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TOTALLY QUASIUMBILICAL SUBMANIFOLDS WITH NONFLAT NORMAL CONNECTION

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Ю. ЛУМИСТЕ, М. ВЯЛЬЯС. ВПОЛНЕ КВАЗИОМБИЛИЧЕСКИЕ ПОДМНОГООБРАЗИЯ С НЕПЛОСКОЙ НОРМАЛЬНОЙ СВЯЗНОСТЬЮ

(*Presented by H. Keres*)

B.-Y. Chen and T. H. Teng have stated (see [1], Theorem 1) that a totally quasiumbilical submanifold in a conformally flat Riemannian space has always flat normal connection. This statement is used in [2], where Theorem 4 (the main theorem) says that an n-dimensional submanifold in an $(n+m)$-dimensional conformally flat space $(m<n-2)$ is totally quasiumbilical if and only if it is a conformally flat submanifold with flat normal connection. Remark that Errata corrige of [2] doesn't concern this main theorem.

In this note we show that the Theorem 1 in [1] and consequently the corresponding part (given above in the spaced type) of the Theorem 4 in [2] are not true. We detect the mistake made in [1] and give a counterexample (see § 1 and the theorem in § 4.)

§ 1. In the «proof» of Theorem 1 in [1] a general formula concerning the Weyl conformal curvature tensor $C_{jj,kl}$ of a submanifold in a conformally flat space is used. It is known (see [3] p. 150) that in this case $C_{ij,kl}=\sum_{\alpha} C^{\alpha}_{ij,kl}$ and thus $\sum_{i,j,k,l} h^{\beta}_{ik} h^{\beta}_{jl} C_{ij,kl}=\sum_{\alpha}\sum_{i,j,k,l} h^{\beta}_{ik} h^{\beta}_{jl} C^{\alpha}_{ij,kl}$. Straightforward computation gives that by $\alpha\neq\beta$ we have

$$(n-1)(n-2)\sum_{i,j,k,l} h^{\beta}_{ik} h^{\beta}_{jl} C^{\alpha}_{ij,kl}=(n-1)(n-2)\left[\left((H^{\beta}H^{\alpha})_{[1]}\right)^2-\right.$$
$$\left.-(H^{\beta}H^{\alpha})_{[2]}\right]+\left[H^{\beta}_{[2]}-(H^{\beta}_{[1]})^2\right]\left[H^{\alpha}_{[2]}-(H^{\alpha}_{[1]})^2\right]+$$
$$+2(n-1)\left[H^{\alpha}_{[1]}(H^{\beta}H^{\beta}H^{\alpha})_{[1]}+H^{\beta}_{[1]}(H^{\beta}H^{\alpha}H^{\alpha})_{[1]}-\right.$$
$$\left.-H^{\beta}_{[1]}H^{\alpha}_{[1]}(H^{\beta}H^{\alpha})_{[1]}-(H^{\beta}H^{\alpha}H^{\alpha}H^{\beta})_{[1]}\right],$$

where $H_{[p]}=\operatorname{trace} H^p$ for a matrix H and $H^{\alpha}=\|h^{\alpha}_{ij}\|$. In [1] the last term is replaced by $(H^{\beta}H^{\alpha})_{[2]}$ and added to corresponding term in the first square bracket. Therefore in [1] the formulas (3.6) and (3.9) are not valid and the proof of the theorem 1 is not correct. Instead of the decision inequality we have an identity.

§ 2. In fact this theorem is not true. In the following we give an example of a totally quasiumbilical submanifold V^n in E^{n+2} with nonflat normal connection. Note that a normal vector $\vec{\xi}$ is called quasiumbilical if $\sum_{\alpha}\xi^{\alpha}h^{\alpha}_{ij}$ has an $(n-1)$-fold eigenvalue and a V^n in E^{n+m} is called totally quasiumbilical if it has m mutually orthogonal quasiumbilical unit normal vector fields (see [3], Chap. 5).

§ 3. Let us start with two lemmas.

