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# INHOMOGENEOUS WORLD MODELS AND HIGH-FREQUENCY APPROXIMATION

Cosmological models, homogeneous only on the average, are studied by using the high-frequency approximation. It is shown that the density inhomogeneities give raise to an additional energy-momentum tensor, realizing the feedback between the evolution of the structure and the overall dynamics of the universe. This tensor can be described in terms of a negative "gravitational pressure" and a phenomenological "internal energy". The possible consequences of the results are discussed in brief.

**1. Introduction.** In his 1968 papers [1, 2] R. A. Isaacson studied gravitational waves of short wavelength, picturing them as small-amplitude ripples on a smoothly changing background geometry. His approach was a success — he managed to show that energy can be ascribed to gravitational waves, though nonlocal, but working to curve up the background as the usual energy. Thus small-scale perturbations of metric can act to change the background itself.

A similar problem does arise in cosmology. As the exact picture of the real complicated universe is very difficult to realize mathematically, one supposes matter to be distributed smoothly and takes the geometry corresponding to this average distribution of matter as the cosmological background. Considering now the structure of matter as being manifested by the perturbations of the metric tensor, a natural question arises — whether and to what extent do the fluctuations influence the background geometry? So far it has been supposed that no such influence occurs. While the assumption is evidently valid for small density perturbations, then for a universe with the fully developed structure and large density perturbations it seems not to be justified. In fact, R. A. Isaacson's results suggest that there exists a kind of "interaction" between the evolution of the structure and the large-scale properties of world models. As his high-frequency condition is well satisfied in the real universe, it will be worth using his approach to clarify the problem in cosmology, too. This is the subject of the present paper. In order not to overcharge the results and their consequences with details, the methods used are only outlined here. A detailed discussion will be given in the papers [3–5] by the author, appearing elsewhere. For the notation and sign conventions adopted we refer to the paper [3]. In general they coincide with the "spacelike" conventions of C. W. Misner et al. [6].

**2. High-frequency approximation.** In the following we suppose that the space-time admits "steady" coordinate systems with the metric

$$g_{ij}=\gamma_{ij}+\varepsilon h_{ij}, \tag{2.1}$$

where $\gamma_{ij}$ is the slowly changing background metric tensor and $h_{ij}$ a rapidly fluctuating tensor. As shown by R. A. Isaacson [1], the ratio of the characteristic lengths of variation of $\gamma_{ij}$ and $h_{ij}$ is of the order $\varepsilon$, too. Thus, only one dimensionless parameter $\varepsilon$ remains, and we can write

$$\gamma_{ij,kl} \sim \gamma_{ij,k} \sim \gamma_{ij} = O(1),$$
$$h_{ij,kl} \sim \varepsilon^{-1} h_{ij,k} \sim \varepsilon^{-2} h_{ij} = O(\varepsilon^{-2}). \tag{2.2}$$

Such order-of-magnitude estimates can be written down for all tensors, the principal point being that while differentiation does not alter the order of magnitude of the background quantities, the high-frenquency quantities get larger by the factor $O(\varepsilon^{-1})$. Expanding now the Ricci tensor into a series with respect to $\varepsilon$, we obtain

$$R_{ij} = \overset{(0)}{R}_{ij} + \varepsilon \overset{(1)}{R}_{ij} + \varepsilon^2 \overset{(2)}{R}_{ij} + \dots \tag{2.3}$$

Indices are raised and lowered by the background metric, and the coefficients $\overset{(k)}{R}_{ij}$ are tensors in the background geometry. As second-order derivatives enter $R_{ij}$, the order of the terms can be estimated as

$$\overset{(0)}{R}_{ij} = O(1), \quad \varepsilon \overset{(1)}{R}_{ij} = O(\varepsilon^{-1}), \quad \varepsilon^2 \overset{(2)}{R}_{ij} = O(1). \tag{2.4}$$

To study the effect of density inhomogeneities in its pure form, we suppose that no pressure is present, and write

$$T^{ij} = (\bar{\varrho} + \varepsilon^{-1}\tilde{\varrho})(\bar{u}^i + \varepsilon \tilde{u}^i)(\bar{u}^j + \varepsilon \tilde{u}^j). \tag{2.5}$$

The decomposition of the four-velocity follows directly from the high-frequency assumption, and the term $\varepsilon^{-1}\tilde{\varrho}$ represents large-amplitude density fluctuations. Using now the identity

$$g^{ij} = \gamma^{ij} - \varepsilon h^{ij} + \dots, \tag{2.6}$$

the Einstein equations without the cosmological term can be splitted up into two equations — one governing the evolution of inhomogeneities (the wave equation)

$$\overset{(1)}{G}_{ij} = \frac{\varkappa}{\varepsilon^2} \tilde{\varrho} \bar{u}_i \bar{u}_j \tag{2.7}$$

and the other, which is the Einstein equation for the background plus some fluctuating terms of order $O(1)$. In order to obtain an equation describing the dynamics of the background and to bring it into accord with our "average" geometry conception, we average now the $O(1)$ equation over three-dimensional space slices. The averaging procedure used is analogous to that of R. A. Isaacson. It preserves the tensor character of the averaged quantities and, as there exist intrinsically defined three-spaces in the case of homogeneous world models, we confine ourselves to homogeneous backgrounds in this sense.

