Proc. Estonian Acad. Sci. Phys. Math., 1990, **39**, N 4, 372—375
https://doi.org/10.3176/phys.math.1990.4.08

УДК 621.391.176

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OPTIMAL QUANTIZATION FOR A DENSITY POWER FUNCTION

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Виктор ОЛЬМАН. ОПТИМАЛЬНОЕ РАЗБИЕНИЕ СТЕПЕННЫХ ПЛОТНОСТЕЙ ПРИ КВАДРАТИЧНОЙ ПОТЕРЕ

(Presented by N. Alumäe)

The problem of optimal n-quantization of one-dimensional density function $g(y)$ under quadratic loss is reduced to the determination of the points of quantization $b_1^*, b_2^*, \ldots, b_{n-1}^*$, which minimize the mean loss

$$\Psi(b_1, b_2, \ldots, b_{n-1}) = \sum_{i=1}^{n} \inf_x \int_{b_{i-1}}^{b} (y-x)^2 g(y)\,dy, \tag{1}$$

$$b_0 = -\infty, \qquad b_n = \infty,$$

presuming that the two first moments of the density $g(y)$ are finite.

As a rule the difficulty of solving this problem lies in the multi-extremality of the mean loss (1), i. e. in the non-uniqueness of the decision of the system of equations representing the necessary conditions of extremum [1] in (1):

$$b_i - x_i = x_{i+1} - b_i, \qquad i = 1, 2, \ldots, n-1, \tag{2}$$

$$x_i = \arg\inf_x \int_{b_{i-1}}^{b_i} (y-x)^2 g(y)\,dy,$$

$$x_i = \int_{b_{i-1}}^{b_i} y g(y)\,dy \Big/ \int_{b_{i-1}}^{b_i} g(y)\,dy, \qquad i = 1, 2, \ldots, n-1,$$

$$b_1 < b_2 < \ldots < b_{n-1}.$$

It is shown in [2] that in case of log-concave density $g(y)$, i. e. if $\ln g(y)$ is a concave function, the system of equations (2) has a unique solution and therefore the necessary conditions of extremum (2) become sufficient. In the present paper a class of power density functions involving both log-concave and log-convex functions in considered. It is proved that system (2) has a unique solution in spite of this essential difference.

Let us consider density power functions, i. e. those that have the following form

$$g(y)=c_k/y^k, \quad y\in(a,b), \quad b>a\geqslant 0,$$
$$g(y)=0, \quad y\notin(a,b),$$

where c_k is a normalizing constant.

It is obvious that for $k<0$ these functions are log-concave and therefore by virtue of [2] system (2) has a unique solution. In case of $k>0$ we have log-convex functions, which contradict the sufficient conditions of uniqueness, formulated in [2], and are for that reason of interest. The natural restrictions are such correlations between parameters that the problem still has a meaning, or in other words $\inf\Psi(b_1, b_2, \ldots, b_{n-1})<<\infty$. It is easy to understand that in case $0<k<1$ it is necessary that $b<\infty$, in case $3\geqslant k\geqslant 1$ $a>0$, $b<\infty$, and for $k>3$ $a>0$. The basic result of this paper consists in the following theorem.

Theorem. *For the density power function with finite first two moments the system of necessary conditions* (2) *has only one solution, and it is the one which supplies the minimum of function* $\Psi(b_1, b_2, \ldots, b_{n-1})$.

The general proof of the theorem falls into several cases, the analysis of which is carried out according to one and the same scheme. Namely, with the help of simple transformations the system of equations (2) is reduced to the system

$$\varphi_k(t_{i-1})=H_k(t_i), \quad i=1, 2, \ldots, n-1,$$
$$t_i=b_{i-1}/b_i, \quad b_0=a, \quad b_n=b, \tag{3}$$

where k is the parameter of density power function. For different $k>0$ the functions φ_k and H_k have the following form

for $$(k-1)(k-2)\neq 0, \quad \varphi_k(t)=-\frac{k-1}{k-2}\frac{1-t^{k-2}}{1-t^{k-1}}\cdot t,$$

$$H_k(t)=\frac{k-1}{k-2}\frac{1-t^{k-2}}{1-t^{k-1}}-2,$$

$$\varphi_1(t)=\frac{1-t}{\ln t}, \quad H_1(t)=\frac{t-1}{t\ln t},$$

$$\varphi_2(t)=\frac{t\ln t}{1-t}, \quad H_2(t)=\frac{\ln t}{t-1}, \quad 0<t<1.$$

Let us now consider each of the three possible cases separately.