Lemma 1. *Let V^2 be a surface with nonflat normal connection in E^4 and $V^n = V^2 \times E^{n-2}$ its product by E^{n-2} in E^{n+2}, $n>2$. Then the unit normal vector of V^n in the point $X \in V^n$, orthogonal to the tangent line from X to the normal curvature ellipse of the component V^2 in the plane of this ellipse, form a quasiumbilical unit vector field on V^n (real if every $X \in V^n$ is not inside this ellipse).*

Proof. We take the orthonormal frame vectors in the point $X \in V^n$ so that $\vec{e}_1$ and $\vec{e}_2$ are tangent to the V^2 and $\vec{e}_3, \ldots, \vec{e}_n$ lie in E^{n-2}. Then (see [4]) $d\vec{e}_u = \omega_u^v \vec{e}_v$; $u, v = 3, \ldots, n$ and in $d\vec{e}_1 = \omega_1^2 \vec{e}_2 + \omega_1^\alpha \vec{e}_\alpha$, $d\vec{e}_2 = -\omega_1^2 \vec{e}_1 + \omega_2^\alpha \vec{e}_\alpha$; $\alpha = n+1, n+2$, we have

$$\omega_1^\alpha = h_{11}^\alpha \omega^1 + h_{12}^\alpha \omega^2, \qquad \omega_2^\alpha = h_{12}^\alpha \omega^1 + h_{22}^\alpha \omega^2.$$

The normal curvature vector in the direction of the unit tangent vector $\vec{t} = t^i \vec{e}_i$; $i = 1, \ldots, n$, is

$$\vec{h}_{ij} t^i t^j = \lambda (\vec{h}_{11} \cos^2 \varphi + 2\vec{h}_{12} \cos \varphi \sin \varphi + \vec{h}_{22} \cos^2 \varphi),$$

where $\vec{h}_{ij} = h_{ij}^{n+1} \vec{e}_{n+1} + h_{ij}^{n+1} \vec{e}_{n+2}$, $\vec{h}_{1u} = \vec{h}_{2v} = \vec{h}_{uv} = 0$, $t^1 = \lambda \cos \varphi$, $t^2 = \lambda \sin \varphi$, $0 \leqslant \lambda \leqslant 1$, $\lambda^2 + (t^3)^2 + \ldots + (t^n)^2 = 1$. Thus the normal curvature indicatrix of the submanifold V^n in the point $X \in V^n$ (see [5], p. 109) is the convex hull of this point X and of the normal curvature ellipse, which corresponds to $\lambda = 1$ and whose point has the radius vector

$$\frac{1}{2} (\vec{h}_{11} + \vec{h}_{22}) + \frac{1}{2} (\vec{h}_{11} - \vec{h}_{22}) \cos 2\varphi + \vec{h}_{12} \sin 2\varphi$$

from the origin X (see [6], p. 253). In the frame with origin in the end-point of mean curvature vector and with basic vectors $\frac{1}{2} (\vec{h}_{11} - \vec{h}_{22})$ and $\vec{h}_{12}$ (if normal curvature is nonflat, they are noncollinear) this ellipse has parametric equations

$$x = \cos 2\varphi, \qquad y = \sin 2\varphi$$

and thus these basic vectors have conjugated directions with respect to the ellipse. If the point $X \in V^n$ is not in the interior of this ellipse, then the frame vector $\vec{e}_1$ can be chosen so that the corresponding normal curvature vector $\vec{h}_{11}$ goes from X in the tangent direction to the ellipse and ends in the point of tangency. The normal curvature vector $\vec{h}_{22}$, corresponding to $\vec{e}_2$, goes from X to the diametrial point of the ellipse, so that $\frac{1}{2} (\vec{h}_{11} - \vec{h}_{22})$ has diametrial direction. Thus $\vec{h}_{11}$ and $\vec{h}_{12}$ are collinear. Taking $\vec{e}_3$ in their common direction we have $h_{11}^4 = h_{12}^4 = 0$ and so

$$\operatorname{diag} \| h_{ij}^4 \| = (0, h_{22}^4, 0, \ldots, 0).$$

The frame vector $\vec{e}_4$ forms now a quasiumbilical unit vector field. The lemma is proved.

L e m m a 2. *There exist locally euclidean surfaces V^2 with nonflat normal connection in E^4.*

P r o o f. If V^2 in E^4 has Gaussian curvature $K=0$ and its normal connection is nonflat, then $X \in V^2$ lives on the orthooptical circle of the normal curvature ellipse (see [6], p. 257). We can take the orthonormal frame as in the proof of Lemma 1 and have then $h^4_{11}=h^4_{12}=0$, $h^4_{22}=\sigma\neq 0$. The well-known formula $K=\sum_\alpha [h^\alpha_{11}h^\alpha_{22}-(h^\alpha_{12})^2]$ gives now that $h^3_{11}h^3_{22}=(h^3_{12})^2$, and there exist ϱ and Θ so that $0<\Theta<\pi$ and

$$h^3_{11}=\varrho\cos^2\Theta,\qquad h^3_{12}=\varrho\sin\Theta\cos\Theta,\qquad h^3_{22}=\varrho\sin^2\Theta.$$

