EESTI NSV TEADUSTE AKADEEMIA TOIMETISED, 26. KÖIDE
FÜÜSIKA * MATEMAATIKA. 1977, NR. 4

ИЗВЕСТИЯ АКАДЕМИИ НАУК ЭСТОНСКОЙ ССР. ТОМ 26
ФИЗИКА * МАТЕМАТИКА. 1977, № 4

512.25/.26+519.3:330.115

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ON g-CONCAVE FUNCTIONS AND PROBABILITY MEASURES

The notions of convexity and concavity play an important role in many aspects of mathematical programming. To extend the classes of convex and concave functions, various concepts of generalized convexity and concavity have been introduced and investigated by several authors, e. g., [1–7]. In particular, Avriel and Zang [1] introduced the notion of G-concave function. In the present paper the concept of G-concavity is, in a slightly modified form, applied to probability measures and investigated from the stochastic programming point of view. The results generalize those of Prékopa [2].

Let $g: G_1 \to G_2$, where $G_1, G_2 \subseteq R^1$, be a strictly monotone increasing function.

D e f i n i t i o n 1. *A function $f: F_1 \to F_2$, where $F_1 \subseteq R^n$ is a convex set and $F_2 \subseteq G_1$, is called g-concave (g-convex) on F_1 if $g(f(x))$ is concave (convex) on F_1.*

In other words, $f(x)$ is g-concave on F_1 if

$$g(f(\lambda x_1 + (1-\lambda)x_2)) \geqslant \lambda g(f(x_1)) + (1-\lambda) g(f(x_2)) \tag{1}$$

and g-convex if

$$g(f(\lambda x_1 + (1-\lambda)x_2)) \leqslant \lambda g(f(x_1)) + (1-\lambda) g(f(x_2)) \tag{2}$$

for all $x_1, x_2 \in F_1$ and $\lambda \in [0, 1]$.

It is obvious that if $f(x)$ is g-convex on some set F then $-f(x)$ is g-concave on this set.

The concept of g-concavity differs from that of G-concavity introduced by Avriel and Zang [1] only by the fact that g is a strictly monotone increasing function, whereas G is supposed to be only strictly monotone.

The logarithmic concavity [3] is a particular case of the g-concavity. In this case $g(t) = \ln t$. On the other hand, there are functions which are not logarithmic concave but are g-concave. For example, $f(x) = x^3$, $x \in R^1$ is negative on the set and hence it cannot be logarithmic concave but it is g-concave if $g(t) = \sqrt[3]{t}$.

Let us establish some properties of g-concave functions.

T h e o r e m 1. *A function which is concave on F_1 is g-concave on F_1 with respect to every $g(t)$, concave on F_2.*

P r o o f is based on the fact that a monotone increasing concave function of a concave function is concave.

T h e o r e m 2. *Let $g(t)$ be a differentiable function on F_2. If $f(x)$ is differentiable and g-concave on F_1 then it is pseudo-concave* on F_1.*

* A differentiable function $h: F_1 \to F_2$ is called pseudo-concave on F_1 if for any $x_1, x_2 \in F_1$ $h'(x_1)(x_2 - x_1) \leqslant 0$ implies $h(x_2) \leqslant h(x_1)$.

Theorem 3. *A function which is g-concave on some set is quasi-concave ** on that set.*

Proofs of theorems 2 and 3 are readily obtained from [1].

The converses of the theorems 2 and 3 do not hold. One can find pseudo-concave and, all the more, quasi-concave functions $f(x)$ for which no such strictly monotone increasing function $g(t)$ exists that $g(f(x))$ is concave. For example, let us consider the function

$$\varphi(x)=\begin{cases}-\frac{1}{4}x^2+\frac{1-\pi}{4}, & \text{if } x\leqslant 1,\\ -\operatorname{arctg} x, & \text{if } x>1,\end{cases}$$

where $x\in F_1=R^1$. It can be easily shown that this function is pseudo-concave. To show that $\varphi(x)$ is not g-concave we shall prove the following lemma.

Lemma. *On the set $[a,\infty]$ concave, monotone decreasing and bounded function $h(x)$ is constant on this set.*

To prove this statement, let us denote $m=\inf\limits_{x\in[a,\infty)} h(x)$ and $M=\sup\limits_{x\in[a,\infty)} h(x)=h(a)$. If $h(x)$ is not constant then $M>m$. Let us choose $\varepsilon>0$, $\bar{x}$ so that $h(\bar{x})<m+\varepsilon$, $x_1=a$ and x_2 so that $\bar{x}=\frac{x_1+x_2}{2}$. Because of the concavity of $h(x)$ we have $m+\varepsilon>h(\bar{x})=h\left(\frac{x_1+x_2}{2}\right)\geqslant\frac{1}{2}h(x_1)+\frac{1}{2}h(x_2)=\frac{1}{2}M+\frac{1}{2}h(x_2)\geqslant\frac{1}{2}M+\frac{1}{2}m$. Therefore, $m+\varepsilon>\frac{1}{2}(M+m)$. If we take now $\varepsilon=\frac{M-m}{2}>0$, we come to the contradiction $\frac{m+M}{2}>\frac{m+M}{2}$ and, hence, $h(x)$ must be constant.

