EESTI NSV TEADUSTE AKADEEMIA TOIMETISED.
FÜÜSIKA * MATEMAATIKA

ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ФИЗИКА * МАТЕМАТИКА
PROCEEDINGS OF THE ACADEMY OF SCIENCES OF THE ESTONIAN SSR.
PHYSICS * MATHEMATICS

1986, 35, 4
https://doi.org/10.3176/phys.math.1986.4.01

УДК 517.944+539.12

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HODOGRAPH TRANSFORMATIONS

(Presented by H. Keres)

A method of performing hodograph transformations of partial derivatives of any order for arbitrary dimensions is worked out. Hodograph transformation means a change of the roles of arguments and functions. The formulas obtained for first derivatives can easily be applied to any dimensions. Several explicit formulas for lower orders and dimensions are also given. Application to partial differential equations is discussed.

Introduction

Hodograph transformations link different adequate descriptions of a geometrical object in one and the same co-ordinate system. A geometrical object can be treated as a set of dependence relations between the co-ordinates. In a fixed co-ordinate system one can present this dependence by m relations between N co-ordinates

$$F_i(X_1, \ldots, X_N)=0, \quad i=1, \ldots, m, \tag{1}$$

where N is the dimension of the space; our geometrical object forms a hyper-surface of a dimension $n=N-m$ in that space. One can choose the values of n co-ordinates arbitrarily and hyper-surface conditions (1) determine the values of the remaining m co-ordinates. The choice of the independent co-ordinates — the arguments — is free and fixes the r e p r e s e n t a t i o n of the hyper-surface.

In order to define a partial derivative of a dependent co-ordinate (function) with respect to the arguments on the hyper-surface one needs to know the whole set of arguments. Taking a new set changes the derivatives. These transformations of partial derivatives are known as hodograph transformations [1].

For a co-ordinate variable in an equation hodograph transformation means a change of the role of the variable from an argument to a function or vice versa. For partial derivatives new derivatives must be expressed by means of the old ones.

There are physical quantities such as space co-ordinates and time that are usually considered as arguments. However, in some cases it is of interest to treat them as functions. An additional symmetry — the field-space-time symmetry can be revealed at that. The physical basis of this symmetry is considered in [2]. The theory of hodograph transformations in a restricted form is developed in [2, 3], the ideas of hodograph transformations are considered also in [4, 5].

In this paper, a general theory of hodograph transformations of partial derivatives is developed with a more detailed consideration being paid to lower orders and dimensions.

1. Transformations of first derivatives

An n-dimensional hyper-surface in an N-dimensional space can be presented, instead of (1), by N equations that give the co-ordinates of the space by means of n parameters

$$X_i = X_i(u_k). \quad \begin{matrix} i=1, \ldots, N, \\ k=1, \ldots, n. \end{matrix} \tag{2}$$

If one takes certain n co-ordinates as parameters, i.e. chooses a representation, there remain $m = N - n$ relations, which determine the remaining m co-ordinates (functions).

By fixing the representation one also provides a definition to a partial derivative of an expression R of the co-ordinates

$$_A\frac{\partial R}{\partial X_p} = \frac{dR}{dX_p}\bigg|_{dX_1=dX_2=\ldots=dX_{p-1}=dX_{p+1}=\ldots=dX_n=0}, \tag{3}$$

where in the representation A the arguments are chosen as the co-ordinates number 1, ..., n. Let us call it set α.

Our aim is to transform (3) into another representation B with the arguments β. For that purpose we calculate all the differentials present in (3). In general,

$$dX_q = \sum_{k=1}^{n} \frac{\partial X_q}{\partial u_k} du_k. \tag{4}$$

Since the choice of the parameters u is arbitrary, one can take the arguments of the B-representation (set β) for them —

$$dX_q = \sum_{k\in\beta} {}_B\frac{\partial X_q}{\partial X_k} dX_k = \sum_{k\in\beta} {}_BD^q_k\, dX_k, \tag{5}$$

where

$$_BD^q_k \equiv {}_B\frac{\partial X_q}{\partial X_k} \tag{6}$$

denotes the derivative of the co-ordinate X_q with respect to the co-ordinate (argument) X_k in the B-representation. Generally, X_q is a function, but its being an argument of B is not ruled out either. In the latter case,

