EESTI NSV TEADUSTE AKADEEMIA TOIMETISED.
FÜÜSIKA * MATEMAATIKA
ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ФИЗИКА * МАТЕМАТИКА
PROCEEDINGS OF THE ACADEMY OF SCIENCES OF THE ESTONIAN SSR.
PHYSICS * MATHEMATICS

1987, 36, 4

УДК 517.944+519.46+539.12

T. HALLAP, K. KIIRANEN

LIE POINT TRANSFORMATIONS ADMITTED BY E. WITTEN EQUATIONS

T. HALLAP, K. KIIRANEN. E. WITTENI VÕRRANDITE POOLT LUBATUD LIE PUNKTTEISENDUSED

Т. ХАЛЛАП, К. КИИРАНЕН. ТОЧЕЧНЫЕ ПРЕОБРАЗОВАНИЯ ЛИ, ДОПУСКАЕМЫЕ УРАВНЕНИЯМИ Э. ВИТТЕНА

(Presented by H. Keres)

The Yang—Mills equations are nonlinear second order partial differential equations for 12 scalar functions $\overset{a}{A}_\mu(x)$ [1]:

$$\partial_\mu \overset{a}{F}_{\mu\nu} + \varepsilon^{abc} \overset{b}{A}_\mu \overset{c}{F}_{\mu\nu} = 0, \tag{1}$$

where $\overset{a}{F}_{\mu\nu} \equiv \partial_\mu \overset{a}{A}_\nu - \partial_\nu \overset{a}{A}_\mu + \varepsilon^{abc} \overset{b}{A}_\mu \overset{c}{A}_\nu$.*

If the functions $\overset{a}{A}_\mu(x)$ satisfy the so-called self-dual Yang—Mills equations

$$\overset{a}{F}_{\mu\nu} - \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \overset{a}{F}_{\alpha\beta} = 0, \tag{2}$$

then they satisfy equations (1) as well. Note that the system of nonlinear first order partial differential equations (2) contains 9 equations and is thereby an underdeterminate system.

E. Witten has searched for possible solutions for equations (2) in cylindrical form [2]:

$$\overset{a}{A}_0 = \frac{A_0 x_a}{r},$$

$$\overset{a}{A}_i = \frac{1+\varphi_2}{r^2} \varepsilon_{iak} x_k + \frac{\varphi_1}{r^3} (\delta_{ia} r^2 - x_i x_a) + A_1 \frac{x_1 x_a}{r^2}, \tag{3}$$

where scalar functions A_0, A_1, φ_1, φ_2 depend only on two variables t and r, where $r^2 \equiv x_k x_k$. This ansatz gives the model which is very similar to the Abelian Higgs model in space with constant negative curvature.

When we take

$$A_0 = u_3 + \frac{1}{r}, \quad A_1 = u_4,$$
$$\varphi_1 = r u_1, \quad \varphi_2 = -r u_2 \tag{4}$$

* Here the sum convention is observed. The Greek indices take values from 1 to 4, the Latin ones — from 1 to 3. Only SU(2) gauge group is considered and constant $g \equiv 1$ for simplicity's sake.

and note $t\equiv x_0$, $r\equiv x_1$ and $\partial_0\equiv\frac{\partial}{\partial t}$, $\partial_1\equiv\frac{\partial}{\partial r}$, then we get the following system of equations for functions u_1, u_2, u_3, u_4:

$$\begin{aligned}\overset{1}{u}_{,0}+\overset{2}{u}_{,1}+u^1u^4-u^2u^3=0\\ \overset{2}{u}_{,0}-\overset{1}{u}_{,1}+u^1u^3+u^2u^4=0\\ \overset{4}{u}_{,0}-\overset{3}{u}_{,1}+u^1u^1+u^2u^2=0.\end{aligned}\tag{5}$$

We call these E. Witten equations. Such equations arise also in the theory of minimal surfaces as Gauss-Codazzi equations [3]. In fact, in this case there is one more equation:

$$\overset{3}{u}_{,0}+\overset{4}{u}_{,1}=0 \quad \text{or} \quad \partial_0A_0+\partial_1A_1=0.\tag{6}$$

E. Witten uses this equation in searching for possible solutions of system (5). The system of equations (5)—(6) is a determinate one.