Let us assume now that there exists a strictly monotone increasing function $g(t)$ so that $g(\varphi(x))$ is a concave function. Then $g(t)$ must be defined on the set $\left(-\infty,\frac{1-\pi}{4}\right]$. Therefore $g_0=g(\varphi(0))=g\left(\frac{1-\pi}{4}\right)$ and $g_1=g(\varphi(+\infty))=g\left(-\frac{\pi}{2}\right)$ are finite. So $g(\varphi(x))$ is a concave monotone decreasing and bounded function on the set $[0,\infty)$. By the lemma, $g(\varphi(x))$ is constant. But this contradicts the condition that $g(t)$ is strictly monotone.

Theorem 4. *If $f(x)$ is g-concave on F_1, $X\subseteq F_1$ is a convex set and $\max\limits_{x\in X} f(x)$ exists then $g(\max\limits_{x\in X} f(x))=\max\limits_{x\in X} g(f(x))$.*

Proof. Let $\max\limits_{x\in X} f(x)=f(x^*)$, $x^*\in X$. Then $f(x)\leqslant f(x^*)$ for every $x\in X$. As $g(t)$ is a strictly increasing function, $g(f(x))\leqslant g(f(x^*))$. Consequently, $\max\limits_{x\in X} g(f(x))=g(f(x^*))=g(\max\limits_{x\in X} f(x))$.

Theorem 4 can be used to replace the problem $\max\limits_{x\in X} f(x)$ with an equivalent concave problem $\max\limits_{x\in X} g(f(x))$ as soon as $f(x)$ is g-concave.

In stochastic programming problems containing probabilistic cost functions or restrictions it is important to have conditions under which these

** A function $h(x)$ is called quasi-concave on a convex set $F_1\subseteq R^n$ if $h(\lambda x_1+(1-\lambda)x_2)\geqslant\min[h(x_1),\,h(x_2)]$ for every $x_1,\,x_2\in F_1$ and every $\lambda\in[0,\,1]$.

functions are g-concave. For this purpose we shall introduce the following notion.

Definition 2. *A probability measure P, defined on a σ-algebra Σ of sets from R^m, is called g-concave if for every pair of convex sets* $A, B \in \Sigma$ and $\lambda \in (0, 1)$ *the inequality*

$$g\{P[\lambda A+(1-\lambda)B]\} \geqslant \lambda g\{P[A]\}+(1-\lambda)g\{P[B]\} \tag{3}$$

holds.

This notion is a generalization of the logarithmic concave probability measure considered in [2].

Theorem 5. *If $f(x, y)$ is quasi-concave on R^{n+m}, $x \in R^n$ and $y \in R^m$ is a random vector corresponding to a g-concave probability measure P then the probabilistic function $v(x) = P[f(x, y) \geqslant 0]$ is g-concave.*

Proof. Let us denote $H(x) = \{y : f(x, y) \geqslant 0\}$. Then

$$v(x) = P[H(x)]. \tag{4}$$

As the function $f(x, y)$ is quasi-concave, so

$$f(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2) \geqslant \min[f(x_1, y_1), f(x_2, y_2)]. \tag{5}$$

Any element of the set $\lambda H(x_1)+(1-\lambda)H(x_2)$ can be expressed in the form $\lambda y_1+(1-\lambda)y_2$, where $y_1 \in H(x_1)$, $y_2 \in H(x_2)$, i.e., $f(x_1, y_1) \geqslant 0$ and $f(x_2, y_2) \geqslant 0$. Because of (5) $f(\lambda x_1+(1-\lambda)x_2, \lambda y_1+(1-\lambda)y_2) \geqslant 0$, that is $\lambda y_1+(1-\lambda)y_2 \in H(\lambda x_1+(1-\lambda)x_2)$. Therefore, $H(\lambda x_1+(1-\lambda)x_2) \supseteq \lambda H(x_1)+(1-\lambda)H(x_2)$ and $P\{H(\lambda x_1+(1-\lambda)x_2)\} \geqslant P\{\lambda H(x_1)+(1-\lambda) \times H(x_2)\}$. Taking into account the monotonity of $g(t)$ and the g-concavity of the measure P we get $g\{P[H(\lambda x_1+(1-\lambda)x_2)]\} \geqslant g\{P[\lambda H(x_1)+(1-\lambda)H(x_2)]\} \geqslant \lambda g\{P[H(x_1)]\}+(1-\lambda)g\{P[H(x_2)]\}$, and hence, because of (4), $g(v(\lambda x_1+(1-\lambda)x_2)) \geqslant \lambda g(v(x_1))+(1-\lambda)g(v(x_2))$.