$$_BD^q_k = \delta_{qk} \quad \text{if} \quad q, k \in \beta. \tag{7}$$

Now substitute (5) for $dX_1, \ldots, dX_n$ in (3):

$$\begin{aligned} &\sum_{k\in\beta} {}_BD^1_k\, dX_k = 0. \\ &\quad \cdots\cdots\cdots \\ &\sum_{k\in\beta} {}_BD^{p-1}_k\, dX_k = 0, \\ &\sum_{k\in\beta} {}_BD^p_k\, dX_k = dX_p, \\ &\quad \cdots\cdots\cdots \\ &\sum_{k\in\beta} {}_BD^n_k\, dX_k = 0. \end{aligned} \tag{8}$$

This is a linear system of n equations for n differentials dX_k. The determinant of the system

$$\det {}_B D^{\alpha}_{\beta} \equiv {}^{A}_{B}D \tag{9}$$

consists of the derivatives of the B-representation enumerated by the sets α and β. The determinant corresponding to an unknown dX_j is obtained by replacing the column $j\,(\in\beta)$ in the matrix ${}_B D^{\alpha}_{\beta}$ by $(0\ 0\ \ldots\ dX_p\ \ldots\ 0)$, where dX_p is on the place $p\,(\in\alpha)$. This equals dX_p multiplied by the cofactor (signed-minor or algebraic complement) of the element ${}_B D^{p}_{j}$ denoted by

$$\operatorname{cof} {}_B D^{\overline{\alpha(p)}}_{\overline{\beta(j)}} \equiv {}^{A}_{B}C^{p}_{j}. \tag{10}$$

Now one gets for dX_j

$$dX_j = {}^{A}_{B}D^{-1}\, {}^{A}_{B}C^{p}_{j}\, dX_p. \tag{11}$$

Next insert dR into (3) in the form

$$dR = \sum_{j\in\beta} {}_B\frac{\partial R}{\partial X_j}\, dX_j. \tag{12}$$

By inserting (11) and (12) into (3) dX_p is cancelled out:

$$ {}_A\frac{\partial R}{\partial X_p} = {}^{A}_{B}D^{-1} \sum_{j\in\beta} {}_B\frac{\partial R}{\partial X_j}\, {}^{A}_{B}C^{p}_{j} \tag{13}$$

which is the general formula for the transformation of a first derivative of any expression between the representations A and B.

It is of immediate interest to get the formula for the transformations of the derivatives of co-ordinate functions. By taking $R = X_s$ one gets

$$ {}_A\frac{\partial X_s}{\partial X_p} \equiv {}_A D^{s}_{p} = {}^{A}_{B}D^{-1} \sum_{j\in\beta} {}_B D^{s}_{j}\, {}^{A}_{B}C^{p}_{j} = {}^{A}_{B}D^{-1} \det {}_B D^{\alpha(p\to s)}_{\beta}, \tag{14}$$

where in the set α of the numerator expression p is replaced by s. This is illustrated in Appendix A. Trying to use this formula for exceptional values of indices one has to note the following:

1) if $p \notin \alpha$, then ${}_A D^{s}_{p}$ is not defined. This case must be excluded.

2) if $s \in \alpha$, then (14) leads to ${}_A D^{s}_{p} = \delta_{sp}$ in accordance with (7).

2. Second derivatives

In order to get the transformation rule of the second derivative on the surface

$$ {}_A\frac{\partial^2 X_s}{\partial X_p\, \partial X_q} \equiv {}_A D^{s}_{pq} = {}_A\frac{\partial}{\partial X_q}\, {}_A D^{s}_{p} = {}_A\frac{\partial}{\partial X_p}\, {}_A D^{s}_{q}, \tag{15}$$

one has to take in (13)

$$R = {}_A D^{s}_{q}. \tag{16}$$

On the left-hand side we get ${}_A D^{s}_{pq}$, on the right-hand side one has to use (14) to change R into the B-representation. One can get several forms of the transformation equation:

$$ {}_A D^s_{qp} = {}^A_B D^{-1} \sum_{i\in\beta} {}^A_B C^p_i \, {}_B \frac{\partial}{\partial X_i} \left({}^A_B D^{-1} \sum_{j\in\beta} {}_B D^s_j \, {}^A_B C^q_j \right), \tag{17} $$

