EESTI NSV TEADUSTE AKADEEMIA TOIMETISED.
FÜÜSIKA * MATEMAATIKA
ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ФИЗИКА * МАТЕМАТИКА
PROCEEDINGS OF THE ACADEMY OF SCIENCES OF THE ESTONIAN SSR.
PHYSICS * MATHEMATICS

1986, 35, 4

УДК 681.511.42

Ülle KOTTA

INVERSION OF DISCRETE-TIME LINEAR-ANALYTIC SYSTEMS

(*Presented by N. Alumäe*)

1. Introduction. There is a considerable amount of literature dealing with construction inverses for linear multivariable systems, see for example [1–3]. Several results have been recently extended to continuous time nonlinear multivariable systems [4–9]. The interest can be explained by the fact that inverse systems have many applications in control system design. It is well known that the determination of right inverse system is related to the problem of constructing an input which will yield some desired output (referred to also as the tracking problem). The left inverse provides a practical method for determining the unknown input of a system from measured output data.

Unfortunately, the inverse system design for nonlinear discrete time systems goes relatively unnoticed. To the author's knowledge, the only contribution is due to S. Monaco and D. Normand-Cyrot [10]. Their paper provides the left inverse for linear analytic systems which satisfy some extra conditions (these conditions will be presented in Section 3). The purpose of this paper is to construct the right inverse for the same class of systems. Mainly the same tools will be used, i. e. tools suitable for representing the evolution of the system which involves compositions of functions [11] and which allow to solve several problems for discrete time nonlinear systems. But unlike [10], in our paper the concept of matrix generalized inverse will also be employed to construct the inverse. For the sake of completeness of presentation, the left inverse will also be written down with the aid of matrix generalized inverse. The case of bilinear systems will be considered separately and the results will be illustrated by means of an example describing a fermentation process.

If the aforementioned conditions are not satisfied, the inverse system is obviously not linear analytic but a polynomial analytic system [12].

2. Problem statement. Consider the nonlinear discrete time system described by equations

$$x(t+1)=x(t)+f_0(x(t))+F(x(t))u(t), \tag{1}$$

$$y(t)=h(x(t)), \tag{2}$$

$$x(0)=x_0,$$

where the state $x(t)\in R^n$, $u(t)$ is the m dimensional input vector, $y(t)$ is the p dimensional output vector, $F(x(t))=[f_1(x(t))\dots f_m(x(t))]$, $f_0, f_1, \dots, f_m: R^n\to R^n$ and $h: R^n\to R^p$ are analytic functions on R^n.

Consider another system

$$\tilde{x}(t+1)=\tilde{x}(t)+\tilde{f}_0(\tilde{x}(t))+\tilde{F}(\tilde{x}(t))\tilde{u}(t),$$
$$\tilde{y}(t)=\tilde{h}(\tilde{x}(t))+\tilde{G}(\tilde{x}(t))\tilde{u}(t), \tag{3}$$
$$\tilde{x}(0)=\tilde{x}_0,$$

where $\tilde{x}(t)\in R^n$, $\tilde{u}(t)\in R^p$ and $\tilde{y}(t)\in R^m$.

Definition 1. *The system* (3) *is called the right inverse for the given system* (1), (2), *if, when driven by a certain desired output with appropriate shift forward of the given system, it computes the required input of the given system as an output.*

Definition 2. *The system* (3) *is called the left inverse for the given system* (1), (2) *if, when driven by the output with appropriate shift forward of the given system, it produces the input of the given system as an output.*

The purpose of this paper is to construct right and/or left inverse for systems (1), (2), if they exist.

3. Preliminaries. This section is devoted to introducing briefly the tools which will be used in the sequel. The more detailed presentation of this material can be found in [11]. Let us introduce the differential operator L_f associated with the function $f(x)$:

$$L_f=\sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} \quad (f_i \text{ is the } i\text{-th component of } f(x)).$$

By $L_f\otimes L_g$ we will denote the tensor product of differential operators defined by

$$L_f\otimes L_g=\sum_{i,j=1}^{n} f_i g_j\frac{\partial^2}{\partial x_i\,\partial x_j} \tag{4}$$

and Δ_f will denote the following operator

$$\Delta_f=I+L_f+\frac{1}{2!}L_f^{\otimes 2}+\ldots+\frac{1}{k!}L_f^{\otimes k}+\ldots, \tag{5}$$

where $L_f^{\otimes k}=L_f\otimes\ldots\otimes L_f$ (k times) and I is the identity operator.

