv}^{m_*+a} X^u Y^v = 0$ [19]. Now this condition is satisfied for every two vectors e_a and e_b $(a \neq b)$.

The property (3) can be proved as follows. From (2.3) and (2.5) it follows that $\nabla' r_{\rho u} = 0$, where $\nabla' r_{\rho u} = dr_{\rho u} - r_{\rho v} \omega_u^v$ is the covariant differential of the covector field on $Q^{m'}$ which has the components $r_{\rho u}$ (ϱ fixed) with respect to the moving frame $\{y, f_u\}$ on $Q^{m'}$. Hence this field is a covariantly constant field on the locally euclidean $Q^{m'}$, thus r_ρ is a linear function with respect to some local affine coordinates in $Q^{m'}$. On the other hand $r_{\rho u}$ are the only components of the gradient of r_ρ with respect to the moving frame $\{x, e_{i_\rho}, e_u\}$ on M^m, which can be nonzero. The orthogonality of these gradients follows from the identity $\sum_u r_{\rho u} r_{\sigma u} = 0$ $(\varrho \neq \sigma)$ which is valid due to (2.2) and (2.5).

It remains to prove that r_ρ can not be constant and $p \leqslant m'$. Suppose that $r_\rho = \text{const}$ for a fixed value $\varrho \in \{1, \ldots, p\}$. Then $r_{\rho u} = 0$ and $\lambda_{\rho u} = 0$, thus $\omega^u_{i_\rho} = \omega^{m'+a}_{m+\rho} = 0$, $n_\rho = r_\rho \varkappa_\rho e_{m+\rho}$. It follows that $r_\rho \varkappa_\rho = 1$,

$$de_{i_\rho} = e_{j_\rho} \omega^{j_\rho}_{i_\rho} + \varkappa_\rho e_{m+\rho} \omega^{i_\rho}, \quad de_{m+\rho} = -\varkappa_\rho e_{i_\rho} \omega^{i_\rho},$$

the differentials of the other frame vectors have zero components on e_{j_ρ} and $e_{m+\rho}$. The submanifold M^m in E^n is the product of a sphere $S^{m_\rho}(r_\rho)$ and the remaining warped product M^{m-m_ρ}, and it can not be irreducible. The inequality $p \leqslant m'$ is also evident since in the opposite case we had in the m'-dimensional linear hull of vectors e_u more than m' mutually orthogonal nonzero vectors $\sum_u \lambda_{\rho u} e_u$.

Next we prove the validity of the converse assertion. The spheres $S^{m_\rho}(r_\rho)$, generating the fibre product of $M^m = Q^{m'} \times_{r_1} S^{m_1}(1) \times_{r_2} \ldots \ldots \times_{r_p} S^{m_p}(1)$ in E^n, lie in $(n-n')$-planes orthogonal to $E^{n'} \supset Q^{m'}$, and for every $\varrho \in \{1, \ldots, p\}$ their $(m_\rho+1)$-planes are parallel and mutually orthogonal in these $(n-n')$-planes. The point $x \in M^m$, its image $y \in Q^{m'}$ and the centres $c_1, \ldots, c_p$ of generating spheres $S^{m_1}(r_1), \ldots, S^{m_p}(r_p)$, going through x, are the vertices of a p-dimensional rectangular parallelepiped. Taking the unit vectors $n_1, \ldots, n_p$ on the sides of this parallelepiped, going from x, and the frame parts $\{x, e_{i_\rho}\}$ freely in the tangent spaces $T_x S^{m_\rho}(r_\rho)$, we have, due to properties of the warped product, the formulae

$$x = y - \sum_\rho r_\rho n_\rho,$$

$$de_{i_\rho} = e_{j_\rho} \omega^{j_\rho}_{i_\rho} + r_\rho^{-1} n_\rho \omega^{i_\rho}, \tag{2.8}$$

$$dn_\rho = -r_\rho^{-1} e_{i_\rho} \omega^{i}.$$

Using the property (2), we can in $T_y Q^{m'}$ take the vectors f_a in the directions of mutually conjugate lines and the remaining vectors among f_u in the $(m'-k+p)$-dimensional plane characteristic of the submanifold $Q^{m'}$ of the rank $p-k$. Then

