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SEMI-SYMMETRIC SUBMANIFOLDS WITH MAXIMAL FIRST NORMAL SPACE

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Ю. ЛУМИСТЕ. ПОЛУСИММЕТРИЧЕСКИЕ ПОДМНОГООБРАЗИЯ С МАКСИМАЛЬНЫМ ПЕРВЫМ НОРМАЛЬНЫМ ПРОСТРАНСТВОМ

(Presented by H. Keres)

1. A Riemannian manifold M^m satisfying $R(X, Y)\cdot R=0$ (i.e. the integrability condition of the system $\nabla R=0$ which characterizes a symmetric Riemannian manifold) is called a semi-symmetric Riemannian manifold [1, 2]. Analogically, a submanifold M^m in a Euclidean E^n satisfying $\bar{R}(X, Y)\cdot h=0$ (i.e. the integrability condition of the system $\overline{\nabla}h=0$ which characterizes a symmetric M^m in E^n [3]), is called a semi-symmetric submanifold M^m in E^n; cf. [4], where, the term «semi-parallel» is used. Intrinsically it is a semi-symmetric Riemannian manifold. Here h is the second fundamental form and $\overline{\nabla}$ is the van der Waerden—Bortolotti connection [5].

A problem is known in the theory of semi-symmetric Riemannian manifolds, [6]: are the only semi-symmetric irreducible manifolds the symmetric ones? The answer is, in general, negative [7], but can be positive through some additional conditions.

A similar problem rises in the theory of semi-symmetric submanifolds M^m in E^n in what conditions the only semi-symmetric submanifolds are symmetric. We are trying to give an answer to this problem in the following.

Note that all semi-symmetric surfaces ($m=2$) and hypersurfaces ($m=n-1$) in E^n are classified in [4] and [8], respectively. All semi-symmetric M^{n-2} in E^n are described in [9]. Most of them are not symmetric.

To formulate our main result, we recall that the linear span of all $h(X, X)$ in a given point $x\in M^m$ for arbitrary $X\in T_xM^m$ is called the first normal space of the submanifold M^m, and is denoted by N_xM^m, The Euclidean subspace, spanned on $x\in M^m$, T_xM^m and N_xM^m, is called the first osculating space of the submanifold M^m in the point x.

Theorem. *Every semi-symmetric submanifold M^m in $E^{1/2m(m+3)}$, $m\geqslant 2$, for which $E^{1/2m(m+3)}$ is the first osculating space in every point $x\in M^m$, is a symmetric one, it has the inner metric of positive constant curvature and, in the case of its completeness, coincides with an orbit of the orthogonal group $O(m+1)$ acting in $E^{1/2m(m+3)}$ by isometries.*

Note that in case $m=2$, this orbit is the classical Veronese surface [10] in a 4-sphere $S^4\subset E^5$ (see also [5], p. 88). In this case our theorem complements the classification theorem in [4]. If $m>2$, this orbit is an elliptic m-space minimally embedded in $S^{\frac{1}{2}m(m+3)-1}$ (i. e. is the «Veronese m-submanifold»; see [11–13]).

We obtain the Theorem as a consequence from a more general result (see Preposition below).

2. Let $\{x; e_1, \ldots, e_m; e_{m+1}, \ldots, e_n\}$ be the moving orthogonal frame adapted to the submanifold M^m in E^n, i. e. let $x \in M^m$ and $e_i \in T_x M^m$, $e_\alpha \in T_x^\perp M^m$; $i, j, \ldots = 1, \ldots, m$; $\alpha, \beta, \ldots = m+1, \ldots, n$. If we identify the point x with its radius vector in E^n, then in the derivation formulae $dx = e_I \omega^I$, $de_I = e_J \omega_I^J$, $\omega_I^J + \omega_J^I = 0$; $I, J = 1, \ldots, n$; we have

$$\omega^\alpha = 0, \tag{1}$$

$$\omega_i^\alpha = h_{ij}^\alpha \omega^j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{2}$$

$$\overline{\nabla} h_{ij}^\alpha := dh_{ij}^\alpha - h_{kj}^\alpha \omega_i^k - h_{ik}^\alpha \omega_j^k + h_{ij}^\beta \omega_\beta^\alpha = h_{ijk}^\alpha \omega^k, \quad h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{3}$$

$$\overline{\nabla} h_{ijk}^\alpha \wedge \omega^k = -h_{kj}^\alpha \Omega_i^k - h_{ik}^\alpha \Omega_j^k + h_{ij}^\beta \Omega_\beta^\alpha. \tag{4}$$