It can be shown that the background terms do not change under averaging, and we finally come to

$$\overset{(0)}{G}_{ij} = \varkappa(\overset{(0)}{T}_{ij} + \overset{P}{T}_{ij}). \tag{2.8}$$

Thus, the inhomogeneities work to introduce an effective energy-momentum tensor. However, this tensor cannot be determined uniquely by the method indicated above. As we use the background metric tensor and not the full $g_{ij}$, as it would be proper, for raising and lowering indices, the equations obtained from the three (covariant, mixed and contravariant) forms of the Einstein equations are not equivalent. The expansion into a series fixes only a part of $\overset{P}{T}_{ij}$. To fix the full tensor we have to consider gauge transformations.

The equations of motion can be expanded in an analogous way. The most significant result is that the fluctuating density evolves with the characteristic time scale equal to the global one. This condition is widely used in the derivation and, as will be seen, it has important consequences.

**3. Gauge invariance.** The equations obtained are tensor equations with respect to the background geometry, and thus manifestly covariant under large-scale coordinate transformations. But when performing a high-frequency (gauge) transformation

$$x'^i = x^i + \varepsilon\xi^i, \tag{3.1}$$

the background does not change, while

$$h'_{ij} = h_{ij} - 2\xi_{(i;j)}. \tag{3.2}$$

As physically relevant quantities and equations cannot depend upon the choice of coordinates, we have to demand our equations to be gauge invariant. Using the conditions put on $h_{ij}$ in order to preserve the "steadiness" of the system and not to introduce fictitious density perturbations, we can show that all the equations but the one obtained previously are gauge-invariant.

This equation (2.8) can now be made gauge-invariant too. It appears that the gauge invariance condition fixes uniquely the expression for $\overset{P}{T}_{ij}$ and it can be written as

$$\overset{P}{T}_{ij} = U^k_{ij}\langle\tilde{\varrho}\tilde{u}_k\rangle + H^{kl}_{ij}\langle\varrho\tilde{h}_{kl}\rangle - \frac{\varepsilon^2}{\varkappa}\langle G^{(2)}_{ij}\rangle, \tag{3.3}$$

where

$$U^k_{ij} = \bar{u}_i\delta^k_j + \bar{u}_j\delta^k_i \tag{3.4}$$

and

$$H^{kl}_{ij} = \bar{u}^l U^k_{ij} - \frac{1}{2}\bar{u}_i\bar{u}_j\gamma^{kl} - \frac{1}{2}\gamma_{ij}\left(\frac{1}{2}\gamma^{kl} + \bar{u}^k\bar{u}^l\right) + \frac{1}{2}\delta^k_i\delta^l_j. \tag{3.5}$$

While gauge invariance is usually used to simplify calculations only, it works here as a means for obtaining the final equations and thus has fundamental significance.

**4. Lorentz gauge.** Fixing now a special (Lorentz) gauge by the condition

$$\psi^k_{i;k} \equiv \left(h^k_i - \frac{1}{2}\delta^k_i h\right)_{;k} = 0, \tag{4.1}$$

the equations can be simplified considerably. The wave equation reduces to

$$\psi_{ij;k}^{\phantom{ij};k} = -\frac{2\varkappa}{\varepsilon^2}\tilde{\varrho}\bar{u}_i\bar{u}_j. \tag{4.2}$$

Hence

$$h_{ij} = 4\varphi\left(\frac{1}{2}\gamma_{ij} + \bar{u}_i\bar{u}_j\right), \tag{4.3}$$

where $\varphi$ is the solution of the generalized Poisson equation

$$\varphi_{;k}^{\phantom{;k};k} = -\frac{\varkappa}{2\varepsilon^2}\tilde{\varrho}. \tag{4.4}$$

The equations of motion can be written as

$$\tilde{\varrho}_{,i}\bar{u}^{i}+\varrho\tilde{\bar{u}}^{i}_{;i}+\varepsilon\tilde{\varrho}_{,i}\tilde{u}^{i}_{T}+\varepsilon\tilde{\varrho}\tilde{u}^{i}_{T;i}=0, \tag{4.5}$$

$$\tilde{u}^{i}_{T;a}\bar{u}^{a}=\varphi^{,i}_{T}, \tag{4.6}$$

where $\tilde{u}^{i}_{T}$ is the component of $\tilde{u}^{i}$, orthogonal to the mean four-velocity $\bar{u}^{i}$. Correspondingly, the derivative of $\varphi$ is taken in the same direction.

