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ON SELF-DUAL BOSONIC MEMBRANES IN THE 3-SPACE

(Presented by V. Hizhnyakov)

In the paper, the self-duality equations for the closed bosonic membranes imbedded in the three-dimensional Euclidean space-time [1] are studied. The symmetry and some classes of solutions of the given system are investigated. The connection of the considered equations with self-duality conditions for a two-dimensional σ-model is shown.

1. Introduction

The self-duality equations for relativistic closed bosonic membranes were introduced in [1] (see also [2]). The action for the membrane embedded in the D-dimensional Euclidean space [3]

$$S \sim \int \{|\det g_{ab}|\}^{1/2} d^3\sigma, \tag{1}$$

where the induced metric

$$g_{ab}=\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b}g_{\mu\nu}\to\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\mu}{\partial\sigma^b}\equiv x^\mu_{,a}\,x^\mu_{,b}, \tag{2}$$

$$a, b=1, 2, 3, \qquad \nu, \mu=1, \ldots, D.$$

σ^a are the parameters of the swept-out surface; two parameters from σ^a (e. g. σ^1, σ^2) parametrize the two-dimensional membrane surface, and the third one ($\sigma^3\equiv\tau$) is the «time» variable.

The self-duality condition [1] is the following:

$$\frac{\delta L}{\delta x^i_{,a}}=\frac{1}{2}\varepsilon^{abc}f_{ijk}x^j_{,b}\,x^k_{,c}, \tag{3}$$

where f_{ijk} are the structure constants of the $O(n)$ group;

$$i, j, k=1, \ldots, n(n-1)/2, \qquad D=n(n-1)/2.$$

In the case of $D=n=3$ ($f_{ijk}=\varepsilon_{ijk}$), the solutions of equations (3) automatically satisfy also the equations of motion as well as the constraints for the relativistic membrane. At the same time (analogously to the situation with the self-dual Yang-Mills field) the action S is proportional to the value of the topological charge

$$Q(x)=\int d^3\sigma\frac{1}{3!}\varepsilon^{abc}f_{ijk}x^i_{,a}\,x^j_{,b}\,x^k_{,c}.$$

However, even in the simple case $D=n=3$ self-duality equations (3) are rather complicated. By choosing the suitable gauge condition

$$g_{ab}\equiv x^i_{,a}\,x^i_{,b}=0, \qquad a\neq b \tag{4}$$

(i.e. by choosing the isothermal coordinates), the self-duality equations become drastically simplified:

$$x^i_{,3}\sqrt{g_{11}g_{22}}=\sqrt{g_{33}}\,\varepsilon^{ijk}x^j_{,1}\,x^k_{,2}. \tag{5}$$

2. The method of finding solutions from the invariance group

Using the standard Lie algorithm (e.g. [4], [5]), the Lie point symmetry group (the group of point transformations) of the system of equations (4)—(5) can be obtained. The operators of the corresponding Lie algebra are

$$X_f = f^a(\sigma^a)\frac{\partial}{\partial \sigma^a}, \quad \forall\, a=1,2,3, \tag{6a}$$

(where f^a are arbitrary functions),

$$X_i = \frac{\partial}{\partial x^i},$$

$$X_d = x^i \frac{\partial}{\partial x^i},$$

$$X_{kl} = x^k \frac{\partial}{\partial x^l} - x^l \frac{\partial}{\partial x^k}, \qquad i,k,l=1,2,3, \tag{6b}$$

$$X_{c_i} = 2x^i x^k \frac{\partial}{\partial x^k} - \bar{x}^2 \frac{\partial}{\partial x^i}, \quad \bar{x}^2 \equiv x^k x^k.$$

Thus, the initial reparametrization invariance is broken up to arbitrary dilatations of the parameters σ^a (owing to «gauge» condition (4)). The existence of subgroup (6a) means that the solutions of the system (4)—(5) contain the arbitrary functions of the parameters σ^a. Subalgebra (6b) corresponds to the conformal group of transformations of the three-dimensional Euclidean space: $C(3)$.

Passing to the finding of the solutions for system (4)—(5) from its invariance group (6b), let us note that the application of the usual scheme for the construction of the invariant solutions (e.g. [4], [5]) encounters here some difficulties. Indeed, the generators of the group are defined in the space of functions (x-space). Therefore, the invariant solutions of ranks 2 and 1 (there is no point in considering the other invariant solutions) are, respectively, as follows:

$$\sigma^a = \sigma^a(f_1(x), f_2(x))$$

or

$$\sigma^a = \sigma^a(f(x)).$$

However, such solutions (even in the simplest case of the invariant solution of rank 1) are rather difficult to find, owing to the fact that the hodograph transformation ($x(\sigma) \to \sigma(x)$) complicates essentially the form of the initial system. Besides, it is not quite clear how the given solution could be used (even if one has its explicit form). The physical quantities (e. g. the action) are expressed through the derivatives $\partial x^k/\partial \sigma^a$, which are impossible to express in the terms of $\partial \sigma^a/\partial x^k$ since here the Jacobian of the hodograph transformation is equal to zero. Therefore, we apply another method to construct exact solutions of system (4)—(5) on the basis of its symmetry group (6b).