Here

$$\Omega_i^j := d\omega_i^j - \omega_i^k \wedge \omega_h^j = -\sum_\alpha h_{i[k}^\alpha h_{l]j}^\alpha \omega^k \wedge \omega^l, \tag{5}$$

$$\Omega_\alpha^\beta := d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta = -\sum_i h_{i[k}^\alpha h_{l]i}^\beta \omega^k \wedge \omega^l \tag{6}$$

are the curvature 2-forms of the Levi-Civita connection ∇ and of the normal connection $\nabla^\perp$, respectively; they all represent the curvature 2-forms of the van der Waerden-Bortolotti connection $\overline{\nabla} = \nabla \oplus \nabla^\perp$. Their last expressions follow from the structure equations $d\omega^I = \omega^J \wedge \omega_J^I$, $d\omega_I^J = \omega_I^K \wedge \omega_K^J$ due to (2). Each of the formulae (2)—(4) can be obtained from the previous one after exterior differentiation and by using the Cartan lemma, if needed.

Denoting $h(X, Y) = h_{ij}^\alpha X^i Y^j e_\alpha$ for $X = e_i X^i$ and $Y = e_j Y^j$, we have the second fundamental form h. Then $h_{ijk}^\alpha = \overline{\nabla}_k h_{ij}^\alpha$ are the components of $\overline{\nabla} h$ and the last equality (3) is the Peterson—Codazzi equation.

3. A submanifold M^m in E^n is said to be symmetric [3] if its second fundamental form h is parallel, i. e. if $\overline{\nabla} h = 0$ or if $h_{ijk}^\alpha = 0$ in (3). Intrinsically it is a locally symmetric space, more exactly, a symmetric R-space, and its immersion in E^n is a standard one [3]. From (4) it follows that in the case of symmetricity we have

$$h_{kj}^\alpha \Omega_i^k + h_{ik}^\alpha \Omega_j^k - h_{ij}^\beta \Omega_\beta^\alpha = 0. \tag{7}$$

This is the component form of $\overline{R}(X, Y) h = 0$ and is, as we see, a system of cubic equations on the components of h. In general, a submanifold M^m in E^n, satisfying (7), is said to be *semi-symmetric*.

4. Let $m_1 = \dim N_x M^m$ be constant on M^m; of course $m_1 \leqslant \frac{1}{2} m(m+1)$. If $n > m + m_1$, the moving orthonormal frame $\{x; e_{m+1}, \ldots, e_{m+m_1}; e_{m+m_1+1}, \ldots, e_n\}$ can be adapted so that $e_\rho \in N_x M^m$, $e_\xi \in N_x^\perp M^m$; $\rho, \sigma, \ldots = m+1, \ldots m+m_1$; $\xi, \eta, \ldots = m+m_1+1, \ldots, n$. Then $h_{ij}^\xi = 0$ and the system of relations $h_{ij} = h_{ij}^\rho e_\rho$ is invertible. From (3) it follows that $h_{ij}^\rho \omega_\rho^\xi = h_{ijk}^\xi \omega^k$, and this gives $\omega_\rho^\xi = \chi_{\rho k}^\xi \omega^k$. Now

$$\tilde{\Omega}^{\sigma}_{\rho} := d\omega^{\sigma}_{\rho} - \omega^{\tau}_{\rho} \wedge \omega^{\sigma}_{\tau} = -\Big(\sum_i h^{\rho}_{i[k} h^{\sigma}_{l]i} + \sum_{\xi}^{\beta} \chi^{\xi}_{\rho[k} \chi^{\xi}_{|\sigma|l]}\Big) \omega^k \wedge \omega^l. \quad (8)$$

Denoting $NM^m = \bigcup_{x \in M^m} N_x M^m$, we have the first normal vector bundle $NM^m \to M^m$ with fibre $N_x M^m$ on every $x \in M^m$. The fact that 2-forms $\tilde{\Omega}^{\sigma}_{\rho}$ are, according to (8), semi-basic, shows that there is a connection in this bundle with connection forms ω^{σ}_{ρ} (see [14], Ch. II, § 1). This connection will be denoted ∇^N and called the *first normal connection.* In the first osculating bundle $\bigcup_{x \in M^m} T_x M^m \oplus N_x M^m \to M^m$ we have the connection $\overline{\nabla}^1 = \nabla \oplus \nabla^N$. If

$$\overline{\nabla}^1 h^{\rho}_{ij} := dh^{\rho}_{ij} - h^{\rho}_{kj} \omega^k_i - h^{\rho}_{ik} \omega^k_j + h^{\sigma}_{ij} \omega^{\rho}_{\sigma} = 0,$$

in short $\overline{\nabla}^1 h = 0$, then h is said to be *first-order parallel.*