The effective energy-momentum tensor becomes now

$$T^{P}_{ij}=\bar{u}_{i}\langle\tilde{\varrho}\tilde{u}_{j}\rangle+\bar{u}_{j}\langle\tilde{\varrho}\tilde{u}_{i}\rangle-\frac{1}{2}(\gamma_{ij}+4\bar{u}_{i}\bar{u}_{j})\langle\tilde{\varrho}\varphi\rangle+\frac{2\varepsilon^{2}}{\varkappa}\langle\varphi_{,i}\varphi_{,j}\rangle. \tag{4.7}$$

**5. Stochastic approach.** As there is no hope presently to describe the complicated structure of the universe in full detail, we assume all our fluctuating fields to be of stochastic nature. To simplify the calculations we assume that no overall rotation is present. Then the three-spaces orthogonal to $\bar{u}^{i}$ coincide with the hypersurfaces where our averaging is performed. Now it can be shown that the mean momentum of the fluctuations

$$\langle\tilde{\varrho}\tilde{u}^{i}_{T}\rangle=0, \tag{5.1}$$

and the correlation

$$\langle\varphi_{,i}\varphi_{,j}\rangle=\frac{1}{3}(\gamma_{ij}+\bar{u}_{i}\bar{u}_{j})\langle\varphi_{,k}\varphi^{,k}\rangle. \tag{5.2}$$

In such a way the effective energy-momentum tensor becomes

$$T^{P}_{ij}=\frac{1}{3}\left(\bar{u}_{i}\bar{u}_{j}-\frac{1}{2}\gamma_{ij}\right)\langle\tilde{\varrho}\varphi\rangle. \tag{5.3}$$

Writing the full energy-momentum tensor in the form

$$T_{ij}=\left(\bar{\varrho}+\frac{1}{3}\langle\tilde{\varrho}\varphi\rangle\right)\bar{u}_{i}\bar{u}_{j}-\frac{1}{6}\langle\tilde{\varrho}\varphi\rangle\gamma_{ij}, \tag{5.4}$$

and comparing it to that of an ideal fluid we see that the effect of inhomogeneities can be interpreted as giving rise to the "cosmological pressure"

$$p^{*}=-\frac{1}{6}\langle\tilde{\varrho}\varphi\rangle \tag{5.5}$$

and the "internal energy"

$$\Pi^{*}=\frac{1}{2}\langle\tilde{\varrho}\varphi\rangle. \tag{5.6}$$

As it can be shown that

$$\langle\tilde{\varrho}\varphi\rangle\geqslant 0, \tag{5.7}$$

the pressure (5.5) is negative.

**6. Discussion.** The present approach demonstrates clearly the reason why the "gravitational pressure" and similar terms occur. They are due to the non-linear character of gravitational field and represent the interaction between the perturbations and the background. The effective energy-momentum tensor, given by (3.3) and in the Lorentz gauge by (4.7) is coordinate-invariant and has thus a definite physical meaning. In connection with the expression (5.3) of this tensor, which has been obtained assuming no rotation of the background, it has to be noted that its form does not resemble this particular condition and hence can be supposed to be valid in the general case. Anyway, most of the known cosmological solutions are non-rotational.

As we see, the feedback between the evolution of structure and the behaviour of the universe in large is realized by the effective energy-momentum tensor. As for the exact time behaviour of this quantity, it has to be calculated yet, but some important consequences can be drawn at once.

When applying the present approach to the stage of structure formation, the formation of the feedback can be described at the first approximation by making use of a switching-on function, this being zero or very small at the epoch of no structure and then going to one, and using our formulae for the effective energy-momentum tensor. As it is known that the negative pressure works to slow down the overall expansion, its forming at the stage of the formation of the structure provides enough time for the structure to develop. In this sense the universe is a self-regulating device and all time-scale difficulties are removed still further. Of course, it is dubious whether there is enough additional time to enlarge the statistical inhomogeneities, but anyway the dynamics of the conventional Friedmann model is changed. As the functions $p^*(t)$ and $\Pi^*(t)$ can be found, in principle, using the equations of motion (5.5), (5.6), it is evident that the "equations of state" $p^* = p^*(\bar{\varrho})$ and $\Pi^* = \Pi^*(\bar{\varrho})$, so badly needed in model determining [7], can be found readily too.

