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DECOMPOSITION AND CLASSIFICATION THEOREMS FOR SEMI-SYMMETRIC IMMERSIONS

Ü. LUMISTE. POOLSÜMMEETRILISTE SISESTUSTE DEKOMPOSITSIOONI- JA KLASSIFIKATSIOONITEOREEMID

Ю. ЛУМИСТЕ. ТЕОРЕМЫ О РАЗЛОЖЕНИИ И КЛАССИФИКАЦИИ ПОЛУСИММЕТРИЧЕСКИХ ПОГРУЖЕНИЙ

(Presented by G. Vainikko)

Let $f: M^m \to E^n$ be an isometric immersion of an m-dimensional Riemannian manifold (M^m, g) into an n-dimensional Euclidean space E^n. Let $\overline{\nabla}$ and α_2 be its van der Waerden — Bortolotti connection and the second fundamental form, respectively. If $\overline{\nabla}\alpha_2=0$, then f is a locally symmetric immersion [1,2] and M^m is a locally symmetric space (by E. Cartan). The integrability condition of the differential system $\overline{\nabla}\alpha_2= =0$ is the symmetricity of $\overline{\nabla}_X\overline{\nabla}_Y\alpha_2$ with respect to tangent vectors X and Y (or, equivalently, the condition $\overline{\Omega}\cdot\alpha_2=0$, where $\overline{\Omega}$ is the curvature form operator of $\overline{\nabla}$). An immersion f satisfying this condition is called semi-symmetric (cf. [3] where the term «semi-parallel» is used); then M^m is a semi-symmetric Riemannian manifold in the sense of [4,5,6]. If, particularly, $\overline{\nabla}_X\overline{\nabla}_Y\alpha_2=0$ for every X and Y, then f is said to be an immersion with parallel third fundamental form $\alpha_3=\overline{\nabla}\alpha_2$ [7,8].

In [9,10] some decomposition theorems were given for the last case. They can be generalized to the case of semi-symmetric immersions as follows.

Let H denote the mean curvature vector of the immersion f and α_2^H the second fundamental form corresponding to H. In every point $x\in M^m$ the form α_2^H is a real symmetric bilinear form.

An immersion f is called a product of immersions $f_\varphi: M^{m_\varphi}\to E^{n_\varphi}$, $\varphi\in\{1, 2, \ldots, s\}$, if (1) $M^m=M^{m_1}\times\ldots\times M^{m_s}$, (2) $E^n=E^{n_1}\times\ldots\times E^{n_s}$, (3) any two distinct E^{n_φ} and E^{n_ψ} are totally orthogonal. If f is a semi-symmetric immersion (resp. an immersion with $\overline{\nabla}\alpha_3=0$), then every f_φ is semi-symmetric (resp. with $\overline{\nabla}\alpha_3=0$), and vice versa.

Theorem 1. *Let $f: M^m \to E^n$ be a semi-symmetric immersion and U^m an open part of (M^m, g), on which α_2^H has distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of constant multiplicities $(r \leqslant m)$. Then the corresponding eigenspaces form r foliations $\Delta_1, \ldots, \Delta_r$ on U^m which are pairwise conjugated with respect to g and α_2 (i.e. $g(\Delta_\rho, \Delta_\sigma) = 0$ and $\alpha_2(\Delta_\rho, \Delta_\sigma) = 0$, if $\rho \neq \sigma$). Let $\Delta'_1, \ldots, \Delta'_s$ be direct sums of $\Delta_1, \ldots, \Delta_r$ such that $\Delta'_\varphi \cap \cap \Delta'_\psi = \{0\}$, if $\varphi \neq \psi$, and $\Delta'_1 \oplus \ldots \oplus \Delta'_s = TU^m$. Let every Δ'_φ be parallel in the Levi-Civita connection ∇ of g. Then around every point $x \in U^m$ the immersion f coincides with a product of semi-symmetric immersions $f_\varphi: U^{m_\varphi} \to E^{m_\varphi}$, where U^{m_φ} is a leaf of Δ'_φ.*

Remark that the first part of Theorem 1 can be deduced from [7], (Theorem 5), stating before that in the case of semi-symmetric immersion f the mean curvature vector H is a commutating vector in the sense of [7]. The second part of Theorem 1 can be considered as a consequence from the first one and from the local version of the fundamental lemma, given in [11] in a global formulation. In fact a direct proof of Theorem 1 exists by straightforward computation using the adapted orthonormal frame bundle and Cartan's method: derivation formulas, exterior differential calculus, curvature forms, etc. Replacing in Theorem 1 the words «semi-symmetric immersion» by «immersion with $\overline{\nabla}\alpha_3 = 0$» (in two places), we get the corresponding theorem in [10].

Theorem 2. *Let f, U^m, and $\Delta'_1, \ldots, \Delta'_s$ be as in Theorem 1 and let $\lambda_r = 0$, $\Delta'_s = \Delta_r$. Suppose there is an orthogonal direct decomposition $\Delta'_s = = \Delta_s^{(1)} \oplus \ldots \oplus \Delta_s^{(s)}$, so that $\Delta_1^*, \ldots, \Delta_s^*$ are parallel in ∇, where $\Delta_\chi^* = = \Delta'_\chi \oplus \Delta_s^{(\chi)}$, if $1 \leqslant \chi \leqslant s-1$, and $\Delta_s^* = \Delta_s^{(s)}$ (the possibility that some of $\Delta_s^{(\varphi)}$, $1 \leqslant \varphi \leqslant s$, are $\{0\}$, is not excluded). Then around every point $x \in U^m$ the immersion f coincides with a product of semi-symmetric immersions $f_\varphi: U^{\bar{m}_\varphi} \to E^{m_\varphi}$, where $U^{\bar{m}_\varphi}$ is a leaf of Δ^*_φ.*

This theorem is a light extension of Theorem 1.