It follows from the definition of Δ_f that for any choice of k analytic functions $g_i: R^n\to R^n$, $i=1,\ldots,k$, one has:

$$\Delta_{g_1+\ldots+g_k}=\Delta_{g_1}\otimes\ldots\otimes\Delta_{g_k}. \tag{6}$$

Moreover, denoting by γ_i any differential operator of the form either (4) or (5), one has:

$$\gamma_1\otimes(\gamma_2+\ldots+\gamma_l)=\gamma_1\otimes\gamma_2+\ldots+\gamma_1\otimes\gamma_l. \tag{7}$$

The differential operator Δ_f can be used to express the composition of functions. More precisely, if $f: R^n\to R^n$ and $h: R^n\to R^p$ are two analytic functions on R^n, then the composition (denoted by «∘») of h with $I+f$, can be expressed as

$$h\circ(I+f)(x)=\Delta_f(h)|_x,$$

where $|_x$ denotes the evaluation at x. More generally —

$$h\circ(I+f_k)\circ\ldots\circ(I+f_1)(x)=\Delta_{f_1}\circ\ldots\circ\Delta_{f_k}(h)|_x. \tag{8}$$

The tools introduced are suitable for representing the evolution of the system (1), (2) which involve compositions of functions. The output evolution in t step can be written as

$$y(t)=\Delta_{f_0+Fu(0)}\circ\ldots\circ\Delta_{f_0+Fu(t-1)}(h)\,|_{x_0}.$$

For each output y_i, $i=1,\ldots,p$, let d_i be the index defined as the smallest integer such that

1) $\Delta_{f_0}\otimes L_{f_{i_\nu}}\otimes\ldots\otimes L_{f_{i_1}}\circ\Delta^r_{f_0}(h_i)\,|_x=0$

$\forall x \quad \forall r<d_i, \quad \forall\nu,\ i_1,\ldots,i_\nu\in\{1,\ldots,m\}$;

2) $\exists_\mu$ and a sequence $i_1,\ldots,i_\mu$ such that

$\Delta_{f_0}\otimes L_{f_{i_\mu}}\otimes\ldots\otimes L_{f_{i_1}}\circ\Delta^{d_i}_{f_0}(h_i)\,|_x\neq0$

$\forall x\in V$, where V is some open subset of R^n. Here $\Delta^r_f=\Delta_f\circ\ldots\circ\Delta_f$ (r times).

It can be shown that $t=d_i+1$ is the first instant of time at which the i-th output is affected by, at least, one input at time $t=0$.

In the sequel we shall consider only the systems for which the condition (9) (the condition mentioned in the introduction) holds:

$$\Delta_{f_0}\otimes L_{f_{i_\nu}}\otimes\ldots\otimes L_{f_{i_1}}\circ\Delta^{d_i}_{f_0}(h_i)\,|_x=0 \tag{9}$$

$$\forall x\in V,\quad \forall\nu\geqslant2,\quad i_1,\ldots,i_\nu\in\{1,\ldots,m\},\quad i=1,\ldots,p.$$

The class of systems we consider includes bilinear systems as well as nonlinear systems with linear output function if all indices $d_i=0$, and it has received considerable attention in the control literature. For example, it is known that this condition is necessary in order that the system (1), (2) would be immersible into a linear system [13] and decomposable by a linear analytic state feedback.