Lemma. *The parallelism of the second fundamental form h (i. e. $\overline{\nabla} h = 0$) yields its first-order parallelism (i. e. $\overline{\nabla}^1 h = 0$). The converse is true iff M^m lies in its first osculating space, i. e. iff $T_x M^m \oplus N_x M^m$ is independent from $x \in M^m$.*

In fact, the last assertion is equivalent to $\omega^{\xi}_{\rho} = 0$, and so is also the converse of the first assertion.

5. Proposition. *If a submanifold M^m in E^n $(m \geqslant 3,\ n \geqslant \frac{1}{2} m(m+3))$ is semi-symmetric and its first normal space $N_x M^m$ has maximal dimension $\frac{1}{2} m(m+1)$ in every point $x \in M^m$, then M^m is intrinsically a space of positive constant curvature and, as a submanifold, it has the first-order parallel second fundamental form.*

Proof. Denoting $\sum_{\rho} h^{\rho}_{ij} h^{\rho}_{kl} = B_{ij,kl}$, we can write the first assumption, i.e. (7), in the form

$$\sum_k (h_{kj} B_{i[p,q]k} + h_{ik} B_{j[p,q]k} - B_{ij,k[p} h_{q]k}) = 0, \quad (9)$$

where $h_{ij} = h^{\alpha}_{ij} e_{\alpha} = h^{\rho}_{ij} e_{\rho}$. The second assumption says that these vectors h_{ij} are linearly independent and thus from (9) we have

$$B_{ij,kl} = \varkappa^2 (2\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (10)$$

In fact, it follovs from (9) that for every three distinct values a, b and c

$$B_{aa,aa} = 2B_{aa,bb} = 4B_{ab,ab} = 4\varkappa^2, \quad B_{aa,ab} = B_{aa,bc} = B_{ab,ac} = 0; \quad (11)$$

if $m \geqslant 4$, then also $B_{ab,cd} = 0$ for distinct a, b, c, d.

Due to (5) and (10)

$$\Omega^j_i = -\varkappa^2 \omega^i \wedge \omega^j. \quad (12)$$

The Schur theorem gives now $\varkappa = \mathrm{const}$ and so the first assertion of the Proposition is proved.

Consequently, $\sum_{\rho} h^{\rho}_{ij} h^{\rho}_{kl} = \mathrm{const}$ and after differentiation, using (3) by $h^{\xi}_{ij} = 0$, we get

$$B_{pj,kl}\omega^p_i + B_{ip,kl}\omega^p_j + B_{ij,pl}\omega^p_k + B_{ij,kp}\omega^p_l + F_{ij,klp}\omega^p + F_{kl,ijp}\omega^p = 0,$$

where $F_{ij,klp} = \sum h^{\rho}_{ij} h^{\rho}_{klp}$ are symmetric with respect to indices in the first pair and also in the second triplet. After substituting here (10), we get $F_{ij,klp}=-F_{kl,ijp}$ and, using this symmetry, we have $F_{ij,klp}=-F_{kl,ipj}=F_{ip,klj}$, so that every index in the first pair can be exchanged by every index in the second triplet. Thus $F_{ij,klp}=F_{kl,ijp}$ and this, together with $F_{ij,klp}=-F_{kl,ijp}$, gives $F_{ij,klp}=0$. It follows that $h^{\rho}_{klp}=0$. This is equivalent to the second assertion of the Proposition.

6. Now, the first two assertions of the Theorem in case $m\geqslant 3$ follow from the Proposition and Lemma.

In case $m=2$ when (12) does not give $\varkappa=\text{const}$, it can be deduced independently from the assumption of the Theorem. Here (11) shows that we can take an orthonormal frame $\{x; e_3, e_4, e_5\}$ in every $T^{\perp}_{x}M^2=N_xM^2$, so that

$$h_{11}=\varkappa(\sqrt{3}\,e_3+e_4),\quad h_{22}=\varkappa(\sqrt{3}\,e_3-e_4),\quad h_{12}=\varkappa e_5.$$

