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APPROXIMATE INPUT-OUTPUT LINEARIZATION OF DISCRETE-TIME NONLINEAR SYSTEMS

Ülle KOTTA. DISKREETSE MITTELINEAARSE SÜSTEEMI SISEND-VÄLJUND KUJUTISE LIGIKAUDNE LINEARISEERIMINE

Юлле КОТТА. ПРИБЛИЖЕННАЯ ЛИНЕАРИЗАЦИЯ ВХОД-ВЫХОДНОГО ОТОБРАЖЕНИЯ НЕЛИНЕЙНОЙ СИСТЕМЫ ДИСКРЕТНОГО ВРЕМЕНИ

(Presented by Ü. Jaaksoo)

1. Introduction

Many researchers have tackled the input-output linearization problem as finding a static state feedback such that the input-output maps (input-dependent part) of the resulting closed-loop system and some linear system coincide. For discrete-time systems this problem has been considered in [1–3]. Generally, the linearizing feedback is not linear in new control, and in [1,3] it has been found in the form of formal power series. In practical situations, of course, one must confine oneself to some finite terms in power series. The question therefore arises: how much does the use of this approximate solution, instead of the exact solution, influence the desired result. It will be shown in this paper that if we take into consideration k terms in this series, then the Volterra kernels of the closed-loop systems up to the kth order will coincide for an approximate and an exact solution.

1. Exact input-output linearization.

Consider the nonlinear discrete-time system described by equations

$$x(t+1)=x(t)+f_0(x(t))+\sum_{i\geqslant 1} f_i(x(t))u^i(t),\ x(0)=x_0, \tag{1}$$

$$y(t)=h(x(t)), \tag{2}$$

where the state $x(t)\in R^n$, the input $u(t)\in R$, the output $y(t)\in R$, $f_i\colon R^n\to R^n$, $i\geqslant 0$ and $h\colon R^n\to R$ are analytic functions on R^n, $f_0(x_0)=0$.

With reference to analytic functions $f(x): R^n\to R^n$ and $g(x): R^n\to R^n$, the following differential operators can be introduced [1,2]:

$$L_f^{\otimes 0}=1,$$

$$L_f^{\otimes r}=\sum_{i_1,\ldots,i_r=1}^{n} f_{i_1}\ldots f_{i_r}\frac{\partial^r}{\partial x_{i_1}\ldots\partial x_{i_r}},\quad r\geqslant 1,$$

$$\Delta_f=\sum_{k\geqslant 0}\frac{1}{k!}L_f^{\otimes k},$$

$$L_f^{\otimes r}\otimes L_g^{\otimes s}=\sum_{i_1,\ldots,j_s=1}^{n} f_{i_1}\ldots f_{i_r}g_{j_1}\ldots g_{j_s}\frac{\partial^{r+s}}{\partial x_{i_1}\ldots\partial x_{j_s}},$$

where f_i, g_j, $i,j=1,\ldots,n$, and I denote the ith component of f, jth component of g and the identity operator, respectively.

Let us introduce the differential operators associated with the system (1)—(2):

$$\delta_s=\sum\frac{1}{n_1!\ldots n_k!\ldots}\Delta_{f_0}\otimes L_{f_1}^{\otimes n_1}\otimes\ldots\otimes L_{f_k}^{\otimes n_k}\otimes\ldots,\qquad s=0,1,2\ldots,$$

where the summation is taken over all sets of nonnegative integers n_i such that the equation $n_1+2n_2+\ldots+kn_k+\ldots=s$ holds.

Moreover, let

$$a_s(x)=\delta_s\circ\delta_0^d h(x)\,|_x,$$

where d is the relative order of the system, i.e. $t=d+1$ is the first instant of time at which the output is affected by the control at $t=0$ [1, 3]; by «$\circ$» is denoted the composition of operators, $\delta^k=\delta\circ\ldots\circ\delta$ (k — multiple composition) and

$$\alpha_k(x)=\sum\frac{(k+c_1+\ldots+c_k-1)!}{k!\,c_2!\ldots c_k!\,a_1^k(x)}\left(-\frac{a_2(x)}{a_1(x)}\right)^{c_2}\ldots\left(-\frac{a_k(x)}{a_1(x)}\right)^{c_k},$$

where the summation is taken over all sets of nonnegative integers c_i such that the equation $c_2+2c_3+\ldots+(k-1)c_k=k-1$ holds. The following theorem holds.*

Theorem [1]. *Consider the system* (1), (2) *for which* $\delta_1\circ\delta_0^d h(x)\,|_x\neq0$ *on some open and dense subset V of R^n. Then the input/ /output linearization problem can be solved on V by the following feedback*

$$u(t)=\sum_{n\geqslant1}\alpha_n(x(t))\,[v(t)-a_0(x(t))]^n. \tag{3}$$

3. Approximate linearization

In this section we shall compare the Volterra kernels of two closed-loop systems, one of which is obtained by using the feedback (3) and the other by the feedback

$$u^*(t)=\sum_{n=1}^{r}\alpha_n(x(t))\,[v(t)-a_0(x(t))]^n, \tag{4}$$

which takes into account only the first r terms in the series (3). The equations of closed-loop systems we get by substituting one formal series into another and they are

$$x(t+1)=x(t)+f_0(x(t))+\sum_{i\geqslant1}\xi_i(x(t))\,[v(t)-a_0(x(t))]^i \tag{5}$$