Let us introduce the matrix $A(x)=[a_{ij}(x)]$, $i=1,\ldots,p$, $j=1,\ldots,m$ and the vector $\delta(x)=[\delta_k(x)]$, $k=1,\ldots,p$ defined by

$$a_{ij}(x)=\Delta_{f_0}\otimes L_{f_j}\circ\Delta^{d_i}_{f_0}(h_i)\,|_x;$$

$$\delta_i(x)=\Delta^{d_i+1}_{f_0}(h_i)\,|_x.$$

4. Construction of right inverse system. As in the linear case we will assume that $p\leqslant m$.

Theorem 1. *Consider the system* (1), (2), *for which the condition* (9) *holds. If the rank of the $p\times m$ matrix $A(x)$ is equal to p in some open and dense subset V of R^n, then the system* (1), (2) *has in V a right inverse which is defined by equations*

$$\begin{aligned}x(t+1)=x(t)+f_0(x(t))-F(x(t))A^R(x(t))\delta(x(t))+\\+F(x(t))[I_m-A^R(x(t))A(x(t))]g(x)(t))+\\+F(x(t))A^R(x(t))Y^*(t),\end{aligned} \tag{10}$$

$$\begin{aligned}u(t)=-A^R(x(t))\delta(x(t))+A^R(x(t))Y^*(t)+\\+[I_m-A^R(x(t))A(x(t))]g(x(t)),\end{aligned} \tag{11}$$

where $A^R=A^T(AA^T)^{-1}$, $Y^*(t)=[y_1(t+d_1+1)\ldots y_p(t+d_p+1)]^T$ *and the vector function* $g(x(t))$ *is arbitrary.*

Proof. For the i-th output, $i=1, \ldots, p$ evolution in d_i+1 step can be written as

$$y_i(t+d_i+1)=h_i[x(t+d_i+1)]=$$
$$=h_i\circ(I+f_0+Fu(t+d_i))\circ\ldots\circ(I+f_0+Fu(t))(x(t))=$$
$$=\Delta_{f_0+Fu(t)}\circ\ldots\circ\Delta_{f_0+Fu(t+d_i)}(h_i)|_{x(t)}.$$

Applying the formula (6), considering the condition (9) and the definition of index d_i, one gets

$$y_i(t+d_i+1)=\Delta_{f_0}^{d_i+1}(h_i)|_{x(t)}+[a_{i1}(x(t))\ldots a_{im}(x(t))]u(t)$$
$$i=1, \ldots, p$$

or in the matrix form

$$Y^*(t)=\delta(x(t))+A(x(t))u(t). \quad (12)$$

To get the equations of right inverse, we have to be able to solve the equation (12) with respect to $u(t)$ (not necessarily uniquely) for arbitrary $Y^*(t)$. The device for computing the solution of this system is the generalized inverse $A^-(x)$ of the matrix $A(x)$, which in general is not uniquely determined. The equation (12) has a solution

$$u(t)=A^-(x(t))[Y^*(t)-\delta(x(t))]+[I_m-A^-(x(t))A(x(t))]g(x(t)),$$

where the vector function $g(x)$ is arbitrary if and only if the consistency condition for the equation (12)

$$[I_p-A(x(t))A^-(x(t))][Y^*(t)-\delta(x(t))]=0 \quad (13)$$

is satisfied ([14], p. 103, 104). By the assumption of the theorem, the rank of $p\times m$ matrix $A(x)$ in V is equal to p, so the uniquely defined right generalized inverse $A^R=A^T(AA^T)^{-1}$ of it exists such that $AA^R=I_p$ ([14], p. 102). Consequently, the consistency condition is satisfied and the system of equations (12) has in V a solution (not unique, if $m>p$)

$$u(t)=-A^R(x(t))\delta(x(t))+A^R(x(t))Y^*(t)+ \quad (14)$$
$$+[I_m-A^R(x(t))A(x(t))]g(x(t)).$$

Substituting (14) into equation (1), we get equation (10).

Arbitrary functions which appear in the equations of the inverse system indicate that there are some degrees of freedom in the choice of an input to produce a desired output. In addition, arbitrary functions may be employed to achieve some desirable properties of the inverse system.

5. Construction of the left inverse system. As in the linear case, we will assume that $m\leqslant p$.

Theorem 2. *Consider the system* (1), (2) *for which the condition* (9) *holds. If the rank of the $p\times m$ matrix $A(x)$ is equal to m in some open and dense subset V of R^n, then the system* (1), (2) *has in V a left inverse, which is defined by equations*

$$x(t+1)=x(t)+f_0(x(t))-F(x(t))A^L(x(t))\delta(x(t))+ \quad (15)$$
$$+F(x(t))A^L(x(t))Y^*(t),$$

$$u(t)=-A^L(x(t))\delta(x(t))+A^L(x(t))Y^*(t), \quad (16)$$

where $A^L=(A^TA)^{-1}A^T$ *and* $Y^*(t)=[y_1(t+d_1+1)\ldots y_p(t+d_p+1)]$.