$$ {}_A D^s_{qp} = {}^A_B D^{-3} \sum_{i\in\beta} {}^A_B C^p_i \left[{}^A_B D \sum_{j\in\beta} \left({}_B D^s_{ji} \, {}^A_B C^q_j + {}_B D^s_j \, {}^A_B C^q_{j,i} \right) - \sum_{j\in\beta} {}_B D^s_j \, {}^A_B C^q_j \, {}^A_B D_{,i} \right], \tag{18} $$

$$ {}_A D^s_{qp} = {}^A_B D^{-3} \, {}^A_B C^p_i \left[{}_B D^t_k \, {}^A_B C^t_k \left({}_B D^s_{ji} \, {}^A_B C^q_j + {}_B D^s_j \, {}^A_B C^q_{j,i} \right) - \right. $$

$$ \left. - {}_B D^s_j \, {}^A_B C^q_j \left({}_B D^u_{ik} \, {}^A_B C^u_k + {}_B D^u_k \, {}^A_B C^u_{k,i} \right) \right]. \tag{19} $$

In (19), t and u are arbitrarily fixed values from α, at lower indices the sum rule is observed, and we have denoted

$$ {}^A_B C^u_{j,i} \equiv {}_B \frac{\partial}{\partial X_i} \, {}^A_B C^u_j, \qquad {}^A_B D_{,i} \equiv {}_B \frac{\partial}{\partial X_i} \, {}^A_B D. \tag{20} $$

In order to get transformation rules for third derivatives one has to take R in (13) equal to a second derivative and apply equations (17), (18) or (19) of the second derivatives, etc. Thus, the procedure can be continued interminably. Since general formulas are too complicated we confine ourselves to some lower dimensions.

One can easily see that the other functions not involved in the change of variables are also not involved in the transformation equations. Thus, neither the number of functions nor the total dimensionality of the space $n+m$ matters in the formulas and only n should be fixed for special cases.

3. Case $n=1$

In this case, there is only one argument in either representation, A and B. Let these be X_a and X_b, respectively. One gets immediately

$$ {}^A_B D = {}_B D^a_b, \qquad {}^A_B C^a_b = 1. \tag{21} $$

From now on the representation index B (but not A) at the derivatives will be omitted.

Equation (14) gives us the formula of the hodograph transformations of the first derivatives

$$ {}_A D^s_a = \frac{D^s_b}{D^a_b}. \tag{22} $$

Second derivatives can be obtained from (19) by taking

$$ q=p=t=u=a, \qquad i=k=j=b. $$

The result is*

$$ {}_A D^s_{aa} = (D^a_b)^{-3} D^{[a}_b D^{s]}_{bb}. \tag{23} $$

If $s=a$, the equation vanishes as expected.

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* Throughout the paper we denote $A^{[ab]} \equiv A^{ab} - A^{ba}$.

In order to get the third derivatives one has to apply (13) adjusted for the case $n=1$:

$$_A\frac{dR}{dX_a}=({}_BD^a_b)^{-1}\,{}_B\frac{dR}{dX_b}\,. \tag{24}$$

Now take $R={}_AD^s_{aa}$ and use (23):

$$_AD^s_{aaa}=(D^a_b)^{-1}\frac{d}{dX_b}\left[(D^a_b)^{-3}D^{[a}_bD^{s]}_{bb}\right]=$$

$$=(D^a_b)^{-5}(-3D^a_{bb}D^{[a}_bD^{s]}_{bb}+D^a_bD^{[a}_bD^{s]}_{bbb}). \tag{25}$$

For the 4th derivatives take $R={}_AD^s_{aaa}$, then

$$_AD^s_{aaaa}=(D^a_b)^{-7}[-3D^a_bD^a_{bbb}D^{[a}_bD^{s]}_{bb}-7D^a_bD^a_{bb}D^{[a}_bD]^s_{bbb}+$$

$$+(D^a_b)^2(D^{[a}_{bb}D^{s]}_{bbb}+D^{[a}_bD^{s]}_{bbbb})+15(D^a_{bb})^2D^{[a}_bD^{s]}_{bb}. \tag{26}$$

These forms of transformation equations are not convenient for practical use. Actually, one has two different cases for each equation. In terms of other notions the meaning of these will be more transparent.