and

$$x(t+1)=x(t)+f_0(x(t))+\sum_{i\geqslant1}\gamma_i(x(t))\,[v(t)-a_0(x(t))]^i, \tag{6}$$

respectively. Here [4]

$$\xi_i=\sum_{l_1+2l_2+\ldots+il_i=i}\frac{|l_1+l_2+\ldots+l_i|!}{l_1!\ldots l_i!}f_{|l_1+\ldots+l_i|}\,\alpha_1^{l_1}\ldots\alpha_i^{l_i}$$

and

$$\gamma_i=\begin{cases}\xi_i, & \text{if } i\leqslant r\\ \displaystyle\sum_{l_1+2l_2+\ldots+rl_r=i}\frac{|l_1+l_2+\ldots+l_r|!}{l_1!\ldots l_r!}f_{|l_1+\ldots+l_r|}\,\alpha_1^{l_1}\ldots\alpha_r^{l_r}, & \text{if } i>r.\end{cases}$$

The output equations for both systems coincide with equation (2).

Let us introduce the notations

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* The result for linear analytic systems (i.e. for case $g_i(x)=0$, $i\geqslant2$) can be obtained as the conclusion from this Theorem. Although the result of Monaco and Normand-Cyrot [3] for such system is given in the different form, it can be shows by little manipulation that both coincide.

$$\tilde{\delta}_s = \sum \frac{1}{n_1! \dots n_k! \dots} \Delta_{f_0} \otimes L_{\xi_1}^{\otimes n_1} \otimes \dots \otimes L_{\xi_k}^{\otimes n_k} \otimes \dots,$$

$$s=0, 1, 2, \dots, \quad (7)$$

$$\delta_s^* = \sum \frac{1}{n_1! \dots n_k! \dots} \Delta_{f_0} \otimes L_{\gamma_1}^{\otimes n_1} \otimes \dots \otimes L_{\gamma_k}^{\otimes n_k} \otimes \dots,$$

$$s=0, 1, 2, \dots,$$

where in both cases the summation is taken over all sets of nonnegative integers n_i such that the equation $n_1+2n_2+\dots+kn_k+\dots=s$ holds.

The closed-loop systems (5), (2) and (6), (2) are analytic in state and polynomial in new control $v(t)$. The input-output behavior of these polynomial-analytic systems around the equilibrium point $v(t)=a_0(x(t))$ can be expressed by Volterra series [5]:

$$y(t+1) = w_0(t+1) + \sum_{\tau_1=0}^{t} w_1(t+1, \tau_1)[v(\tau_1) - a_0(x(\tau_1))] + \dots$$

$$\dots + \sum_{\tau_1=0}^{t} \sum_{\tau_2=\tau_1}^{t} \dots \sum_{\tau_k=\tau_{k-1}}^{t} w_k(t+1, \tau_1, \dots, \tau_k) \times$$

$$\times [v(\tau_1) - a_0(x(\tau_1))] \dots [v(\tau_k) - a_0(x(\tau_k))] + \dots$$

The kernels for the system (5), (2) are

$$w_0(t+1) = \tilde{\delta}_0^{t+1} h|_{x_0},$$

$$w_1(t+1, \tau_1) = \tilde{\delta}_0^{\tau_1} \circ \tilde{\delta}_1 \circ \tilde{\delta}_0^{t-\tau_1} h|_{x_0}, \quad (8)$$

....

$$w_k(t+1, \tau_1, \dots, \tau_k) = \tilde{\delta}_0^{\tau} \circ \tilde{\delta}_1 \circ \tilde{\delta}_1^{\tau_2-\tau_1-1} \circ \tilde{\delta}_1 \circ \tilde{\delta}_0^{\tau_3-\tau_2-1} \circ \dots \circ \tilde{\delta}_1 \circ \tilde{\delta}_0^{t-\tau_k} h|_{x_0}.$$

the following notation has been used for a compact expression of the kernels (here $\tilde{\delta}_0^{-1}$ has not been defined and carries no meaning)

$$\tilde{\delta}_{i_1} \circ \tilde{\delta}_0^{-1} \circ \tilde{\delta}_{i_2} \circ \tilde{\delta}_0^{-1} \circ \dots \circ \tilde{\delta}_0^{-1} \circ \tilde{\delta}_{i_r} = \tilde{\delta}_r, \quad r \geqslant 0, \quad i_1 = \dots = i_r = 1.$$

We get the kernels for the system (6), (2) if we replace «$\sim$» in (8) by «$*$».

Let us now study the Volterra kernels of the closed-loop systems (5), (2) and (6), (2). From (8) it is not difficult to observe that kth kernel w_k of the system (5) (2) will depend on $\tilde{\delta}_1, \tilde{\delta}_2, \dots, \tilde{\delta}_k$ and not upon $\tilde{\delta}_m$, $m>k$. Moreover, $\tilde{\delta}_s$ in turn will depend on $\xi_1, \xi_2, \dots, \xi_s$, and not on ξ_l, $l>s$ (see (7)). Therefore, the kth kernel w_k will depend on $\xi_1, \dots, \xi_k$, and not on ξ_l, $l>k$. Analogous observations can be made for the system (6), (2): the kth kernel will depend on $\gamma_1, \dots, \gamma_k$, and not on γ_l, $l>k$. As $\xi_i=\gamma_i$ for $i \leqslant r$, it follows that the kernels up to kth one will coincide for systems (5), (2) and (6), (2).

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