$$dy = f_u \omega^u,$$

$$df_u = f_v \omega^v_u + \delta^a_u h^{\alpha'}_{aa} \omega^a e_{\alpha'}$$

where all vectors $e_{\alpha'}$ are forming the moving orthonormal frame of $T^\perp_y Q^{m'}$ in $E^{n'}$. As $Q^{m'}$ is locally euclidean due to (1) the identity $0 \equiv \omega_{a'} \wedge \omega^c_{\alpha'} = \sum_{\alpha'} h^{\alpha'}_{aa} \omega^a \wedge (-\sum_b g'^{cb} h^{\alpha'}_{bb} \omega^b)$ is valid and hence $g'^{cb} \sum h^{\alpha'}_{aa} h^{\alpha'}_{bb} = 0$ $(a \neq b)$, i. e. nonzero vectors $h^{\alpha'}_{aa} e_{\alpha'}$ and $h^{\alpha'}_{bb} e_{\alpha'}$ are orthogonal if $a \neq b$. Taking $e_{m_*+a} \parallel h^{\alpha'}_{aa} e_{\alpha'}$, we have $df_u = f_v \omega^v_u + \delta_{ua} e_{m_*+a} \varkappa_a \omega^a$.

As the functions $r_1, \ldots, r_p$ are due to the first part of (3) linear functions on $Q^{m'}$, there hold $dr_\rho = r_{\rho u} \omega^u$, $\nabla' r_{\rho u} = 0$. Now

$$dx = dy - d \sum_\rho r_\rho n_\rho = e_u \omega^u + \sum_{\rho, i_\rho} e_i \omega^{i_\rho}, \tag{2.9}$$

where $e_u = f_u - \sum_\rho r_{\rho u} n_\rho$ and

$$de_u = \sum_\rho r_\rho^{-i} r_{\rho u} e_{i_\rho} \omega^{i_\rho} + e_v \omega_u^v + \delta_{ua} e_{m_*+a} \varkappa_a \omega^a. \tag{2.10}$$

It is seen that the vectors e_a are tangent to the lines on M^m which are originals of the mutually conjugate lines on $Q^{m'}$ by bundle map; as these lines are tangent to $S^1(R_a)$ due to (2), then $\langle e_a, e_b \rangle = 0$ $(a \neq b)$. The remaining vectors among e_u belong to $T_x E^{m_0} \subset T_x M^m$ and, hence, the corresponding f_u in the characteristic of $Q^{m'}$ can be chosen so that $\langle e_u, e_v \rangle = 0$ $(u \neq v)$. The lenghts of the vectors f_u can be taken so that $\|f_u\| = 1 - \sum_\rho r_{\rho u}^2$; then $\|e_u\| = 1$ and the frame $\{x, e_{i_\rho}, e_u\}$ in $T_x M^m$ is orthonormal.

In the linear hull of the vectors e_u, e_{i_ρ} and n_ρ (ϱ is fixed) the vector $n_\rho + \sum r_{\rho u} e_u$ is orthogonal to e_u and e_{i_ρ}, as it is easy to see. We use it in order to choose the unit vector $e_{m+\rho}$ so that $n_\rho + \sum_u r_{\rho u} e_u = r_\rho \varkappa_\rho e_{m+\rho}$. From the second part of (3) it follows that $0 = \langle \sum_u r_{\rho u} e_u, \sum_v r_{\sigma v} e_v \rangle = \sum_u r_{\rho u} r_{\sigma u}$ $(\varrho \neq \sigma)$ and thus $\langle e_{m+\rho}, e_{m+\sigma} \rangle = 0$ $(\varrho \neq \sigma)$, but (2.8) gives

$$de_{i_\rho} = e_{j_\rho} \omega_{i_\rho}^{j_\rho} - \sum_u e_u r_\rho^{-1} r_{\rho u} \omega^{i_\rho} + \varkappa_\rho e_{m+\rho} \omega^{i_\rho}. \tag{2.11}$$

The formulae (2.9)—(2.11) show that M^m is normally flat semi-summetric submanifold i. e. the converse assertion of the Theorem is true.

Remark 1. If $k = p$ or, equivalently, rank $Q^{m'} = 0$ (i. e. if $Q^{m'}$ is a single m'-plane $E^{m'}$), then for the irreducibility of M^m it is necessary that $p = m'$, and that the dimension s of the linear hull of the gradients of functions $r_1, \ldots, r_p$ on M^m is also equal to m'.