Then, in $de_i=e_j\omega_i^j+e_\alpha\omega_i^\alpha$ we have (cf. [4])

$$\omega_1^3=\varkappa\sqrt{3}\,\omega^1,\quad \omega_1^4=\varkappa\omega^1,\quad \omega_1^5=\varkappa\omega^2,$$
$$\omega_2^3=\varkappa\sqrt{3}\,\omega^2,\quad \omega_2^4=-\varkappa\omega^2,\quad \omega_2^5=\varkappa\omega^1.$$

The differential prolongation gives that

$$-\frac{1}{2}d\ln\varkappa=A\omega^1+B\omega^2,\quad \frac{1}{5}(2\omega_1^2-\omega_4^5)=-B\omega^1+A\omega^2,$$
$$\frac{1}{\sqrt{3}}\omega_3^4=A\omega^1-B\omega^2,\qquad \frac{1}{\sqrt{3}}\omega_3^5=B\omega^1+A\omega^2$$

and then

$$dA=B\omega_1^2+\frac{1}{5}(14B^2-11A^2)\omega^1-5AB\omega^2,$$
$$dB=-A\omega_1^2-5AB\omega^1+\frac{1}{5}(14A^2-11B^2)\omega^2.$$

Now the exterior differentiation yields

$$A\left[\varkappa^2+\frac{42}{25}(A^2+B^2)\right]=B\left[\varkappa^2+\frac{42}{25}(A^2+B^2)\right]=0,$$

thus $A=B=0$ and $\varkappa=\text{const}$.

Repeating, by assumptions of the Theorem, the last part in the proof of the Proposition we get in case $m=2$, too, the parallelism of h, i.e. the symmetricity of M^m.

7. It remains to prove the last assertion of the Theorem. The submanifold M^m considered above lies in the hypersphere S^{n-1}, $n=\frac{1}{2}m(m+3)$, because the point c with the radius vector

$$c=x+\frac{m}{2\varkappa^2(m+1)}H$$

is fixed, i.e. $dc=0$, where $H=\frac{1}{m}\sum_i h_{ii}$ is the mean curvature vector, and $\|H\|=\varkappa=\text{const}$ due to (11). Every inner motion of the positive constant curvature space M^m, which is determined infinitesimally by ω^i and ω_i^j, is a rotation in E^n about c. In fact, by this infinitesimal displacement we have $dx=e_i\omega^i$, $de_i=e_j\omega^j_i+h_{ij}\omega^j$, $\omega^j_i+\omega^i_j=0$, $dh_{ij}=-\sum_k B_{ij,kl}e_k\omega^l+$ $+h_{kj}\omega^k_i+h_{ik}\omega^k_j$, where (10) holds, and then $\langle e_i, e_j\rangle$, $\langle e_i, h_{kl}\rangle$ and $\langle h_{ij}, h_{kl}\rangle$ conserve their values, respectively δ_{ij}, 0 and $B_{ij,kl}$, as is easy to see by differentiation.

In the case of completeness of M^m the universal covering group for all these inner motions is $O(m+1)$, which acts on E^n by rotations with centre c, and M^m is its orbit in $S^{n-1}\subset E^n$. So the Theorem is proved.

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CHANGE OF THE ORDER OF FERROELECTRIC PHASE TRANSITION UNDER HYDROSTATIC PRESSURE

P. KONSIN, T. ÖRD. SENJETTELEKTRILISE FAASISIIRDE LIIGI MUUTUMINE HÜDROSTAATILISE RÕHU TOIMEL

П. КОНСИН, Т. ЭРД. ИЗМЕНЕНИЕ РОДА СЕГНЕТОЭЛЕКТРИЧЕСКОГО ФАЗОВОГО ПЕРЕХОДА С ГИДРОСТАТИЧЕСКИМ ДАВЛЕНИЕМ

(Presented by V. Hizhnyakov)

In this paper, the possibilities of changing the order of ferroelectric phase transition under high hydrostatic pressure have been investigated on the phenomenological level. Such effect or a corresponding tendency has been found, e. g., in the monocrystals $BaTiO_3$ [1–3], SbSI [4–7], KDP [8–11], $Sn_2P_2S_6$ [12], TGSe [13–15].

Let us start from the expansion of the free energy in powers of the order parameter y and the strain ε for a liquid-like model of a ferroelectric in which, besides the standard terms, the terms with the coefficients h, f and w have been taken into account:

$$F=\alpha(T)y^2+\beta y^4+\gamma y^6+\frac{1}{2}c\varepsilon^2+g\varepsilon y^2+\frac{1}{2}h\varepsilon^2y^2+f\varepsilon y^4+\frac{1}{3}w\varepsilon^3. \quad (1)$$

Here c and g are the elastic and electrostrictive constants.