For the surface V^2 we have now the next differential system:

$$\omega^3=\omega^4=0,$$
$$\omega^3_1=\varrho\cos\Theta\cdot\omega^1_\Theta,\qquad \omega^3_2=\varrho\sin\Theta\cdot\omega^1_\Theta,$$
$$\omega^4_1=0,\qquad \omega^4_2=\sigma\omega^2,$$

where

$$\omega^1_\Theta=\omega^1\cos\Theta+\omega^2\sin\Theta.$$

By the exterior differentiation we get the covariant system

$$[d\varrho\cdot\cos\Theta-\varrho\sin\Theta\cdot(d\Theta+\omega^2_1)]\wedge\omega^1_\Theta+\varrho\cos\Theta\cdot(d\Theta+\omega^2_1)\wedge\omega^2_\Theta=0,$$
$$[d\varrho\cdot\sin\Theta+\varrho\cos\Theta\cdot(d\Theta+\omega^2_1)+\sigma\sin\Theta\cdot\omega^4_3]\wedge\omega^1_\Theta+$$
$$+[\varrho\sin\Theta\cdot(d\Theta+\omega^2_1)+\sigma\cos\Theta\cdot\omega^4_3]\wedge\omega^2_\Theta=0,$$
$$[\sigma\sin\Theta\cdot\omega^2_1-\varrho\cos\Theta\cdot\omega^4_3]\wedge\omega^1_\Theta+\sigma\cos\Theta\cdot\omega^2_1\wedge\omega^2_\Theta=0,$$
$$[d\sigma\sin\Theta-\sigma\cos\Theta\cdot\omega^2_1+\varrho\sin\Theta\cdot\omega^4_3]\wedge\omega^1_\Theta+$$
$$+[d\sigma\cdot\cos\Theta+\sigma\sin\Theta\cdot\omega^2_1]\wedge\omega^2_\Theta=0,$$

where

$$\omega^2_\Theta=-\omega^1\sin\Theta+\omega^2\cos\Theta.$$

Here we have 5 basic secondary forms: $d\varrho$, $d\Theta+\omega^2_1$, ω^4_3, ω^2_1 and $d\sigma$. The polar matrix has rank $s_1=4$ and $s_2=1$. Using Cartan lemma with basic primary forms ω^1_Θ and ω^2_Θ we get from the first pair of covariants 4 expressions with 6 new coefficients. Left sides of these expressions contain only 3 basic secondary forms and therefore 2 linear dependencies arise for those 6 coefficients. The next two covariants contain each a new basic secondary form and give 2 new coefficients. The general 2-dimensional integral element depends on $s_1+2s_2=6$ arbitrary parameters. By Cartan theory [4] the considered differential system is compatible and the lemma is proved.

R e m a r k 1. The normal curvature 2-form of the V^2 in Lemma 2 is

$$\Omega^4_3=-\varrho\sigma\cos\Theta\sin\Theta\omega^1\wedge\omega^2.$$

R e m a r k 2. The differential system in the proof of Lemma 2 gives the general locally euclidean V^2 with nonflat normal connection in E^4. If we add to its equations the next two new equations:

$$d\varrho=0,\qquad d\Theta+\omega^2_1=0$$

we get a differential system for a special case of those V^2 in E^4 with covariant system

$$\omega_3^4 \wedge \omega^4 = 0,$$

$$\varrho \cos^2\Theta \cdot \omega_3^4 \wedge \omega^1 + \sigma d\Theta \wedge \omega^2 = 0,$$

$$\sigma d\Theta \omega^1 + d\sigma \wedge \omega^2 + \varrho \sin\Theta \cos\Theta \cdot \omega_3^4 \wedge \omega^1 = 0.$$

This system has 3 basic secondary forms and its polar matrix has rank $s_1 = 3$. The compatibility of the differential system follows immediately from the corresponding Cartan theorem ([4], p. 94).

§ 4. Now we give a counter-example to Theorem 1 in [1], which shows that this theorem is false.

Theorem. *There exist totally quasiumbilical submanifolds V^n with nonflat normal connection in E^{n+2}.*

Proof. Let V^2 be a locally euclidean surface with nonflat normal connection in E^4 (see Lemma 2, Remarks 1 and 2). Its point $X \in V^2$ lies on the orthooptical circle of the normal curvature ellipse. Thus tangent lines from X to the ellipse are orthogonal. Due to the Lemma 1 unit vectors in directions of these two lines form two orthogonal quasiumbilical normal vector fields on the product $V^n = V^2 \times E^{n-2}$ in E^{n+2}. Therefore V^n is totally quasiumbilical and has nonflat normal connection. The theorem is proved.

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