There are three essentially different co-ordinates:
1) the argument of the A-representation, let us denote it by a,
2) the argument of the B-representation, further called b,
3) functions in both representations, let the one of interest be f.

Since the representation is determined by the argument, one can drop the representation indices A and B and denote the derivatives as

$$f_a\equiv{}_A\frac{df}{da}\,. \tag{27}$$

Now in equation (22) $X_a=a$ and $X_s=a, b, f$, where $X_s=a$ leads to the trivial case $a_a=1$. We obtain two transformations

$$b_a=\frac{b_b}{a_b}=\frac{1}{a_b}\,,$$

$$f_a=\frac{f_b}{a_b}. \tag{28}$$

Second derivatives from equation (23) take the forms

$$b_{aa}=\frac{1}{a^3_b}(a_bb_{bb}-b_ba_{bb})=-\frac{a_{bb}}{a^3_b}\,, \tag{29}$$

$$f_{aa}=\frac{1}{a^3_b}(a_bf_{bb}-f_ba_{bb}),$$

and higher derivatives from (25) and (26),

$$b_{aaa}=a^{-5}_b(3a^2_{bb}-a_ba_{bbb}),$$

$$f_{aaa}=a^{-5}_b[-3a_{bb}(a_bf_{bb}-f_ba_{bb})+a_b(a_bf_{bbb}-f_ba_{bbb})], \tag{30}$$

$$b_{aaaa}=a^{-7}_b(10a_ba_{bb}a_{bbb}-a^2_ba_{bbbb}-15a^3_{bb}),$$

$$f_{aaaa}=a^{-7}_b(-4a^2_ba_{bbb}f_{bb}+10a_ba_{bb}a_{bbb}f_b-6a^2_ba_{bb}f_{bbb}+ \tag{31}$$

$$+a^3_bf_{bbbb}-a^2_ba_{bbbb}f_b+15a_ba^2_{bb}f_{bb}-15a^3_{bb}f_b).$$

4. General case $n=2$

Here it is convenient to denote the arguments as

$$\alpha_1 = X_\alpha, \quad \alpha_2 = X_{\bar\alpha}, \qquad \beta_1 = X_1, \quad \beta_2 = X_2. \tag{32}$$

The transition determinant is

$$ {}^{A}_{B}D = D_1^{[\alpha} D_2^{\bar\alpha]} \tag{33}$$

and the cofactors are

$$ {}^{A}_{B}C^p_j = (-1)^{(p+j)} D^{\bar p}_{\bar j}, \tag{34}$$

where $(p+j)$ is the sum of the row and column numbers in ${}^{A}_{B}D$, $\bar p$ is the number other than p in α.

The transformation of derivatives can be obtained from (14) and (19):

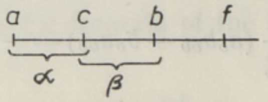
$$ {}_A D^s_p = \frac{D_1^{[s} D_2^{\bar p]}}{D_1^{[p} D_2^{\bar p]}}. \tag{35}$$

$$ {}_A D^s_{qp} = (-1)^{(p+q)}\, {}^{A}_{B}D^{-3} D^{\bar p}_{[2} \left(D_1^{[\alpha} D_2^{\bar\alpha]} D^{[s}_{1][1} D^{\bar q]}_{2} - D_1^{[s} D_2^{\bar q]} D^{[\alpha}_{1][1} D^{\bar\alpha]}_{2]} \right). \tag{36}$$

5. Case $n=2$, $\alpha \cap \beta \neq \theta$.

Here specified cases are discussed. Take first the case where only one argument is changed from one representation to the other one. Let us denote the variables:

$$\begin{aligned} &a \text{ — argument in } A, \text{ function in } B,\\ &c \text{ — argument in both } A \text{ and } B,\\ &b \text{ — argument in } B, \text{ function in } A,\\ &f \text{ — function in both } A \text{ and } B \end{aligned} \tag{37}$$



If there are more functions, they are not involved in each other's transformations and can be treated independently.* The transition determinant is

$$ {}^{A}_{B}D = \begin{vmatrix} {}_B a_c & {}_B a_b \\ {}_B c_c & {}_B c_b \end{vmatrix} = -{}_B a_b; \tag{38}$$

for in the B-representation

$$ {}_B c_b = 0, \quad {}_B c_c = 1. \tag{39}$$

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* See also the remark after (54).