These two decomposition theorems have valuable applications to the problems of local classification of semi-symmetric immersions. All semi-symmetric surface immersions $(m=2)$ are described in [3], where the author of [3] announces that he has a classification of semi-symmetric hypersurface immersions $(n=m+1)$, too. Those of them which have parallel α_3 (i.e. $\overline{\nabla}\alpha_3 = 0$), were found out independently in [8].

The following two theorems give the local classification of all two-codimensional semi-symmetric immersions (and immersions with $\overline{\nabla}\alpha_3 = = 0$ among them).

Firstly we give a list of immersions proved to be semi-symmetric:

(1) plane immersion: $f(M^m) = E^m$ is an m-dimensional Euclidean subspace (m-plane) in E^n,

(2) sphere immersion: $f(M^m) = S^m$ is an orbit (m-sphere) in E^{m+1} described in all rotations around a fixed point (centre) by a point different from it,

(3) round cone immersion: $f(M^m) = C^m$ is a regular part of an m-cone of revolution in E^{m+1} generated in all rotations around a fixed one-dimensional axis by a straight line having a common point (vertex) with it,

(4) rank 1 immersion: $f(M^m)$ is a regular part of the envelope of a one-parameter family of m-planes in E^n,

(5) rank 2 immersion with flat $\overline{\nabla}$: $f(M^m)$ is a regular part of the envelope of a two-parameter family of m-planes in E^n, $n \geqslant m+2$, and f has flat $\overline{\nabla}$ (i.e. $\overline{\Omega} \equiv 0$),

(6) orthogonal canal immersion: $f(M^m)=\tilde{S}^m$ is a regular part of the envelope of a one-parameter family of m-spheres S^m in E^n, $n \geqslant m+2$, and the principal (or the first) normal of the orthogonal trajectory of the family of characteristic $(m-1)$-spheres on $\tilde{S}^m$ is everywhere orthogonal to the $(m+1)$-plane of the family m-sphere S^m,

(7) orthogonal canal cone immersion: $f(M^m)=\tilde{C}^m$ is a regular part of the envelope of a one-parameter family of round cones C^m with a common vertex in E^n, such that the intersection of the envelope and an $(n-1)$-sphere around the vertex is a noneuclidean analogue to the previous case.

The classification theorem for semi-symmetric immersions of codimension $\leqslant 2$ states the following.

Theorem 3. *Every semi-symmetric immersion $f: M^m \to E^{m+2}$ locally coincides*

(*i*) *with one of hypersurface immersions* (2)—(4) *or with its product by an identity plane immersion (i. e. immersion* (1) *in the case $m=n$) superposed by a hyperplane immersion, or*

(*ii*) *with one of* (1), (4)—(7) *in the case $n=m+2$ or*

(*iii*) *with a two-codimensional product of several of these immersions.*

Remark that (*i*) gives a full description of semi-symmetric hypersurface immersions. Note also that plane immersions (1) have $\alpha_2=0$, sphere immersions (2) and their products by identity plane immersions have $\overline{\nabla}\alpha_2=0$. These are only hypersurface immersions with $\overline{\nabla}\alpha_2=0$. In the case of codimension 2 only products of two sphere immersions and their products by identity plane immersions are to be added to get all immersions with $\overline{\nabla}\alpha_2=0$ and codimension $\leqslant 2$.

The next theorem gives the classification of all other two-codimensional immersions with $\overline{\nabla}\alpha_3=0$ (recall that $\alpha_3=\overline{\nabla}\alpha_2$).

Theorem 4. *Any two-codimensional immersion with parallel $\alpha_3 \neq 0$ locally coincides*

(*a*) *with a product of a clothoid immersion by an identity plane immersion superposed by a hyperplane immersion, or*

(*b*) *with a product of a clothoid immersion by clothoid or spherical immersion and by an identity plane immersion, or*

(*c*) *with a product of a spherical clothoid immersion by an identity plane immersion, or*

(*d*) *with a product of a twisted spherical clothoid binormal regulus immersion by an identity plane immersion.*

Here the identity plane immersion can be trivial (i. e. (1) with $m=n=0$) and not essential in products.

The immersions with $\overline{\nabla}\alpha_3=0$ listed in Theorem 4 as the first components of products are found in [8] by classification of surface immersions with $\overline{\nabla}\alpha_3=0$, and are as follows.

In the case of clothoid immersion $f(M^1)$ is a line in E^2 with natural equation $k=as$. In the case of spherical clothoid immersion $f(M^1)$ is a line on S^2 in E^3 with geodesic curvature $k_g=as$. In the case of twisted spherical clothoid binormal regulus immersion $f(M^2)$ is a surface in E^4, generated by great circles of a S^3 binormal to a line on S^3 with the geodesic first curvature $k_g=as$ and the geodesic second curvature $\varkappa_g=\pm\frac{1}{r}$, where r is the radius of S^3.

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