Another consequence that has to be stressed is that the peculiar density changes slowly in time in the same way as the background quantities do. Taking into account that the present hydrodynamical approach can be applied to the systems larger or equal in extent to galaxies, it follows that galaxies and clusters of galaxies are not fully isolated from the cosmological background. This result is in harmony with those of L. Ozernoi and A. Chernin [8], obtained by entirely different considerations.

And finally we note that though the results of the present paper cannot be directly applied to the early stages of the universe (if, of course, there are no large initial inhomogeneities), they are well suited for studying the final stages of the evolution of the universe. Of course, in order to do it adequately one has to include the "physical" pressure into our equations. This can be readily done by using the methods outlined above. The changes of the peculiar density within the cosmological time-scale and the structure-induced "negative pressure" may have important consequences for those late and final stages.

The present approach is well suited for studying the problems where the background and superposed fluctuations can be separated. In cosmology it can be applied to all homogeneous backgrounds, thus allowing to study the effects of rotation and shear on the evolution and properties of the structure. Being an approximate method, it is, nevertheless, simple to handle and has the remarkable property of using only one non-dimensional parameter. This cannot be said about all cosmological approximations. And, finally, it allows us to introduce stochastic fields in a natural way. As the mathematical theory of homogeneous stochastic fields on homogeneous backgrounds is rather well-developed, cosmic turbulence can be studied by using the methods which have been so fruitful in the usual theory of turbulence. And, of course, the use of the method is not limited to cosmology. For example, the problem of turbulence in massive objects (quasars, e. g.) is similar to the problem of structure in cosmology, and the results of the present paper support the opinion that quasars may be stabilized by intensive turbulent motions.

To complete this section, we note that the possibility of a negative phenomenological pressure has been suggested by a number of authors [9–13]. The most rigorous derivations so far were those of W. M. Irvine [9] and D. Layzer [10], but they have used analogies to obtain the final formulae, and so their results coincide with ours only in a qualitative manner. And, in addition, the present approach clearly demonstrates the sources of the effective energy-momentum tensor (see (3.3)).

We hope that we have succeeded in convincing the reader that the approach outlined here has a wide domain of application and that its use may turn out promising in the future.

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## REFERENCES

1. Isaacson R. A., Phys. Rev., **166**, 1263 (1968).
2. Isaacson R. A., Phys. Rev., **166**, 1272 (1968).
3. Saar E., Tartu Astr. Obs. Publ., **39**, 206 (1971).
4. Saar E., Tartu Astr. Obs. Publ., **39**, 234 (1971).
5. Saar E., Tartu Astr. Obs. Publ., **39**, 249 (1971).
6. Misner C. W., Thorne K. S., Wheeler J. A., Preprint, 1968.
7. Saar E., Tartu Astr. Obs. Publ., **39**, 183 (1971).
8. Озерной Л. М., Чернин А. Д., Астр. ж., **45**, 1137 (1968).
9. Irvine W. M., Ann. Phys. (N. Y.), **32**, 322 (1965).
10. Layzer D., Astrophys. J., **138**, 735 (1963).
11. Layzer D., Lectures in Appl. Math., **8**, Relativity Theory and Astrophysics, Providence, 1967, p. 237.
12. Широков М. Ф., Фишер И. З., Астр. ж., **39**, 899 (1962).
13. Зельдович Я. Б., Новиков И. Д., Релятивистская астрофизика, М., 1967, с. 127.

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### MITTEHOMOGEENSED KOSMOLOOGILISED MUDELID JA KÕRGESAGEDUSLIK LÄHENDUS

Kasutades ainejaotuse kirjeldamiseks kõrgesageduslikku lähendust, käsitletakse kosmoloogilisi mudeleid, mis on homogeensed vaid keskmiselt. Näidatakse, et tiheduse häiritused tekitavad täiendava energiaimpulsi tensori, mis põhjustab tagasisidestuse aine struktuuri arengu ja kosmoloogilise mudeli üldise dünaamika vahel. Seda tensorit võib kirjeldada negatiivse gravitatsioonilise rõhu ja fenomenoloogilise siseenergia abil. Lühidalt analüüsitakse esitatud teooriast tulenevaid järeldusi.

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### НЕОДНОРОДНЫЕ КОСМОЛОГИЧЕСКИЕ МОДЕЛИ И ВЫСОКОЧАСТОТНОЕ ПРИБЛИЖЕНИЕ

В высокочастотном приближении исследованы космологические модели, однородные лишь в среднем. Показано, что неоднородности плотности приводят к возникновению дополнительного тензора энергии-импульса, реализующего обратную связь между эволюцией структуры и динамикой космологической модели в целом. Этот тензор можно описывать с помощью отрицательного «гравитационного давления» и феноменологической «внутренней энергии». Обсуждаются возможные следствия полученных результатов.