The results of the calculations following method [4] indicate that the generators of Lie point transformations group admitted by the equations (5) form an infinite-dimensional algebra L_∞:

$$X=f\frac{\partial}{\partial x_0}+g\frac{\partial}{\partial x_1}-u_1f_{,0}\frac{\partial}{\partial u_1}-u_2g_{,1}\frac{\partial}{\partial u_2}+$$

$$+(u_4f_{,1}-u_3f_{,0}-f_{,01})\frac{\partial}{\partial u_3}+(u_3g_{,0}-u_4g_{,1}+g_{,01})\frac{\partial}{\partial u_4},\tag{7}$$

$$Y=h\left(u_2\frac{\partial}{\partial u_1}-u_1\frac{\partial}{\partial u_2}\right)+h_{,0}\frac{\partial}{\partial u_4}+h_{,1}\frac{\partial}{\partial u_4},\tag{8}$$

where $f_{,1}+g_{,0}=0$, $f_{,0}=g_{,1}=0$ and $h(x_0,x_1)$ is an arbitrary function. If we take into account equation (6) as well, then

$$h_{,00}+h_{,11}=0.$$

The generators of Lie point transformations group for equations (1) and (2) were computed in [5]. The result obtained was the algebra $L_\infty=L_{15}\oplus L^0$ in both cases where L_{15} is the algebra of the conformal group and L^0 contains the gauge transformations:

$$\begin{aligned}X&=\xi^\mu\frac{\partial}{\partial x^\mu}-\xi^\nu_{,\mu}\overset{a}{A}_\nu\frac{\partial}{\partial\overset{a}{A}_\mu},\\ Y&=\overset{a}{\zeta}_\mu\frac{\partial}{\partial\overset{a}{A}_\mu},\end{aligned}\tag{9}$$

where

$$\xi_\mu(x)=k_\mu+d\cdot x_\mu+a_{\mu\nu}x_\nu+2c_\nu x_\nu x_\mu-c_\mu x_\nu x_\nu$$

is the Killing vector and

$$\overset{a}{\zeta}_\mu(x,A)=\overset{a}{\theta}_{,\mu}-\varepsilon^{abc}\overset{b}{\theta}\overset{c}{A}_\mu.\tag{10}$$

$\overset{a}{\theta}(x)$ denote arbitrary functions of x_μ and d, k_μ, c_μ, $a_{\mu\nu}=-a_{\nu\mu}$ are arbitrary constants.

In the two-dimensional case $(n=2)$ $f\equiv\xi^0$ and $g\equiv\xi^1$ from (7)—(8) are arbitrary functions of variables $x_0\pm ix_1$, whereas in the general $n>2$ (here $n=4$) the functions $\xi^\mu(x)$ are second order polynoms.

The generator Y describes the gauge transformations in both cases and it is the ideal of the algebra since

$$[X, Y]=Y.$$

However, when the functions $\overset{a}{A}_\mu$ satisfy the Lorentz gauge condition

$$\partial_\mu \overset{a}{A}_\mu=0, \tag{11}$$

then in transformations (9)—(10)

$$\overset{a}{\theta}_{,\mu}=0 \quad \text{and} \quad \overset{a}{A}_\mu c_\mu=0. \tag{12}$$

Consequently, the special conformal transformation will disappear and in place of gauge transformations only rotations $SO(3) \simeq SU(2)$ in isotopic space will remain:

$$Y_{ab}=\overset{a}{A}_\mu \frac{\partial}{\partial \overset{b}{A}_\mu}-\overset{b}{A}_\mu \frac{\partial}{\partial \overset{a}{A}_\mu}. \tag{13}$$

In this case the finite algebra $L_{14}=L_{11} \oplus L_3$ [6].

The authors are grateful to M. Kõiv for discussions.

REFERENCES

1. *Prasad, M. K.* Physica, **1D**, 167—191 (1980); В кн.: Геометрические идеи в физике. М., «Наука», 1983, 64—96.
2. *Witten, E.* Phys. Rev. Lett., 38, № 3, 121—124 (1977).
3. *Comtet, A.* Phys. Rev., **D18**, № 10, 3890—3892 (1978).
4. *Овсянников Л. В.* Лекции по теории групповых свойств дифференциальных уравнений. Новосибирск, «Наука, 1966.
5. *Schwarz, F.* Lett. Math. Phys., **6**, № 5, 355—359 (1982).
6. *Rosenhaus, V., Kiiranen, K.* ENSV TA Toim. Füüs. Matem., **31**, № 3, 304—312 (1982).

Tartu State University

Received
March 20, 1987