The usual notion of the invariant solution under the r-parameter transformation group, H, is connected with the invariant manifold of the group H (H-is a subgroup of a symmetry group of the considered system).

$$\Phi:\ \Phi^k(\sigma^a, x^i) = 0, \tag{7}$$

$$a=1,\ldots,n, \quad i,k=1,\ldots,m,$$

σ^a and x^i are arguments and functions, respectively, where by

$$X_\alpha \Phi^k(\sigma, x)\,|_\Phi = 0, \tag{8}$$

$$\alpha = 1, \ldots, r, \quad k = 1, \ldots, m$$

for all generators X_α of the group H.

$$X_\alpha = \xi^a_\alpha(\sigma, x)\,\partial/\partial\sigma^a + \eta^i_\alpha(\sigma, x)\,\partial/\partial x^i. \tag{9}$$

In fact, Φ^k depend on the invariants of the group H

$$\Phi^k = \Phi^k(I^1, \ldots, I^t), \quad k = 1, \ldots, m,$$
$$X_\alpha I^\tau = 0, \quad \tau = 1, \ldots, t, \quad t = n + m - R, \tag{10}$$
$$R = \operatorname{rank} \|\xi^a_\alpha, \eta^i_\alpha\|, \quad (R < n + m).$$

The case when

$$\operatorname{rank} \left\| \frac{\partial I^\tau}{\partial x^i} \right\| = m, \tag{11}$$

$$i = 1, \ldots, m, \quad \tau = 1, \ldots, t$$

i. e. the form of the invariant manifold allows us to express all the sought quantities x^i as functions of the independent variables σ^a, leads to the H-invariant solution. The case when equation (11) is not satisfied ($\operatorname{rank} \|\partial I^\tau/\partial x^i\| < m$) leads to a partially invariant solution (for rigorous definitions and details, see [4], [5]).

In the present paper, we construct the manifold that will be called «semi-invariant». Let us represent a group H as a direct sum of two of its subgroups (if possible)

$$H = H_1 \oplus H_2.$$

For the generators we have respectively

$$X_\alpha = (X_a, X_b)$$

$$\alpha = 1, \ldots, r, \quad \begin{matrix} a = 1, \ldots, r_1, \\ b = r_1 + 1, \ldots, r. \end{matrix}$$

We call the obtained manifold Φ-semi-invariant if

$$\Phi = \Phi_1 \cup \Phi_2,$$

where Φ_i is the H_i-invariant manifold, i. e. (symbolically)

$$X_a \Phi_1\,|_\Phi = 0, \quad X_b \Phi_2\,|_\Phi = 0. \tag{12}$$

We call the obtained manifold Φ-semi-invariant if

$$X_a X_b \Phi_1\,|_\Phi = X_b X_a \Phi_2\,|_\Phi = 0. \tag{13}$$

It is obvious that to satisfy conditions (13) it suffices to satisfy

$$[X_a, X_b] = 0. \tag{14}$$

If, besides, the subgroup $H_1(H_2)$ is an Abelian one, then it is the center of the group H.

Additionally we demand that the system of equations defining the semi-invariant manifold should allow one to express all the functions x^i through the arguments σ^a:

$$\operatorname{rank} \|\partial I^\tau/\partial x^i\| = m,$$
$$\tau = 1, \ldots, t, \quad i = 1, \ldots, m, \tag{15}$$
$$t \leqslant t_1 + t_2,$$

where t is the number of all (various) invariants.

3. Semi-invariant solutions of the equations for self-dual membranes

To construct the semi-invariant solutions of system (4)—(5) with respect to the subgroups of conformal group (6b), we choose the one-parameter subgroup H_1 (with the generator X_a) and the two-(or more) parameter subgroup H_2 (with generators X_b). (Only the number t of the invariants I^τ of the group H_2 depending on x^j (in the given case $t=1$) is important:

$$I^\tau, \tau=1, \ldots, t, \ t=m-R, \ R=\operatorname{rank}\|\eta^i_\alpha\|.$$

The generators of the invariance group (6b) have the form

$$X_\alpha=\eta^i_\alpha(x)\frac{\partial}{\partial x^i}).$$

We demand that

$$[X_a, X_b]=0.$$

Thus, the group H_1 is the center of the group H.