In fact, in the opposite case, taking among the vectors e_u first s vectors $e_{u'}$, $s \leqslant p < m'$, in this hull, we have $r_{\rho u''} = 0$, where u'' runs the remaining values of u. Then $\lambda_{\rho u''} = 0$ and thus $\omega_{i_\rho}^{u''} = 0$; on the other hand, however, $\omega_{u''}^{v'} = 0$. Now $de_{u''} = e_{v''} \omega_{u''}^{v''}$ but the differentials of other frame vectors, except $e_{u''}$, have zero components on these $e_{u''}$. This shows that M^m is the product

$$(E^s \times_{r_1} S^{m_1}(1) \times_{r_2} \ldots \times_{r_p} S^{m_p}(1)) \times E^{m'-s},$$

and it is not irreducible.

3. Second-order envelopes of Veronese cylinders. The remark above states for a particular case $p = 1$ that the 2nd-order envelope of round (or spherical) cylinders $S^{m_1}(R_1) \times E^{m'}$ in E^n is either a single such cylinder or a product $(E^1 \times_{r_1} S^{m_1}(1)) \times E^{m'-1}$. Here r_1 is a linear function on the straight line E^1, thus the warped product $E^1 \times_{r_1} S^{m_1}(1)$ is a round cone in a E^{m_1+2} (cf. [9]).

Our task is to investigate now what will happen if we take a Veronese submanifold $V^{m_1}(R_1)$ (i. e. an image of the 2nd standard immersion of $S^{m_1}(R_1)$) instead of sphere $S^{m_1}(R_1)$. Here $V^{m_1}(R_1)$ can be characterized as the symmetric submanifold in E^n, the first osculating space of which has the maximal dimension $\frac{1}{2} m_1(m_1+3) = n$ at every point $x \in V^{m_1}(R_1)$ (see [13, 14]).

Let M^{m+q} be the 2nd-order envelope of the so-called Veronese cylinders $V^m(R)\times E^q$ in E^n, $n\geqslant\frac{1}{2}m(m+3)+q$. If we restrict the freedom of the moving orthonormal frame in E^n so that $x\in M^{m+q}$, $e_i\in T_xV^m(R)\subset\subset T_xM^{m+q}$, $e_u\in T_xE^q\subset T_xM^{m+q}$, $e_\alpha\in T_x^\perp M^{m+q}$ (here the indices $i, j, \ldots$ run the set $\{1, \ldots, m\}$ and $u, v, \ldots$ the set $\{m+1, \ldots, m+q\}$), then

$$\omega^\alpha=0$$

$$\omega_i^\alpha=h_{ij}^\alpha\omega^j, \quad \omega_u^\alpha=0, \tag{3.1}$$

where $h_{ij}^\alpha=h_{ji}^\alpha$. The first normal space of $V^m(R)$ (and also of M^{m+q}) at x is the linear hull of $\frac{1}{2}m(m+1)$ vectors $h_{ij}=h_{ij}^\alpha e_\alpha$, therefore these vectors are linear independent at every point $x\in M^{m+q}$. If we denote

$$\langle h_{ij}, h_{kl}\rangle=B_{ij,kl}, \tag{3.2}$$

then

$$B_{ij,kl}=\varkappa^2(2\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \quad \varkappa=R^{-1} \tag{3.3}$$

(see [13]). The second group of equations (3.1) gives, after exterior differentiation, that $\omega_u^i\wedge h_{ij}^\alpha\omega^j=0$ and thus $h_{ij}^\alpha\omega_u^i=h_{ujk}^\alpha\omega^k$, where $h_{ujk}^\alpha=h_{ukj}^\alpha$. As the vectors h_{ij} are linear independent we have $\omega_u^i=\lambda_{uk}^i\omega^k$ and $h_{ujk}^\alpha=\lambda_{uk}^i h_{ij}^\alpha$, hence $\lambda_{uk}^i h_{ij}=\lambda_{uj}^i h_{ik}$, and so $\lambda_{uk}^k=\lambda_{uj}^j$ (not to sum!), $\lambda_{uk}^j=0$ $(j\neq k)$. It follows that $\lambda_{uk}^i=\lambda_u\delta_k^i$ and

$$\omega_u^i=\lambda_u\omega^i. \tag{3.4}$$

By exterior differentiation we get now

$$d\lambda_u=\lambda_v\omega_u^v-\lambda_u\lambda_v\omega^v \tag{3.5}$$

and the next step gives the identity. Consequently,

$$d(\sum_u\lambda_u e_u)=-(\sum_u\lambda_u e_u)(\lambda_v\omega^v)+(\sum_u\lambda_u^2)(e_i\omega^i),$$

i.e. the vector $\sum_u\lambda_u e_u\in T_xE^q$ is invariantly connected to an arbitrarily fixed point $x\in M^{m+q}$ because $dx=0\Rightarrow\omega^i=\omega^v=0\Rightarrow d(\sum_u\lambda_u e_u)=0$.