Proof. The proof is analogous to the proof of Theorem 1. The difference is that now we have to be able to solve the equation (12) uniquely with respect to $u(t)$ for corresponding $Y^*(t)$. By assumption of the theorem, the rank of $p \times m$ matrix $A(x)$ in V is equal to m, so the uniquely determined left generalized inverse $A^L = (A^T A)^{-1} A^T$ of it exists such that $A^L A = I_m$ ([14], p. 102). The consistency condition (13) is satisfied because $Y^*(t)$ corresponds to $u(t)$. Consequently the system of equations has in V a unique solution

$$u(t) = -A^L[x(t)]\delta[x(t)] + A^L[x(t)]Y^*(t). \tag{17}$$

Substituting (17) into equation (1), we get equation (15).

If both right inverse and left inverse exist, then they are identical and the above distinction need not be made. This happens only when the system has the same number of inputs and outputs. In that case $A^R = A^L = A^{-1}$.

If $p < m$, then the left inverse cannot be constructed, because equation (12) has no unique solution. On the other hand, if $p > m$, then $I_p \neq AA^L$, and so the consistency conditions for equation (12) will not be satisfied by any y. Consequently, the system (12) is not solvable without imposing conditions on y.

6. Application to bilinear systems. Let us consider a bilinear system

$$x(t+1) = Ax(t) + \sum_{i=1}^{m} B_i x(t) u_i(t),$$

$$y(t) = Cx(t),$$

where $A, B_1, \ldots, B_m$ are $n \times n$ matrices and $C = [c_1^T \ldots c_m^T]^T$ is $p \times n$ matrix. In the case of bilinear systems, the condition (9) holds automatically, because $\Delta_{f_0}^r (c_i^T x) = c_i^T A^r x$ is a linear function and therefore its second and higher-order derivatives are equal to zero. Now the index d_i, $i = 1, \ldots, p$ is defined as the smallest integer such that the following conditions hold

1) $c_i^T A^r B_j = 0 \quad \forall r < d_i, \quad \forall j \in \{1, \ldots, m\}$,

2) $\exists k$ such that

$$c_i^T A^{d_i} B_k \neq 0.$$

The matrix $A(x)$ and vector $\delta(x)$ are now defined by

$$a_{ij}(x) = c_i^T A^{d_i} B_j x,$$

$$\delta_k(x) = c_k^T A^{d_k+1} x.$$

7. Example. As an example, consider the process which produces single-cell protein from yeast grown on methanol. This process is described by the following equations [15]

$$x_1(t+1) = x_1(t) + H\mu_m \frac{x_2(t)}{K + x_2(t)} x_1(t) - Hx_1(t)u_1(t),$$

$$x_2(t+1) = x_2(t) - \frac{H}{R}\mu_m \frac{x_2(t)}{K + x_2(t)} x_1(t) - Hx_2(t)u_1(t) + Hu_2(t),$$

$$y_1(t) = x_1(t),$$

$$y_2(t) = x_2(t).$$

The indices of the system are $d_1=0$, $d_2=0$. The matrix $A(x)$ is given by

$$A(x)=\begin{bmatrix} -Hx_1 & 0 \\ -Hx_2 & H \end{bmatrix}$$

and is nonsingular in the subspace $V=R^2-\{x|x_1=0\}$. Note that by x_1 is denoted the biomass concentration and therefore we may assume that it is not equal to zero. The inverse system is given by the equations:

$$x(t+1)=x(t)+[y_1(t+1)\ y_2(t+1)]^T$$

$$u(t)=\mu_m\frac{x_2}{K+x_2}\begin{bmatrix} 1 \\ x_2+\frac{x_1}{R} \end{bmatrix}-\frac{1}{H}\begin{bmatrix} \frac{y_1(t+1)}{x_1} \\ \frac{x_2}{x_1}y_1(t+1)+y_2(t+1) \end{bmatrix}.$$