First derivatives can be obtained from (35) (index B is omitted on the right)

$$ {}_A b_a = \frac{1}{a_b}, \qquad (40) $$

$$ {}_A b_c = -\frac{a_c}{a_b}, $$

$$ {}_A f_a = \frac{f_b}{a_b}, \qquad (41) $$

$$ {}_A f_c = \frac{f_c a_b - a_c f_b}{a_b}. $$

The equation for second derivatives (36) takes the form

$$ {}_A D^s_{qp} = (-1)^{(p+q)} a_b^{-3} D^{\bar{p}}_{[b} \left(D^a_b D^{[s}_{c][c} D^{\bar{q}]}_{b]} - D^{[s}_c D^{\bar{q}]}_b D^a_{c]b} \right), \qquad (42) $$

where the indices a, b, c stand for the corresponding variables. In particular, (42) leads to

$$ {}_A b_{aa} = -a_b^{-3} a_{bb}, $$

$$ {}_A b_{ac} = -a_b^{-3} (a_b a_{bc} - a_c a_{bb}), \qquad (43) $$

$$ {}_A b_{cc} = -a_b^{-3} (a_b^2 a_{cc} - 2 a_b a_c a_{bc} + a_c^2 a_{bb}), $$

$$ {}_A f_{aa} = a_b^{-3} (a_b f_{bb} - a_{bb} f_b), $$

$$ {}_A f_{ac} = a_b^{-3} [a_b (a_b f_{bc} - a_c f_{bb}) + f_b (a_c a_{bb} - a_b a_{bc})], \qquad (44) $$

$$ {}_A f_{cc} = a_b^{-3} [a_b (a_b^2 f_{cc} - 2 a_b a_c f_{bc} + a_c^2 f_{bb}) - $$

$$ - f_b (a_b^2 a_{cc} - 2 a_b a_c a_{bc} + a_c^2 a_{bb})]. $$

The higher derivatives being cumbersome to present, we confine ourselves to the second derivatives. Yet for further reference we present also some of the simplest of higher ones:

$$ {}_A b_{aaa} = a_b^{-5} (3 a_{bb}^2 - a_b a_{bbb}), \qquad (45) $$

$$ {}_A b_{ccc} = a_b^{-5} (a_b a_c^3 a_{bbb} - 3 a_b^2 a_c^2 a_{bbc} + 3 a_b^3 a_c a_{bcc} - $$

$$ - a_b^4 a_{ccc} - 3 a_c^3 a_{bb}^2 + 9 a_b a_c^2 a_{bb} a_{bc} - \qquad (46) $$

$$ - 3 a_b^2 a_c a_{bb} a_{cc} - 6 a_b^2 a_c a_{bc}^2 + 3 a_b^3 a_{bc} a_{cc}), $$

$$ {}_A b_{aaaa} = a_b^{-7} (-a_b^2 a_{bbbb} + 10 a_b a_{bb} a_{bbb} - 15 a_{bb}^3). \qquad (47) $$

6. Case $n=2$, $\alpha \cap \beta = \theta$

In this case, the arguments do not coincide in A and B. Let us denote for convenience

$$ \begin{aligned} &a, \bar{a} \text{ — arguments in } A, \\ &b, \bar{b} \text{ — arguments in } B, \qquad (48) \\ &f \text{ — a function in both } A \text{ and } B. \end{aligned} $$

Equation (35) for the first derivatives reads now

$$ {}_AD^s_p = \frac{D^{[s}_{b} D^{\bar{p}]}_{\bar{b}}}{D^{[p}_{b} D^{\bar{p}]}_{\bar{b}}} \tag{49} $$

which, applied to different functions, gives

$$ \begin{aligned} {}_Ab_a &= \bar{a}_{\bar{b}} / (a_b \bar{a}_{\bar{b}} - a_{\bar{b}} \bar{a}_b), \\ {}_Af_a &= (f_b \bar{a}_{\bar{b}} - f_{\bar{b}} \bar{a}_b) / (a_b \bar{a}_{\bar{b}} - a_{\bar{b}} \bar{a}_b). \end{aligned} \tag{50} $$

Since a and $\bar{a}$ as well as b and $\bar{b}$ are interchangeable, one can easily get the remaining equations by $a \leftrightarrow \bar{a}$ and/or $b \leftrightarrow \bar{b}$.