Theorem 2. *The 2nd-order envelope M^{m+q} of Veronese cylinders $V^m(R)\times E^q$ in E^n, $n\geqslant\frac{1}{2}m(m+3)+q$, is a product $M^{m+1}\times E^{q-1}$, where M^{m+1} is the 2nd-order envelope of $V^m(R)\times E^1$ in E^{n-q+1}, and it can be either*

(i) a product $M^m\times E^1$ where M^m is the 2nd-order envelope of $V^m(R)$ in E^{n-q} (and, hence, $M^{m+q}=M^m\times E^q$) or

(ii) a cone with a point-vertex and 1-dimensional generators, the directrix M_s^m of which on the sphere S^{n-q} around the vertex is the 2nd-order envelope of $V^m(R)$ in S^{n-q}.

The case (ii) is impossible if $m\geqslant 3$.

Proof. If $\sum_u\lambda_u e_u=0$, then $\lambda_u=0$, $\omega_u^i=0$, the systems $\omega^u=0$ and $\omega^i=0$ are both completely integrable, because $d\omega^u=\omega^v\wedge\omega_v^u$, $d\omega^i=\omega^i\wedge\omega_j^i$. As $de_u=e_v\omega^v$, $de_i=e_j\omega^j+h_{ij}\omega^j$ and dh_{ij} have zero components on e_u, so $M^{m+q}=M^m\times E^q$. This is the case (i).

Let $\sum_u \lambda_u \varrho_u \neq 0$. Then we can adapt the frame so that $e_{m+1} \parallel \sum_u \lambda_u e_u$. It gives

$$\lambda_{m+1} = \lambda \neq 0, \quad \lambda_{u'} = 0; \quad u', v', \ldots \text{ run}\{m+2, \ldots m+q\}. \tag{3.6}$$

From (3.4) and (3.5) it follows, respectively,

$$\omega^i_{m+1} = \lambda \omega^i \qquad \omega^i_{u'} = 0, \tag{3.7}$$

$$d\lambda = -\lambda^2 \omega^{m+1}, \quad \omega^{m+1}_{u'} = 0. \tag{3.8}$$

Now $d\omega^{u'} = \omega^{v'} \wedge \omega^{u'}_{v'}$, $d\omega^i = \omega^j \wedge \omega^i_j + \omega^{m+1} \wedge \lambda \omega^i$, $d\omega^{m+1} = 0$, so the systems $\omega^{u'} = 0$ and $\omega^i = 0$ are both completely integrable. As

$$de_i = e_j \omega^j_i - \lambda e_{m+1} \omega^i + h_{ij} \omega^j, \qquad de_{u'} = e_{v'} \omega^{v'}_{u'} \tag{3.9}$$

and de_{m+1}, dh_{ij} have zero components on $e_{u'}$, the envelope M^{m+q} is a product $M^{m+1} \times E^{q-1}$. As (3.1) are satisfied for M^{m+1} (only instead of u there is a single value $m+1$) this M^{m+1} is the 2nd-order envelope of $V^m(R) \times E^1$.

For M^{m+1} (i.e. by $\omega^{u'} = 0$)

$$d\left(x - \frac{1}{\lambda} e_{m+1}\right) = \omega^i e_i + \omega^{m+1} e_{m+1} + \frac{1}{\lambda^2} (-\lambda^2 \omega^{m+1}) e_{m+1} - \frac{1}{\lambda} (\lambda \omega^i e_i) = 0.$$

It follows that the point with radius vector $y = x - \frac{1}{\lambda} e_{m+1}$ is fixed for M^{m+1} and all integral straight lines of the system $\omega^i = 0$ go through this fixed point. Hence M^{m+1} is a cone with point-vertex. For integral submanifolds of the equation $\omega^{m+1} = 0$ on this cone we have $\lambda = \text{const}$, thus each of them lies on a sphere $S^{n-q}(\lambda^{-1})$ around this vertex. With respect to the inner geometry of constant curvature λ^2 of $S^{n-q}(\lambda^{-1})$ this integral submanifold M^m_S is semi-symmetric and its first osculating space has the maximal dimension $\frac{1}{2} m(m+3)$, thus M^m_S is the 2nd-order envelope of $V^m(R)$ in $S^{n-q}(\lambda^{-1})$ (see [7, 15]). This gives (ii).