In conclusion we shall demonstrate that *the probability measure corresponding to the one-dimensional Cauchy distribution is not logarithmic concave but it is g-concave if* $g(t) = \operatorname{tg}[\pi(t-1/2)]$.

In one-dimensional case a convex set is a segment, a semi-segment or an interval, i.e., it is determined by a couple $\langle u, v\rangle$, where u, v are either numbers or $+\infty$, or $-\infty$ and $u<v$. If $A=\langle u_1, v_1\rangle$, $B=\langle u_2, v_2\rangle$ then $\lambda A+(1-\lambda)B=\langle \lambda u_1+(1-\lambda)u_2, \lambda v_1+(1-\lambda)v_2\rangle$ and (3) takes the form

$$g\{P\langle \lambda u_1+(1-\lambda)u_2, \lambda v_1+(1-\lambda)v_2\rangle\} \geqslant \lambda g\{P\langle u_1, v_1\rangle\}+(1-\lambda)g\{P\langle u_2, v_2\rangle\}$$

that coincides with the condition of g-concavity of a function of two variables.

Let us show that the function $P\{\langle u, v\rangle\} = \frac{1}{\pi}(\operatorname{arctg} v - \operatorname{arctg} u)$ corresponding to the one-dimensional Cauchy distribution is g-concave if $g(t) = \operatorname{tg}[\pi(t-1/2)]$ but is not logarithmic concave. To demonstrate it we shall evaluate the matrices of second order derivatives of the functions $\ln[1/\pi(\operatorname{arctg} v - \operatorname{arctg} u)]$ and $\operatorname{tg}[\pi(1/\pi(\operatorname{arctg} v - \operatorname{arctg} u)-1/2)]$

$$\frac{1}{(\operatorname{arctg} v - \operatorname{arctg} u)^2}\begin{pmatrix} \dfrac{-1+2u(\operatorname{arctg} v - \operatorname{arctg} u)}{(1+u^2)^2} & \dfrac{1}{(1+u^2)(1+v^2)} \\ \dfrac{1}{(1+u^2)(1+v^2)} & \dfrac{-1-2v(\operatorname{arctg} v - \operatorname{arctg} u)}{(1+v^2)^2} \end{pmatrix} \tag{6}$$

and

$$\frac{2}{(u-v)^3}\begin{pmatrix} 1+v^2 & -uv-1 \\ -uv-1 & 1+u^2 \end{pmatrix}, \tag{7}$$

respectively.

If $-1+2u(\operatorname{arctg} v-\operatorname{arctg} u)>0$ (e. g. $u=1$, $v=4$) then the element in the first row and first column

$$\frac{-1+2u(\operatorname{arctg} v-\operatorname{arctg} u)}{(\operatorname{arctg} v-\operatorname{arctg} u)^2(1+u^2)^2}>0.$$

Consequently, (6) is not a negative semi-definite matrix for every $u, v\in R^1$.

But it is easy to see that the matrix (7) is negative semi-definite for every $u, v\in R^1$ and hence the function $\operatorname{tg}[\pi(1/\pi(\operatorname{arctg} v-\operatorname{arctg} u)-1/2)]$ is concave.

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Received
Nov. 23, 1976

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g-NÕGUSATEST FUNKTSIOONIDEST JA TÕENÄOSUSMÕÕTUDEST

Defineeritakse g-nõgusa funktsiooni ja g-nõgusa tõenäosusmõõdu mõisted kui vastavält logaritmiliselt nõgusa funktsiooni ja logaritmiliselt nõgusa tõenäosusmõõdu üldistused. Vaadeldakse g-nõgusate funktsioonide omadusi ja esitatakse tingimused, millal tõenäosusfunktsioon $v(x)=P[f(x, y)\geqslant 0]$ on g-nõgus. Näidatakse, et ühedimensionaalsele Cauchy jaotusele vastav tõenäosusmõõt on g-nõgus, kuid ei ole logaritmiliselt nõgus.

Эбу ТАММ

О g-ВОГНУТЫХ ФУНКЦИЯХ И ВЕРОЯТНОСТНЫХ МЕРАХ

Определяются понятия g-вогнутой функции и g-вогнутой вероятностной меры как обобщения логарифмически вогнутой функции и логарифмически вогнутой вероятностной меры соответственно. Рассматриваются некоторые свойства g-вогнутых функций и даются условия, при которых функция вероятности $v(x)=P[f(x, y)\geqslant 0]$ g-вогнута. Показывается, что вероятностная мера, соответствующая одномерному распределению Коши, g-вогнута, но логарифмически невогнута.