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ALGEBRAS OF INVARIANT CONNECTIONS ON
REDUCTIVE SPACES

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A. ФЛЯЙШЕР. АЛГЕБРЫ ИНВАРИАНТНЫХ СВЯЗНОСТЕЙ НА РЕДУКТИВНЫХ ПРОСТРАНСТВАХ

(Presented by G. Vainikko)

Let $M=G/H$ be a reductive homogeneous space* with fixed Lie algebra decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. For $X, Y \in \mathfrak{m}$ let $[X, Y]=X \circ Y + \mathfrak{h}(X, Y)$ where $X \circ Y = [X, Y]_{\mathfrak{m}}$ and $\mathfrak{h}(X, Y) = [X, Y]_{\mathfrak{h}}$ are the projections of $[X, Y]$

into \mathfrak{m} and \mathfrak{h} , respectively. By A. Sagle [1] \mathfrak{m} with multiplication $X \circ Y$ becomes a nonassociative anticommutative algebra denoted by (\mathfrak{m}, \circ) . The structure of the algebra (\mathfrak{m}, \circ) is related to the geometry of M as follows

Theorem 1. (Sagle [2]) *Let $M=G/H$ be a reductive homogeneous space with a natural torsion-free connection and fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. If M is holonomy irreducible, then (\mathfrak{m}, \circ) is simple. Conversely, if M is pseudo-Riemannian and (\mathfrak{m}, \circ) is simple, then M is holonomy irreducible.*

Of special interest is the dependence between the structure of the basic group G and the structure of (\mathfrak{m}, \circ) . In the Riemannian case we have here the following

Theorem 2. *Let $M=G/H$ be a Riemannian nonsymmetric homogeneous space. Then the simplicity of G implies the simplicity of (\mathfrak{m}, \circ) where \mathfrak{m} is the orthogonal complement of \mathfrak{h} with respect to the Killing form of \mathfrak{g} .*

From Theorems 1 and 2 follows a well-known

Theorem 3. (Kostant [3]) *Let $M=G/H$ be a Riemannian naturally reductive homogeneous space. If G is simple, then M is holonomy irreducible.*

Now we define a class of reductive homogeneous spaces for which the analogous result holds in the pseudo-Riemannian case.

Definition. *A reductive homogeneous space G/H with fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ is called special reductive if the algebra (\mathfrak{m}, \circ) contains no ideals with zero multiplication.*

Example. *Let G be a semisimple connected Lie group and H be its semisimple closed subgroup. Then there is a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where $\mathfrak{m}=\mathfrak{h}^\perp$ relative to the Killing form K of \mathfrak{g} and $M=G/H$ is a reductive space [4]. If the restriction $K_{\mathfrak{m}}$ is equal to the Killing form of the algebra (\mathfrak{m}, \circ) , then M is special reductive.*

* Throughout this paper all the homogeneous spaces are supposed to be simply connected.

Theorem 4. For every Riemannian reductive homogeneous space $M=G/H$ holds the decomposition $M=M_1 \times M_2$, where M_1 is a symmetric space and M_2 is a special reductive homogeneous space.

Theorem 5. Let $M=G/H$ be a homogeneous space and the Lie algebra \mathfrak{g} admit an $\text{ad}(G)$ -invariant non-degenerate symmetric bilinear form B such that its restriction $B|_{\mathfrak{h}}$ to \mathfrak{h} is non-degenerate. Then

(1) The homogeneous space M is naturally reductive with respect to the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ defined by

$$\mathfrak{m} = \{X \in \mathfrak{g}; B(X, Y) = 0 \text{ for all } Y \in \mathfrak{h}\} \quad (\text{see [5], p. 203}).$$

(2) If M is special reductive, then M is either holonomy irreducible or there holds a direct sum decomposition

$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \dots + \mathfrak{m}_r,$$

where all \mathfrak{m}_i are mutually orthogonal and holonomy irreducible, moreover

- (a) $[\mathfrak{h}, \mathfrak{m}_i] \subset \mathfrak{m}_i$;
- (b) $\mathfrak{m}_i \circ \mathfrak{m}_i \subset \mathfrak{m}_i$;
- (c) $[\mathfrak{m}_i, \mathfrak{m}_j] = 0, i \neq j$;
- (d) $\mathfrak{g}_i = \mathfrak{m}_i + \mathfrak{h}(\mathfrak{m}_i, \mathfrak{m}_i)$ are ideals in \mathfrak{g} for $i, j = 0, 1, \dots, r$.

Corollary. Let $M=G/H$ be a special naturally reductive homogeneous space. If G is simple, then M is holonomy irreducible.

The algebra (\mathfrak{m}, \circ) is not the unique algebra connected with $M=G/H$. In [6] a correspondence was established between G -invariant connections on M with fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and certain nonassociative algebras (\mathfrak{m}, α) , where α is the bilinear multiplication function on \mathfrak{m} . In the case when this connection is the natural torsion-free connection we have $\alpha(X, Y) = \frac{1}{2} X \circ Y$ and therefore the algebra (\mathfrak{m}, \circ) can be called algebra of natural torsion-free connection. Similarly as above we give here some results about the connection between the structure of G and (\mathfrak{m}, α) .

Theorem 6. Let G/H be a Riemannian reductive nonsymmetric homogeneous space with decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and the Riemannian connection induced by (\mathfrak{m}, α) . Then the simplicity of G implies the simplicity of (\mathfrak{m}, α) .

Theorem 7. Let G/H be a special reductive homogeneous space with decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and the pseudo-Riemannian connection induced by (\mathfrak{m}, α) . Then the simplicity of G implies the simplicity of (\mathfrak{m}, α) .

In assumption that all the connections on G/H are induced by pseudo-Riemannian metrics we receive

Theorem 8. Every invariant pseudo-Riemannian connection on reductive isotropy irreducible homogeneous space is the natural torsion-free connection.

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