1. Let

$$\begin{gathered} X_a=X_{12}, \\ X_b=X_d, X_{c_3}. \\ [X_{12}, X_d]=[X_{12}, X_{c_3}]=0, \\ [X_d, X_{c_3}]=X_{c_3}. \end{gathered} \tag{16}$$

The invariants of the group H_1 (depending on x) are

$$x^{1^2}+x^{2^2}, \ x^3,$$

and the invariant of the group H_2 is x^1/x^2.

$$\begin{gathered} X_d I^u=0 \Rightarrow I^u=x^1/x^2, \quad x^3/x^2, \\ X_{c_3} I^v=0 \Rightarrow I^v=x^1/x^2, \quad \bar{x}^2/x^2, \\ (\bar{x}^2\equiv x^i x^i). \end{gathered}$$

Thus, we have

$$\begin{aligned} \Phi_2: &\quad \sigma^1=\sigma^1(x^1/x^2), \\ \Phi_1: &\quad \begin{aligned} \sigma^2&=\sigma^2(x^{1^2}+x^{2^2}, x^3), \\ \sigma^3&=\sigma^3(x^{1^2}+x^{2^2}, x^3). \end{aligned} \end{aligned} \tag{17}$$

The sequence of the indices of σ^a is of no importance, since the initial system of equations (4)—(5) is symmetric under the change $\{\sigma^\alpha\}\to\{\sigma^b\}$. From equations (17) we get the form of the $(X_{12}; X_d, X_{c_3})$-semi-invariant solution

$$\begin{aligned} x^1&=\cos m(\sigma^1) f(\sigma^2, \sigma^3), \\ x^2&=\sin m(\sigma^1) f(\sigma^2, \sigma^3), \\ x^3&=g(\sigma^2, \sigma^3). \end{aligned} \tag{18}$$

Inserting equations (18) into system (4)—(5), we get the only equation for the functions $f(\sigma^2, \sigma^3)$, $g(\sigma^2, \sigma^3)$:

$$f_{,2}f_{,3}+g_{,2}g_{,3}=0. \tag{19}$$

Consider some special cases of the solutions of the equation (19).

$$f=R(\sigma^3), \qquad g=\Psi(\sigma^2) \tag{i}$$

and we obtain a cylindrical membrane [3]

$$x^1=R(\sigma^3)\cos m(\sigma^1), \quad x^2=R(\sigma^3)\sin m(\sigma^1), \quad x^3=\Psi(\sigma^2). \tag{20}$$

$$f=r_1+R(\sigma^3)\Phi(\sigma^2), \tag{ii}$$

$$g=r_2+\sqrt{(\Phi^2(\sigma^2)+a)(b-R^2(\sigma^3))}, \tag{21}$$

where a, b, r_1, r_2 are arbitrary constants. The class of the solutions (21) in the case of

$$a=-1, \quad b=0$$

leads to two solutions obtained in [1]: a sphere $(r_1=r_2=0)$

$$x^1=\cos m(\sigma^1)\Phi(\sigma^2)R(\sigma^3), \quad x^2=\sin m(\sigma^1)\Phi(\sigma^2)R(\sigma^3), \quad x^3=\sqrt{1-\Phi^2(\sigma^2)}\,R(\sigma^3) \tag{22}$$

and a torus $(r_1\neq 0, r_2=0)$

$$x^1=\cos m(\sigma^1)(r_1+\Phi(\sigma^2)R(\sigma^3)), \quad x^2=\sin m(\sigma^1)(r_1+\Phi(\sigma^2)R(\sigma^3)), \quad x^3=\sqrt{1-\Phi^2(\sigma^2)}\,R(\sigma^3), \tag{23}$$

$$g(\sigma^2,\sigma^3)=\pm if(\sigma^2,\sigma^3). \tag{iii}$$

$$g=g_1(\sigma^2+i\lambda\sigma^3)+g_2(\sigma^2-i\lambda\sigma^3), \tag{iiii}$$

$$f=\pm i[g_1(\sigma^2+i\lambda\sigma^3)-g_2(\sigma^2-i\lambda\sigma^3)]. \tag{24}$$

2. Let

$$X_a=X_d, \quad X_b=X_{kl}, \quad k,l=1,2,3, \quad [X_a,X_b]=0, \quad \operatorname{rank}\|\eta^i_\alpha\|=2. \tag{25}$$

Then

$$\Phi_1:\ \sigma^1=\sigma^1(x^1/x^3, x^2/x^3), \quad \sigma^2=\sigma^2(x^1/x^3, x^2/x^3), \tag{26}$$

$$\Phi_2:\ \sigma^3=\sigma^3(\bar{x}^2).$$

Thus, the form of the $(X_d;\, X_{kl})$-semi-invariant solution is

$$x^a=f(\sigma^3)F^a(\sigma^1,\sigma^2), \quad F^aF^a=1, \quad a=1,2,3. \tag{27}$$