1. Let $b<\infty$, $a>0$. Let us assume that there are two different sets of points, which are solutions of system (2): $\{t'_1, t'_2, \ldots, t'_n\}$, and $\{t''_1, t''_2, \ldots, t''_n\}$, where $t'_n\neq t''_n$, so that in case of $t'_n=t''_n$ both solutions will coincide. We suppose for definiteness that $t'_n<t''_n$. Then by virtue of the decrease of the functions $\varphi_k(t)$ and $H_k(t)$ proved in the application for $k>0$, we get

$$t'_i<t''_i, \quad i=1, 2, \ldots, n. \tag{4}$$

But by virtue of the definition of the variables we have

$$\prod_{i=1}^{n} t'_i=\prod_{i=1}^{n} t''_i=a/b,$$

which contradicts inequality (4), and hence system (2) has only one solution.

2. Let $b=\infty$. Then inequalities $k>3$, $a>0$ are the necessary conditions for the existence of the second moment. The system of equations (3) is reduced to the system

$$\varphi_k(t_i)=H_k(t_{i+1}), \quad i=1, 2, \ldots, n-2,$$

$$\varphi_k(t_{n-1})=\frac{k-1}{2-1}-2.$$

By virtue of the monotonicity of the function $\varphi_k(t)$ we get only one sequence of the roots $t^*_{n-1}, \ldots, t^*_1$, and therefore $b^*_i=a/(t^*_1 t^*_2 \ldots t^*_i)$, $i=1, 2, \ldots, n-1$ represent a unique solution of system (2).

3. Let $a=0$. Then it is necessary that $0<k<1$ and $b<\infty$. The system of equations (2) is reduced to the system

$$\varphi_k(t_i)=H_k(t_{i+1}), \quad i=2, 3, \ldots, n-1,$$

$$H_k(t_2)=-\frac{k-1}{k-2}.$$

By virtue of the monotonicity of the function $H_k(t)$ we get the unique sequence of the roots $t^*_2, t^*_3, \ldots, t^*_n$, and $b^*_i=bt^*_{i+1}t^*_{i+2}\ldots t^*_n$ $i=1, 2, \ldots, n-1$, represent the unique solution of system (2).

Application

Lemma. *For $k>0$ the functions $\varphi_k(t)$ and $H_k(t)$ are decreasing over the interval* $(0, 1)$.

Proof. With the help of simple transformations we get

$$\varphi'_k(t)=\frac{k-1}{k-2}[-t^{k-1}(k-2)+(k-1)t^{k-2}-1]/(t^{k-1}-1)^2,$$

$$H'_k(t)=\frac{k-1}{k-2}t^{k-3}[-t^{k-1}+(k-1)(t-1)+1]/(t^{k-1}-1),$$

$$(k-1)(k-2)\neq 0, \tag{5}$$

$$\varphi'_1(t)=\frac{-\ln t+\left(1-\frac{1}{t}\right)}{\ln^2 t}, \quad H'_1(t)=[\ln t-(t-1)]/(t\ln t)^2,$$

$$\varphi_2(t)=-1/H_1(t), \qquad H_2(t)=-1/\varphi_1(t).$$

Let us consider the following functions:

$$\tilde{\varphi}_k(t)=\varphi'_k(t)(t^{k-1}-1)^2, \quad \tilde{H}_k(t)=H'_k(t)t^{3-k}(t^{k-1}-1)^2,$$
$$(k-1)(k-2)\neq 0.$$

As

$$\tilde{\varphi}'_k(t)=(k-1)^2 t^{k-3}(1-t)>0,$$

$$\tilde{H}'_k(t)=(k-1)^2(1-t^{k-2})/(k-2)>0, \quad 1>t>0,$$

then

$$\tilde{\varphi}_k(t)<\tilde{\varphi}_k(1)=0, \quad \tilde{H}_k(t)<\tilde{H}_k(1)=0, \quad 0<t<1,$$

and therefore $\varphi'_k(t)<0$, $H'_k(t)<0$, $(k-1)(k-2)\neq 0$.

For $k=1$ and $k=2$ by virtue of obvious inequalities $\ln t > 1 - \frac{1}{t}$ and $\ln t < t-1$, $0 < t < 1$, we get $\varphi_1'(t) < 0$, $H_1'(t) < 0$, and by virtue of (5) analogous inequalities for $\varphi_2'(t)$ and $H_2'(t)$ that ends the proof of lemma.

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Received
Jan. 11, 1990