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ON A CLASS OF SOLUTIONS OF YANG-MILLS EQUATION

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В. РОЗЕНГАУЗ. ОБ ОДНОМ КЛАССЕ РЕШЕНИЙ УРАВНЕНИЯ ЯНГА—МИЛЛСА

(*Presented by V. Hizhnyakov*)

Since the time instantons have been discovered [1], the attempts to understand their role in the vacuum of the theory have not stopped (see, e. g. [2–5]). At the same time, the search of other classical configurations of the gauge field is also of great interest due to the possible contribution made by these solutions or their combinations to a functional integral [6–7].

In the present paper, we investigate the classical solutions of the Yang-Mills equation which are neither self-dual nor antidual.

So, we seek the extremum of the Yang-Mills action functional

$$S=\frac{1}{4}\int G^a_{\mu\nu}\, G^a_{\mu\nu}\, d^4x,$$

$$G^a_{\mu\nu}=A^a_{\nu,\mu}-A^a_{\mu,\nu}+g\varepsilon_{abc}A^b_\mu A^c_\nu, \tag{1}$$

in the known form ([5])

$$A^a_\mu=\frac{2}{g}\,\eta_{a\mu\nu}\frac{x_\nu}{x^2}f(x^2), \tag{2}$$

where, keeping in mind the translational invariance, under x_ν we mean $(x-x_0)_\nu$ ($x_{0\mu}$ is the free parameter); $x^2\equiv x_\mu x_\mu$, g is the coupling constant and $\eta_{a\mu\nu}$ are the t'Hooft symbols: $\eta_{amn}=\varepsilon_{amn}$, $\eta_{a\mu 4}=-\eta_{a4\mu}=\delta_{a\mu}$; $a, m, n=1, 2, 3$; $\mu, \nu=1, 2, 3, 4$.

Substituting expression (2) into equation (1), we find

$$S=\frac{24\pi^2}{g^2}\int\frac{dx^2}{x^2}\,[x^4f'^2+f^2(1-f)^2]. \tag{3}$$

It is convenient to introduce a new variable $t=\ln x^2$. Then S has a simple form

$$S=\frac{24\pi^2}{g^2}\int \mathrm{d}t[\dot f^2+f^2(1-f)^2],\quad \dot f\equiv\frac{df(t)}{\mathrm{d}t}. \tag{4}$$

Now let us minimize $S:\ \frac{\delta S}{\delta f}=0$, hence

$$\dot f^2=c_1+f^2(1-f)^2, \tag{5}$$

where c_1 is an arbitrary constant.

Besides the trivial solutions $f=c=0, 1, 1/2$ (the case $c=1/2$ corresponds to merons [6]), the solutions of equation (5) can be found from

$$\pm\int\frac{df}{\sqrt{c_1+f^2(1-f)^2}}=\bar{t}, \qquad \bar{t}\equiv t+c_2. \tag{6}$$

The existence of $c_2=\ln\frac{1}{\varrho^2}=\text{const}$ is the reflection of the dilatational invariance of the problem. Another kind of symmetry is also obvious, $f\to 1-f$ as is the inversion $x^2/\varrho^2\to\varrho^2/x^2$ $(\bar{t}\to-\bar{t})$. Both last symmetry transformations mean the transition to a solution with the same action value S and the opposite sign of topological charge $Q=\frac{g^2}{32\pi^2}\int G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\,d^4x=6\int \mathrm{d}t\,\dot{f}f(1-f)$.

Taking into account the symmetry discussed and introducing $u=f-1/2$, consider the equation

$$\int\frac{du}{\sqrt{(u^2-1/4)^2+c_1}}=\bar{t}. \tag{6'}$$

1. $c_1\leqslant -1/16$.
The solution of the equation (6′) is of the form

$$F\left(\arccos\frac{b}{u},\, s\right)=\sqrt[4]{-4c_1}\,\bar{t}, \tag{7}$$

where

$$b^2=1/4+\sqrt{-c_1},\qquad s=a/\sqrt{a^2+b^2},\qquad a^2=-1/4+\sqrt{-c_1};$$

and

$$F(\varphi,k)=\int_0^{\varphi}d\alpha/\sqrt{1-k^2\sin^2\alpha}$$

is the elliptic integral of the first kind.
(We have used the well-known integral relation

$$\int_b^u\frac{dx}{\sqrt{(x^2+a^2)(x^2-b^2)}}=\frac{1}{a^2+b^2}F(\varepsilon,s),\qquad u>b>0,$$

$$\varepsilon=\arccos\frac{b}{u}).$$

Note that for $c_1=-1/16$ the solution has a simple form

$$u=1/[\sqrt{2}\cos(\bar{t}/\sqrt{2})]. \tag{8}$$

2. $-1/16<c_1\leqslant 0$.

$$F(\arcsin(a/u),\, b/a)=a\bar{t}; \tag{9}$$

$$a^2=1/4+\sqrt{-c_1},\qquad b^2=1/4-\sqrt{-c_1}.$$

When $c_1=0$, expression (9) leads to

$$f=x^2/(x^2-\varrho^2);\qquad x^2>\varrho^2. \tag{10}$$

The solution with the given profile function is self-dual (the corresponding antidual solution: $f=\varrho^2/(\varrho^2-x^2)$, $x^2<\varrho^2$).

3. $c_1 \geqslant 0$.

$$F(2\,\mathrm{arctg}\,(u/\tilde{c}_1),\ p) = 2\tilde{c}_1\bar{t}, \qquad (11)$$

$$\tilde{c}_1 = \sqrt[4]{c_1 + 1/16}, \qquad p = \sqrt{1/2 + 1/(8\tilde{c}_1^2)}\ .$$

When $c_1 = 0$, equation (11) gives two solutions: $f = x^2/(x^2 - \varrho^2)$ and $f = x^2/(x^2 + \varrho^2)$. The last solution corresponds to the instanton in the regular gauge [5].

The given solutions can be expressed through the Jacobi elliptic functions

$$u = \pm 1/[b\,\mathrm{cn}\,(r\bar{t}, s)], \quad r \equiv \sqrt[4]{-4c_1}, \quad T = 2K(s)/r; \qquad (7')$$

$$u = \pm 1/[a\,\mathrm{sn}\,(a\bar{t}, b/a)], \quad T = 2K(b/a)/a; \qquad (9')$$

$$u = \pm\tilde{c}_1\,\mathrm{tg}\,[am(2\tilde{c}_1\bar{t}, p)/2], \quad T = 2K(p)/\tilde{c}_1; \qquad (11')$$

$K(s) \equiv F(\pi/2, s)$, (T is the correspondence period).

Solutions (7′), (9′), (11′) diverge on the boundaries of their periods. It is not clear whether it is possible to regularize any of the given solutions, for instance, by the dimensional regularization procedure. For $-1/16 \leqslant c_1 \leqslant 0$ there exists another kind of solution

$$F(\arcsin\,(u/b),\ b/a) = a\bar{t}, \qquad (12)$$

$$a^2 = 1/4 + \sqrt{-c_1}; \quad b^2 = 1/4 - \sqrt{-c_1},$$

or

$$u = \pm b\,\mathrm{sn}\,(a\bar{t},\ b/a), \quad T = 2K(b/a)/a, \qquad (12')$$

which is continuous in the interval $-K(b/a)/a < \bar{t} < K(b/a)/a$. Configurations (12′) in this interval lead to a finite action. The instanton corresponds to the case $c_1 = 0$ and the meron, to $c_1 = -1/16$.

Note added in proof. Solutions (12′) were obtained in the paper by G. Z. Baseyan and S. G. Matinyan [8].

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