REFERENCES

1. *Silverman, L. M.* IEEE Trans. Automat. Contr., **14**, № 3, 270—276 (1969).
2. *Silverman, L. M., Payne, H. J.* SIAM J. Contr., **9**, № 2, 199—233 (1971).
3. *Lovass-Nagy, V., Miller, R. J., Powers, D. L.* Int. J. Contr., **24**, № 5, 733—739 (1976).
4. *Hirschorn, R.* M. IEEE Trans. Automat. Contr., **24**, № 6, 855—865 (1979).
5. *Hirschorn, R. M.* IEEE Trans. Automat. Contr., **26**, № 2, 593—595 (1981).
6. *Singh, S. N.* IEEE Trans. Automat. Contr., **26**, № 2, 595—598 (1981).
7. *Singh, S. N.* IEEE Trans. Automat. Contr., **27**, № 1, 270—272 (1982).
8. *Singh, S. N.* IEEE Trans. Automat. Contr., **27**, № 4, 958—960 (1982).
9. *Fliess, M.* Lect. Notes Contr. and Inf. Sci., **58**, 323—330 (1984).
10. *Monaco, S., Normand-Cyrot, D.* Some remarks on the invertibility of nonlinear discrete-time systems. American Control Conference, 1983, **1**, 324—328.
11. *Normand-Cyrot, D.* An algebraic approach to the input/output description of nonlinear discrete time systems. American Control Conference, 1982, 466—471.
12. *Kotta, Ü.* Preprints of AFCET of Congress «Automatique 85». Toulouse, 1985.
13. *Monaco, S., Normand-Cyrot, D.* Int. J. Contr., **38**, № 1, 245—261 (1983).
14. *Graybill, F. A.* Introduction to Matrices with Applications in Statistics. Belmont, Wadsworth Publ. Comp., 1969.
15. *Alvarez Gallegos, Joaquin, Alvarez Gallegos, Jaime.* Trans. ASME. J. of Dyn. Syst., Meas. and Contr., **104**, № 4, 212—217 (1982).

Academy of Sciences of the Estonian SSR,
Institute of Cybernetics

Received
Dec. 27, 1985

Ülle KOTTA

DISKREETSE LINEAAR-ANALÜÜTILISE SÜSTEEMI PÖÖRAMINE

On vaadeldud mitmemõõtmelise diskreetse lineaar-analüütilise süsteemi pööramise ülesannet juhul, kui see süsteem rahuldab veel ühte lisatingimust. On tuletatud piisavad tingimused nimetatud süsteemi parem- ja vasakpoolseks pööratavuseks. Nende tingimuste täidetuse korral on konstrueeritud parem- ja vasakpoolsed pöördsüsteemid, mis samuti kuuluvad lineaar-analüütiliste süsteemide klassi.

Юлле КОТТА

ОБРАЩЕНИЕ ЛИНЕЙНО-АНАЛИТИЧЕСКИХ СИСТЕМ С ДИСКРЕТНЫМ ВРЕМЕНЕМ

Рассматривается задача обращения многомерной нелинейной дискретной системы, описываемой уравнениями

$$x(t+1) = x(t) + f_0(x(t)) + F(x(t))u(t),$$

$$y(t) = h(x(t)),$$

$$x(0) = x_0,$$

где $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$, $F(x(t)) = [f_1(x(t)) \ldots f_m(x(t))]$, $f_0, \ldots, f_m : R^n \to R^n$ и $h : R^n \to R^p$ являются аналитическими функциями. Предполагается, что система удовлетворяет некоторому дополнительному условию. Отметим, что этому дополнительному условию удовлетворяют, например, билинейные, а также нелинейные системы с линейным выходом, если $d_1 = \ldots = d_p = 0$. Здесь $t = d_i + 1$ является первым моментом времени, в котором на i-й координате выхода y_i проявляется действие хотя бы одного входа в момент $t = 0$.

Получены достаточные условия правой и левой обратимости. При выполнении этих условий построены правые и левые обратные системы, которые также принадлежат к классу линейно-аналитических систем.