Putting equations (27) into system (4)—(5), we obtain

$$F^a_{,1}F^a_{,2}=0, \tag{28a}$$

$$F^a_{,1}\sqrt{F^{\bar a}_{,2}F^{\bar a}_{,2}}=-\sqrt{F^{\bar a}_{,1}F^{\bar a}_{,1}}\,\varepsilon^{abc}F^bF^c_{,2},$$

$$F^a_{,2}\sqrt{F^{\bar a}_{,1}F^{\bar a}_{,1}}=\sqrt{F^{\bar a}_{,2}F^{\bar a}_{,2}}\,\varepsilon^{abc}F^bF^c_{,1}. \tag{28b}$$

If

$$F^a_{,1}F^a_{,1}=F^a_{,2}F^a_{,2}\,, \tag{29}$$

then (28b) reduces to

$$F^a_{,\mu}=-\varepsilon_{\mu\nu}\varepsilon^{abc}F^bF^c_{,\nu}\,. \tag{30}$$

Equations (30) (with (28a) and (29)) coincide with the well-known (anti-) self-duality equations for the two-dimensional σ-model [6]. The corresponding solutions look as follows:

$$\frac{F^1+iF^2}{1-F^3}=f(\sigma^1-i\sigma^2). \tag{31}$$

(f is arbitrary function).

Thus, the self-duality equations for closed bosonic membranes in the three-dimensional space lead to self-dual equations for the two-dimensional σ-model. Concerning the connection between the membranes and the σ-models of arbitrary dimensions, see [7].

3. Let

$$\begin{aligned} X_a&=X_{c_3},\\ X_b&=X_{c_1},X_{c_2}. \end{aligned} \tag{32}$$

The subgroup of special conformal transformations is an Abelian one

$$[X_{c_i},X_{c_j}]=0.$$

We have:

$$\begin{aligned} \Phi_1:\quad &\sigma^1=\varphi(x^1/x^2,\ \bar{x}^2/x^1),\\ &\sigma^2=\Phi(x^1/x^2,\ \bar{x}^2/x^1),\\ \Phi_2:\quad &\sigma^3=\Psi(\bar{x}^2/x^3). \end{aligned} \tag{33}$$

(The only invariant of the group H_2: $\{X_{c_1},X_{c_2}\}$ depending on x, is $\bar{x}^2/x^3$). The corresponding semi-invariant solution has the form

$$\begin{aligned} x^1&=f(\sigma^1,\sigma^2)/[f^2(\sigma^1,\sigma^2)+g^2(\sigma^1,\sigma^2)+m^2(\sigma^3)],\\ x^2&=g(\sigma^1,\sigma^2)/[f^2(\sigma^1,\sigma^2)+g^2(\sigma^1,\sigma^2)+m^2(\sigma^3)],\\ x^3&=m(\sigma^3)(\sigma^3)/[f^2(\sigma^1,\sigma^2)+g^2(\sigma^1,\sigma^2)+m^2(\sigma^3)]. \end{aligned} \tag{34}$$

Putting (34) into system (4)—(5), we get

$$f_{,1}f_{,2}+g_{,1}g_{,2}=0. \tag{35}$$

Equation (35) coincides with equation (19), which enables us to write down the corresponding solutions.

4. Concluding remarks

In the present paper we have found some classes of solutions of self-duality equations for closed bosonic membranes in a three-dimensional space from the invariance group of the system. We hope that the given solutions turn out to be useful in investigating more important multidimensional cases. Note also that the solutions of the considered model are classified not by their topological numbers (due to the arbitrariness of all the functions of the parameters in the solution) but by the form of the corresponding metric tensor g_{mn}.

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OMADUAALSETEST BOSONMEMBRAANIDEST KOLMEMÕÕTMELISES RUUMIS

Artikkel käsitleb kolmemõõtmelisse eukleidilisse ruumi sisestatud bosonmembraanide omaduaalsuse võrrandeid. On uuritud vaadeldava süsteemi sümmeetriat ja mõningaid lahendite klasse ning näidatud seost süsteemi ja kahedimensioonilise σ-mudeli omaduaalsuse võrrandite vahel.

В. РОЗЕНГАУЗ

О САМОДУАЛЬНЫХ БОЗОННЫХ МЕМБРАНАХ В ТРЕХМЕРНОМ ПРОСТРАНСТВЕ

Рассматриваются уравнения самодуальности для замкнутых бозонных мембран, вложенных в трехмерное евклидово пространство. Изучается симметрия и некоторые классы решений данной системы. Показана связь с уравнениями самодуальности для двумерной σ-модели.