The second derivatives are given by (36):

$$ {}_AD^s_{qp} = (-1)^{(p+q)} (D^{[a}_{b} D^{\bar{a}]}_{\bar{b}})^{-3} D^{\bar{p}}_{[b} (D^{[a}_{b} D^{\bar{a}]}_{\bar{b}} D^{[s}_{b][b} D^{\bar{q}]}_{\bar{b}]} - D^{[s}_{b} D^{\bar{q}]}_{\bar{b}} D^{[a}_{b][b} D^{\bar{a}]}_{\bar{b}]}) \tag{51} $$

or, explicitly

$$ \begin{aligned} {}_Ab_{aa} &= D^{-3}[\bar{a}_{\bar{b}} (D\bar{a}_{\bar{b}b} - D_b \bar{a}_{\bar{b}}) - \bar{a}_b (D\bar{a}_{\bar{b}\bar{b}} - D_{\bar{b}} \bar{a}_{\bar{b}})], \\ {}_Ab_{a\bar{a}} &= D^{-3}[a_b (D\bar{a}_{\bar{b}\bar{b}} - D_{\bar{b}} \bar{a}_{\bar{b}}) - a_{\bar{b}} (D\bar{a}_{\bar{b}b} - D_b \bar{a}_{\bar{b}})], \end{aligned} \tag{52} $$

$$ \begin{aligned} {}_Af_{aa} &= D^{-3} (\bar{a}_{\bar{b}} F_b - \bar{a}_b F_{\bar{b}}), \\ {}_Af_{a\bar{a}} &= D^{-3} (a_b F_{\bar{b}} - a_{\bar{b}} F_b), \end{aligned} \tag{53} $$

where

$$ \begin{aligned} D &= a_b \bar{a}_{\bar{b}} - a_{\bar{b}} \bar{a}_b; \qquad D_q \equiv (D)_q, \\ F_q &= D (f_b \bar{a}_{\bar{b}} - f_{\bar{b}} \bar{a}_b)_q - D_q (f_b \bar{a}_{\bar{b}} - f_{\bar{b}} \bar{a}_b). \end{aligned} \tag{54} $$

As for the other second derivatives, see the remark after (50). The formulas for higher derivatives and higher dimensions are considerably more complicated, but it should be noted that in some cases the formulas obtained can be applied also to higher dimensions. The point is that as the additional functions do not matter in the transformations, so the arguments common to both representations, A and B, not included in the derivative under discussion, are not involved in the transformation formulas and may be omitted. The result is that the effective n is less than n was originally. The fact is reflected in the similarities within the pairs of equations (28_1), (40_1); (28_2), (41_1); (29_1), (43_1); (29_2), (44_1); (30_1), (45); (31_1), (47) and, for instance, one can easily find f_{aaa} and f_{aaaa} in this case by using (30_2) and (31_2), respectively.

The first derivatives for any dimension do not present any problem.

7. Hodograph transforms of physical equations

Hodograph transformations applied to equations of mathematical physics link different equivalent descriptions of the same phenomena. In general, different representations do not resemble each other (see Appendix B). Some equations, however, are hodograph invariant, i. e. the expression equated to zero retains the same form that may in general be multiplied by a non-zero factor. One of these is the Lagrange equation of minimal surfaces (B1). In physics, hodograph invariant equations leading, for instance, to an equivalence of fields and space-time co-ordinates have a specific role [2]. Proceeding from that we refer to a way of the formation of hodograph invariant equations — one has to equate to zero an expression consisting of the variables and their derivatives of all representations in a fully symmetric way. Thus, the invariance being achieved, as usual, a choice should be made between arguments and functions, i. e. the ingredient of the expression should be transformed into a certain representation. The invariance, as a matter of course, remains. Some examples of the procedure and the results have been given in [2], though the notations are different from those of the present paper.

The general characteristic features of hodograph transforms are:

1) The order of the expression in preserved.
2) The degree of the expression with respect to every variable is preserved.
3) If the expression consists only of derivatives and constants, the same applies also to its hodograph transforms. In other cases, the inclusion of functions can be converted into that of arguments or vice versa, which may sometimes be of use in solving the equation.
4) The symmetry group of the equation preserves [6].