To prove the last assertion, we consider the curvature 2-forms Ω^j_i of M^m_S. Here

$$\Omega^j_i = d\omega^j_i - \omega^k_i \wedge \omega^j_k = \omega^{m+1}_i \wedge \omega^j_{m+1} + \omega^\alpha_i \wedge \omega^j_\alpha$$

and from (3.7), (3.1) and (3.2) it follows that

$$\Omega^j_i = -(\lambda^2 + \varkappa^2) \omega^i \wedge \omega^j. \tag{3.10}$$

This expression shows that M^m_S in the case $m \geqslant 3$ is intrinsically the Riemannian manifold of constant curvature and the Schur theorem gives that $\lambda^2 + \varkappa^2 = \text{const}$. Hence $2\lambda\, d\lambda + 2\varkappa\, d\varkappa = 0$, and due to (3.8)

$$d\varkappa = \varkappa^{-1} \lambda^3 \omega^{m+1}.$$

On the other hand the first group of equations (3.1) gives, after exterior differentiation, that

$$dh^\alpha_{ij} = h^\alpha_{kj} \omega^k_i + h^\alpha_{ik} \omega^k_j - h^\beta_{ij} \omega^\alpha_\beta - h^\alpha_{ij} \lambda \omega^{m+1} + h^\alpha_{ijk} \omega^k, \tag{3.11}$$

where h^α_{ijk} are symmetric with respect to indices i, j, k. If now to differentiate (3.2), we get

$$B_{pj,kl} \omega^p_i + B_{ip,kl} \omega^p_j + B_{ij,pl} \omega^p_k + B_{ij,kp} \omega^p_l - 2\lambda(\varkappa^{-2}\lambda^2 + 1) B_{ij,kl} \omega^{m+1} +$$
$$+ F_{ij,klp} \omega^p + F_{kl,ijp} \omega^p = 0,$$

where $F_{ij,klp}=\sum_{\alpha} h^{\alpha}_{ij} h^{\alpha}_{klp}$ are symmetric with respect to indices in the first pair and also the second triplet. After substituting here (3.3), we obtain

$$F_{ij,klp}\omega^p+F_{kl,ijp}\omega^p-2\lambda(\lambda^2+\varkappa^2)(2\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\omega^{m+1}=0, \quad (3.12)$$

but this conrtadicts to $\lambda\neq 0$.

4. On the 2nd-order envelope of $V^m(R)$. Theorem 2 reduces the investigation of the 2nd-order envelope M^{m+q} of Veronese cylinders $V^m(R)\times E^q$ to the case of $q=1$, and shows that if $m\geqslant 3$ then only products $M^m\times E^1$ are to be considered, where M^m is the 2nd-order envelope of $V^m(R)$.

Relating to this envelope M^m Theorem C is known (see Section 1). For the sake of completeness we give here a new proof of this theorem in the context of Section 3, more compact than the argumentation in [13, 15, 16].

Proof of Theorem C. We start with the case $m\geqslant 3$.

While considering the first assertion we have to use the formulae of Section 3 by $n=\frac{1}{2}m(m+3)$ and $\lambda=0$. Then (3.12) reduces, due to the symmetricity of $F_{ij,klp}$, to $F_{ij,klp}=0$, and thus the vector $h^{\alpha}_{klp}\, e_{\alpha}$ is orthogonal to $\frac{1}{2}m(m+1)$ linearly independent vectors $h^{\alpha}_{ij}\, e_{\alpha}$ in their linear hull, hence $h^{\alpha}_{klp}=0$. Now M^m has the parallel second fundamental form and is $V^m(R)$.

For the second assertion we have $n=\frac{1}{2}m(m+3)+1$ and $\lambda=0$. Then, denoting the $\frac{1}{2}m(m+1)$ vectors among e_{α}, which belong to this hull, by e_{ρ} and the remaining one by e_{ξ}, we have $h^{\xi}_{ij}=0$ and, as above, it follows $h^{\rho}_{ijk}=0$. Now by $\alpha=\xi$ from (3.11) it follows $\omega_{ij}=h^{\xi}_{ijk}\omega^k$, where $\omega_{ij}=h^{\rho}_{ij}\omega^{\xi}_{\rho}$. Taking (3.11) by $\alpha=\varrho$ and using the exterior differentiation (by $\lambda=0$ and thus $\varkappa=\text{const}$), we get

$$h^{\rho}_{kj}\,\Omega^k_i+h^{\rho}_{ik}\,\Omega^k_j-h^{\sigma}_{ij}\,\Omega^{\rho}_{\sigma}=0,$$

