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ALGEBRAS OF INVARIANT CONNECTIONS ON REDUCTIVE SPACES

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А. ФЛЯЙШЕР. АЛГЕБРЫ ИНВАРИАНТНЫХ СВЯЗНОСТЕЙ НА РЕДУКТИВНЫХ ПРОСТРАНСТВАХ

(*Presented by G. Vainikko*)

Let $M=G/H$ be a reductive homogeneous space* with fixed Lie algebra decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. For $X, Y\in\mathfrak{m}$ let $[X, Y]=X\circ Y+\mathfrak{h}(X, Y)$ where $X\circ Y=[X, Y]_{\mathfrak{m}}$ and $\mathfrak{h}(X, Y)=[X, Y]_{\mathfrak{h}}$ are the projections of $[X, Y]$ into $\mathfrak{m}$ and $\mathfrak{h}$, respectively. By A. Sagle [1] $\mathfrak{m}$ with multiplication $X\circ Y$ becomes a nonassociative anticommutative algebra denoted by $(\mathfrak{m},\circ)$. The structure of the algebra $(\mathfrak{m},\circ)$ is related to the geometry of M as follows

T h e o r e m 1. (Sagle [2]) *Let $M=G/H$ be a reductive homogeneous space with a natural torsion-free connection and fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. If M is holonomy irreducible, then $(\mathfrak{m},\circ)$ is simple. Conversely, if M is pseudo-Riemannian and $(\mathfrak{m},\circ)$ is simple, then M is holonomy irreducible.*

Of special interest is the dependence between the structure of the basic group G and the structure of $(\mathfrak{m},\circ)$. In the Riemannian case we have here the following

T h e o r e m 2. *Let $M=G/H$ be a Riemannian nonsymmetric homogeneous space. Then the simplicity of G implies the simplicity of $(\mathfrak{m},\circ)$ where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{g}$.*

From Theorems 1 and 2 follows a well-known

T h e o r e m 3. (Kostant [3]) *Let $M=G/H$ be a Riemannian naturally reductive homogeneous space. If G is simple, then M is holonomy irreducible.*

Now we define a class of reductive homogeneous spaces for which the analogous result holds in the pseudo-Riemannian case.

D e f i n i t i o n. *A reductive homogeneous space G/H with fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ is called special reductive if the algebra $(\mathfrak{m},\circ)$ contains no ideals with zero multiplication.*

E x a m p l e. *Let G be a semisimple connected Lie group and H be its semisimple closed subgroup. Then there is a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, where $\mathfrak{m}=\mathfrak{h}^{\perp}$ relative to the Killing form K of $\mathfrak{g}$ and $M=G/H$ is a reductive space* [4]. *If the restriction $K_{\mathfrak{m}}$ is equal to the Killing form of the algebra $(\mathfrak{m},\circ)$, then M is special reductive.*

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* Throughout this paper all the homogeneous spaces are supposed to be simply connected.

Theorem 4. *For every Riemannian reductive homogeneous space $M=G/H$ holds the decomposition $M=M_1\times M_2$, where M_1 is a symmetric space and M_2 is a special reductive homogeneous space.*

Theorem 5. *Let $M=G/H$ be a homogeneous space and the Lie algebra $\mathfrak{g}$ admit an ad(G)-invariant non-degenerate symmetric bilinear form B such that its restriction $B_{\mathfrak{h}}$ to $\mathfrak{h}$ is non-degenerate. Then*

(1) *The homogeneous space M is naturally reductive with respect to the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ defined by*

$$\mathfrak{m}=\{X\in\mathfrak{g};\ B(X,Y)=0 \text{ for all } Y\in\mathfrak{h}\}$$ *(see* [5], p. 203).

(2) *If M is special reductive, then M is either holonomy irreducible or there holds a direct sum decomposition*

$$\mathfrak{m}=\mathfrak{m}_0+\mathfrak{m}_1+\ldots+\mathfrak{m}_r,$$

where all $\mathfrak{m}_i$ are mutually orthogonal and holonomy irreducible, moreover
(a) $[\mathfrak{h},\mathfrak{m}_i]\subset\mathfrak{m}_i$;
(b) $\mathfrak{m}_i\circ\mathfrak{m}_i\subset\mathfrak{m}_i$;
(c) $[\mathfrak{m}_i,\mathfrak{m}_j]=0,\ i\neq j$;
(d) $\mathfrak{g}_i=\mathfrak{m}_i+\mathfrak{h}(\mathfrak{m}_i,\mathfrak{m}_i)$ *are ideals in* $\mathfrak{g}$
for $i, j=0, 1, \ldots, r$.

Corollary. *Let $M=G/H$ be a special naturally reductive homogeneous space. If G is simple, then M is holonomy irreducible.*

The algebra $(\mathfrak{m},\circ)$ is not the unique algebra connected with $M=G/H$. In [6] a correspondence was established between G-invariant connections on M with fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and certain nonassociative algebras $(\mathfrak{m},\alpha)$, where α is the bilinear multiplication function on $\mathfrak{m}$. In the case when this connection is the natural torsion-free connection we have $\alpha(X,Y)=\frac{1}{2}X\circ Y$ and therefore the algebra $(\mathfrak{m},\circ)$ can be called algebra of natural torsion-free connection. Similarly as above we give here some results about the connection between the structure of G and $(\mathfrak{m},\alpha)$.

Theorem 6. *Let G/H be a Riemannian reductive nonsymmetric homogeneous space with decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and the Riemannian connection induced by $(\mathfrak{m},\alpha)$. Then the simplicity of G implies the simplicity of $(\mathfrak{m},\alpha)$.*

Theorem 7. *Let G/H be a special reductive homogeneous space with decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ and the pseudo-Riemannian connection induced by $(\mathfrak{m},\alpha)$. Then the simplicity of G implies the simplicity of $(\mathfrak{m},\alpha)$.*

In assumption that all the connections on G/H are induced by pseudo-Riemannian metrics we receive

Theorem 8. *Every invariant pseudo-Riemannian connection on reductive isotropy irreducible homogeneous space is the natural torsion-free connection.*

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