One should remember that the solving of one hodograph transform immediately gives the corresponding solutions of the other transforms — simply by taking inverse functions.

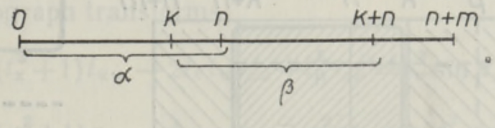
Our considerations do not include metrics. So the results are valid for any co-ordinate system without modifications. The hypersurfaces referred to in the beginning that correspond to the solutions of equations in this chapter, should only be correspondingly differentiable. Care should be taken when the transformations are applied to certain points where a first derivative vanishes, which is equivalent to a local reduction of the set of arguments, and transformed derivatives tend to infinity. Such points, if needed, should be treated as limits.

The author is grateful to M. Kõiv for fruitful discussions.

APPENDIX A

Structure of the determinants in equation (14)

In order to illustrate the formation of the determinants in (14), it is convenient to arrange all variables on a line in the following way



The matrices of the first derivatives ${}_AD_t^u$ and ${}_BD_t^u$ are shown in the Figure.

The replacement made to get the numerator in (14) is shown. Note that ${}_B^AD$ is reducible to a k-row determinant. It can also be seen that the transformation equations turn simpler if $n<s\leqslant k+n$ and/or $0<p\leqslant k$.

APPENDIX B

Hodograph transforms of some soliton equations

Let us take some physical evolution equations as examples.

1. The Lagrange equation of minimal surfaces [7]

$$(1+u_x^2)\,u_{tt}-2u_xu_tu_{xt}+(1+u_t^2)\,u_{xx}=0 \quad \text{(B1)}$$

is known as hodograph invariant [2]. It has solutions

$$u=F(x\pm it), \quad \text{(B2)}$$

where F denotes an arbitrary function. From the hodograph invariance of (B1) it follows that the functions

$$u=F(x)\pm it \quad \text{(B3)}$$

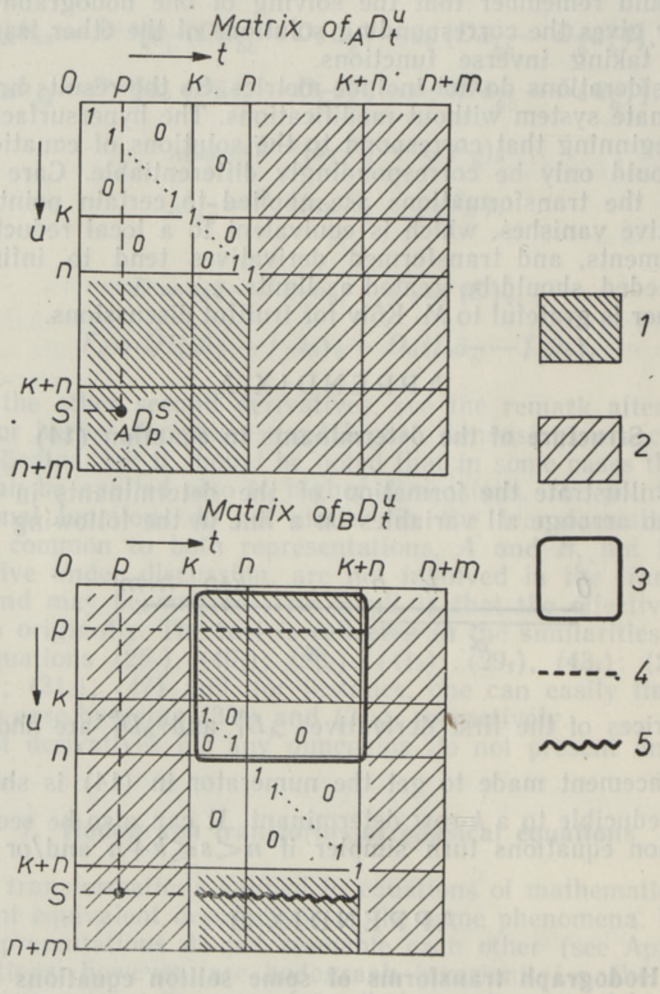
resulting from the form of the inverse functions of (B2), are also solutions of (B1).