where Ω^j_i are given by (3,10; $\lambda=0$) and

$$h^{\sigma}_{ij}\Omega^{\rho}_{\sigma}=\sum_{\sigma} h^{\sigma}_{ij}\Big(-\sum_{p} h^{\sigma}_{pk}h^{\rho}_{pl}\,\omega^k\wedge\omega^l+\omega^{\xi}_{\sigma}\wedge\omega^{\rho}_{\xi}\Big)=$$

$$=-\varkappa^2(h^{\rho}_{il}\,\omega^j\wedge\omega^l+h^{\rho}_{jl}\,\omega^i\wedge\omega^l)+\omega_{ij}\wedge\omega^{\rho}_{\xi}.$$

Hence $\omega_{ij}\wedge\omega^{\rho}_{\xi}=0$ and, after contracting with h^{ρ}_{kl}, we have

$$\omega_{ij}\wedge\omega_{kl}=0,$$

i.e. the 1-forms $\omega_{ij}=h^{\xi}_{ijk}\,\omega^k$ are mutually proportional. So the matrix of coefficients in right hand sides has proportional rows and thus its columns are also proportional. For a nontrivial envelope M^m this leads to

$$\omega_{11}=\varrho(\omega^1+\mu_v\omega^v), \qquad \omega_{1u}=\mu_u\omega_{11}, \qquad \omega_{uv}=\mu_u\mu_v\omega_{11},$$

where now $u, v, \ldots$ run the set of values $\{2, \ldots, m\}$. The system of covariants is

$$\theta\wedge\omega_{11}+\varrho\psi_u\wedge\omega^u=0, \qquad \psi_u\wedge\omega_{11}=0, \qquad (\mu_u\psi_v+\mu_v\psi_u)\wedge\omega_{11}=0,$$

where $\theta = d\ln\varrho - 3\mu_u\omega_1^u$, $\psi_u = d\mu_u - \mu_v\omega_u^v + \mu_u\mu_v\omega_1^v + \omega_1^u$. Due to the Cartan theory $s_1 = m$ and, as from these covariants $\theta = \tau\omega_{11} + \varrho\tau_u\omega^u$, $\psi_u = \tau_u\omega_{11}$, so here we have also m new coefficients i.e. $N = m$, so $s_1 = N$ and the system is involutory. Considered nontrivial envelope M^m exists with the arbitrariness of m functions of one variable.

Now remains the case $m = 2$. The first assertion is proved (separately from the case $m \geqslant 3$) in [13], but we can deduce it also from the formulae of Section 5 below (see Remark 2). The second assertion is proved in [16], but it can be also obtained in the same way as above for $m \geqslant 3$, only the complementary request $\varkappa = \text{const}$ must be added. Note that the problem of existence of the 2nd-order envelope M^2 of $V^2(R)$ in E^n, $n \geqslant 6$, with $R \neq \text{const}$ is still open.

5. On the case (ii) by $m = 2$. Theorem 2 leaves open the problem of existence of cones M^3, 2nd-order enveloping the Veronese cylinders $V^2(R) \times E^1$ in E^n. The attempts to solve this problem in general meet with great technical difficulties (see Remark 3 below).

Here we can exclude some cases of small codimensions

Proposition. *In E^6 there is no cones M^3, 2nd-order enveloping the Veronese cylinders $V^2(R) \times E^1$; more exactly, the only 2nd-order envelope of $V^2(R) \times E^1$ in E^6 is a single $V^2(R) \times E^1$. In E^7 there are no cones M^3, 2nd-order enveloping the Veronese cylinders $V^2(R) \times E^1$, so that R at $x \in M^3$ depends only on the distance between x and the vertex of the cone M^3.*

Proof. Let M^3 be the 2nd-order envelope of $V^2(R) \times E^1$ in E^n. As above, we take the frame vectors at $x \in M^3$ so that $e_i \in T_x V^2(R) \subset T_x M^3$; $i = 1, 2$; $e_3 \in T_x E^1 \subset T_x M^3$, but e_4, e_5 and e_6 in $T_x^\perp V^2(R)$ should be taken so that $e_4 \parallel h_{11} + h_{22}$, $e_5 \parallel h_{11} - h_{22}$, $e_6 \parallel h_{12}$; the other vectors in $T_x^\perp M^3$ we denote by e_ξ. Then

$$\begin{array}{llll}
\omega^4 = 0, & \omega_1^4 = \varkappa\sqrt{3}\,\omega^1, & \omega_2^4 = \varkappa\sqrt{3}\,\omega^2, & \omega_3^4 = 0, \\
\omega^5 = 0, & \omega_1^5 = \varkappa\omega^1, & \omega_2^5 = -\varkappa\omega^2, & \omega_3^5 = 0 \\
\omega^6 = 0, & \omega_1^6 = \varkappa\omega^2, & \omega_2^6 = \varkappa\omega^1, & \omega_3^6 = 0, \\
\omega^\xi = 0, & \omega_1^\xi = 0, & \omega_2^\xi = 0, & \omega_3^\xi = 0.
\end{array}$$

The equations of the first column give identities by exterior differentation, due to the equations of the other columns, but the equations of the last column give

$$\omega_3^1 = \lambda\omega^1, \quad \omega_3^2 = \lambda\omega^2, \quad d\lambda = -\lambda^2\omega^3$$

(see (3.7) and (3.8)). The middle two columns lead to

$$d\ln\varkappa = -2(K_1\omega^1 + K_2\omega^2) - \lambda\omega^3, \tag{5.1}$$

$$\frac{1}{\sqrt{3}}\,\omega^5 = K_1\omega^1 - K_2\omega^2, \tag{5.2}$$

$$\frac{1}{\sqrt{3}}\,\omega_4^6 = K_2\omega^1 + K_1\omega^2, \tag{5.3}$$

$$\frac{1}{5}(\omega_5^6 - 2\omega_1^2) = K_2\omega^1 - K_1\omega^2, \tag{5.4}$$

$$\sqrt{3}\,\omega_4^\xi + \omega_5^\xi = p^\xi\omega^1 + q^\xi\omega^2, \tag{5.5}$$

$$\omega_6^\xi = q^\xi \omega^1 + r^\xi \omega^2, \tag{5.6}$$

$$\sqrt{3}\,\omega_4^\xi - \omega_5^\xi = r^\xi \omega^1 + s^\xi \omega^2. \tag{5.7}$$

Let $n=6$, i. e. let there be no e_ξ. From Eqs. (5.1)—(5.4) it follows that

$$dK_1 = K_2\omega_1^2 + \frac{1}{5}(14K_2^2 - 11K_1^2 - \lambda^2)\omega^1 - 5K_1K_2\omega^2 - K_1\lambda\omega^3, \tag{5.8}$$

$$dK_2 = -K_1\omega_1^2 - 5K_1K_2\omega^1 + \frac{1}{5}(14K_1^2 - 11K_2^2 - \lambda^2)\omega^2 - K_2\lambda\omega^3. \tag{5.9}$$

Now after exterior differentation we get

$$K_i[25\varkappa^2 + 42(K_1^2 + K_2^2) - 28\lambda^2] = 0. \tag{5.10}$$

If here $[\ldots]=0$, then the differentiation leads to $K_i\pi=0$, where $\pi = 115\varkappa^2 + 14\cdot 15(K_1^2+K_2^2) \neq 0$, due to $\varkappa \neq 0$, and thus $K_i=0$, but then $\lambda=0$ from (5.8) and (5.9). If $[\ldots]\neq 0$, we get directly that $K_i=0$ (see (5.10)), and also $\lambda=0$. The first assertion is proved.

Let $n=7$; as $\varkappa = R^{-1}$ and λ^{-1} is the distance between x and the vertex of the cone (see the proof of Theorem 2), the assumption of the second part of the Proposition on R means that in (5.1) we have $K_1=K_2=0$, and so (5.2)—(5.4) reduce to

$$\omega_4^5 = \omega_4^6 = \omega_5^6 - 2\omega_1^2 = 0.$$

After the exterior differentiation we get

$$\omega_4^\xi \wedge \omega_5^\xi = 0, \quad \omega_4^\xi \wedge \omega_6^\xi = 0, \quad \omega_5^\xi \wedge \omega_6^\xi = 2\lambda^2 \omega^1 \wedge \omega^2.$$

Together with (5.5)—(5.7) this gives

$$p^\xi s^\xi - q^\xi r^\xi = 0, \quad p^\xi r^\xi - q^\xi q^\xi = 2\lambda^2, \quad s^\xi q^\xi - r^\xi r^\xi = 2\lambda^2. \tag{5.11}$$

The index ξ takes now a single value 7 which we can omit, so that we have

$$ps = qr, \quad pr = 2\lambda^2 + q^2, \quad sq = 2\lambda^2 + r^2, \quad \lambda \neq 0,$$

but here is a contradiction. In fact, from the last two it follows that p, q, r, s are nonzero; thus a σ exists so that $r=\sigma p$, $s=\sigma q$. Hence $\sigma p^2 = 2\lambda^2 + q^2$, $\sigma q^2 = 2\lambda^2 + \sigma^2 p^2$, but this leads to $2\lambda^2(\sigma+1)=0$, $\sigma>0$, which is impossible.