The pseudo-Euclidean version of the Lagrange equation — the Born—Infeld equation

$$(1-u_t^2)\,u_{xx}+2u_xu_tu_{xt}-(1+u_x^2)\,u_{tt}=0 \quad \text{(B4)}$$

has soliton solutions of an arbitrary form but fixed speed

$$u=F(x\pm t). \quad \text{(B5)}$$



1 — regular region; 2 — undefined region; 3 — the transition determinant ^A_BD; forming the numerator in (14), line 4 is to be replaced by line 5.

Equation (B4) is no more hodograph invariant. Instead, it leads to

$$(t_u^2-1)\,t_{xx}-2t_xt_ut_{xu}+(t_x^2-1)\,t_{uu}=0 \qquad \text{(B6)}$$

and

$$(x_t^2-1)\,x_{uu}-2x_ux_tx_{ut}+(x_u^2+1)\,x_{tt}=0. \qquad \text{(B7)}$$

Soliton solution (B5) leads to «vertical soliton» solutions for (B6) and (B7)

$$t=F(u)\pm x, \quad x=F(u)\pm t. \qquad \text{(B8)}$$

2. The Korteweg de Vries equation [8]

$$u_t+cuu_x+u_{xxx}=0 \qquad \text{(B9)}$$

takes rather a complicated form in the t-representation

$$t_u^4-cut_u^4t_x+t_ut_x^3\,t_{uuu}-3t_u^2t_x^2\,t_{uux}+3t_u^3t_xt_{uxx}-$$

$$-\,t_ut_{xxx}-3t_x^3\,t_{uu}^2+9t_ut_x^2\,t_{uu}t_{ux}- \qquad \text{(B10)}$$

$$-\,3t_u^2t_xt_{uu}t_{xx}-6t_u^2t_xt_{ux}^2+3t_u^3t_{ux}t_{xx}=0,$$

but in the x-representation it is more acceptable —

$$x_u^4x_t-cux_u^4+x_ux_{uuu}-3x_{uu}^2=0. \qquad \text{(B11)}$$

To a known soliton solution of (B9)

$$u=\frac{12}{c}\,\mathrm{sch}^2\,(x-4t+\delta) \qquad \text{(B12)}$$

one can easily find corresponding solutions of (B10) and (B11), for instance

$$x=\mathrm{arsh}\,\sqrt{cu/12}-\delta+4t. \qquad \text{(B13)}$$

3. The sine-Gordon equation in the form [7]

$$u_{xx}-u_{tt}=\sin u \qquad \text{(B14)}$$

leads to the hodograph transforms

$$\begin{aligned}(t_x^2+1)\,t_{uu}-2t_x\,t_u\,t_{xu}+t_u^2\,t_{xx}&=t_u^3\sin u,\\(x_t^2+1)\,x_{uu}-2x_tx_ux_{tu}+x_u^2x_{tt}&=-x_u^3\sin u\end{aligned} \qquad \text{(B15)}$$

with some resemblance to the Born—Infeld equation. Starting from the other form of the sine-Gordon equation

$$u_{xt}=\sin u, \qquad \text{(B16)}$$

one reaches the simpler transforms

$$t_xt_{uu}-t_u\,t_{xu}=t_u^3\sin u,$$

$$x_tx_{uu}-x_ux_{ut}=-x_u^3\sin u.$$

The known solutions can be transformed trivially.

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Received
Nov. 27, 1985

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HODOGRAAFTEISENDUSED

On välja töötatud suvalist järku osatuletiste hodograafteisenduste meetod mis tahes dimensiooni puhul. Hodograafteisendus on argumentide ja funktsioonide osade vahetus. Saadud valemid esimeste tuletiste jaoks on kergesti rakendatavad iga dimensiooni korral. On antud ka hulk erijuhulisi valemeid madalamate järkude ja dimensioonide tarvis ning analüüsitud tulemuste rakendatavust osatuletistega diferentsiaalvõrranditele.

А. АЙНСААР

ГОДОГРАФ-ПРЕОБРАЗОВАНИЯ

Выработан метод осуществления годограф-преобразований частных производных любого порядка при произвольной размерности. Годограф-преобразование — это перестановка аргументов и функций. Полученные формулы для первых производных легко применимы при любой размерности. Приведены также некоторые явные соотношения для нижних порядков и размерностей. Обсуждены применения при уравнениях с частными производными.