Remark 2. The proof of the first assertion of Theorem C for the case $m=2$ primarly given in [13], can be obtained from (5.8)—(5.10), if we substitute $\lambda=0$. It follows that $\varkappa=\text{const}$, and now the argumentation of Section 4 can be used.

Remark 3. For the most general case of the cone M^3 in E^n, $n\geqslant 7$, 2nd-order enveloping the Veronese cylinders $V^2(R)\times E^1$, the system of equations is the same as in the proof of the Proposition, till (5.7). Equations (5.8) and (5.9) are to be replaced by

$$dK_1 = K_2\omega_1^2 + \left[\frac{1}{5}(14K_2^2 - 11K_1^2 - \lambda^2) + \frac{1}{30}(4T_1 - T_3)\right]\omega^1 - \\ - 5(K_1K_2 + T_2)\omega^2 - K_1\lambda\omega^3,$$

$$dK_2 = -K_1\omega_1^2 - 5(K_1K_2 + T_2)\omega^1 + \\ + \left[\frac{1}{5}(14K_1^2 - 11K_2^2 - \lambda^2) + \frac{1}{30}(4T_3 - T_1)\right]\omega^2 - K_2\lambda\omega^3,$$

where T_2, T_3 and T_1 denote the left sides of the relations (5.11), respectively (to be summed by ξ).

These last two equations can be reduced to (5.8) and (5.9) if $T_1 = T_2 = T_3 = 0$, and to (5.11) if $K_1 = K_2 = 0$. In general they must be prolonged by exterior differentiation as well as (5.5)—(5.7), but this leads to great technical difficulties.

Based on the investigation of some particular cases we formulate the conjecture that the considered cone M^3 does not exist.

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Ülo LUMISTE

MÕNEDE SÜMMEETRILISTE SILINDRILISTE ALAMMUUTKONDADE TEIST JÄRKU MÄHKIJAD

Eukleidilises ruumis E^n oleva poolsümmeetrilise alammuutkonna M^m analüütiline tunnus on $\bar{R}(X, Y)h = 0$, geomeetriliselt on ta iseloomustatav kui sümmeetriliste alammuutkondade teist järku mähkija. Üldine normaaltasane poolsümmeetriline alammuutkond on sfääride (*incl.* ringjoonte) ja tasandi korrutiste $S^{m_1}(R_1) \times \ldots \times S^{m_k}(R_k) \times E^{m_0}$ teist järku mähkija, mille täielik geomeetriline kirjeldus on artiklis esitatud. On iseloomustatud erijuhtu, mil $k=1$ ja seega mähitakse silindreid $S^{m_1}(R) \times E^{m_0}$; ning uuritud teist järku mähkijaid silindrite puhul $V^m(R) \times E^q$, kus $V^m(R)$ on Veronese alammuutkond. Lõplikud tulemused on saadud üldjuhul, kui $m \geqslant 3$; erijuht $m=2$ on komplitseeritum, osa probleeme jääb siin lahtiseks.

Юло ЛУМИСТЕ

ПОЛУСИММЕТРИЧЕСКИЕ ОГИБАЮЩИЕ НЕКОТОРЫХ СИММЕТРИЧЕСКИХ ЦИЛИНДРИЧЕСКИХ ПОДМНОГООБРАЗИЙ

Аналитическим признаком полусимметрического подмногообразия M^m в евклидовом пространстве E^n является $R(X, Y)h = 0$, геометрически оно характеризуется как огибающее второго порядка симметрических подмногообразий. Общее нормально плоское полусимметрическое подмногообразие является огибающим второго порядка произведений $S^{m_1}(R_1) \times \ldots \times S^{m_k}(R_k) \times E^{m_0}$ сфер (включая окружности) и плоскости. Дается полное геометрическое описание такого огибающего. Характеризуется случай, когда $k=1$ и огибаются цилиндры $S^{m_1}(R) \times E^{m_0}$. Исследуются огибающие второго порядка цилиндров $V^m(R) \times E^q$, где $V^m(R)$ есть подмногообразие Веронезе. Окончательные результаты получены для общего случая, когда $m \geqslant 3$; частный случай $m=2$ более сложный, часть проблем остаются здесь открытыми.