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RADIATIVE TRANSFER IN THE ATMOSPHERES OF ROTATING STARS

The theory of radiative transfer in stellar atmospheres has been worked out basically for plane-parallel medium. In the case of extended atmospheres where the thickness of the layers cannot be assumed to be negligibly small as compared to the value of the local curvature radius, one has to consider a spherically symmetrical atmosphere. Such stars are non-rotating supergiants. However, according to both theoretical predictions and observational data, the effects of gravitational and radiative interactions in close binaries as well as the rapid rotation of single stars result in conspicuous distortion of stellar bodies. The problem of constructing models of distorted stars is one of the urgent tasks of modern astrophysics. To start with its solution, one has to investigate in detail physical processes governing both the stellar interiors and atmospheres. Adequate methods to solve the equations of radiative transfer and of the state of stellar matter in non-spherical case are to be developed as well. A number of papers concerning various aspects of the problem of rapidly rotating stars have been published recently (see, for example, [1–4]). All of the authors engaged with the problem discussed here have used a locally plane-parallel model and assumed, as far as we know, the applicability of the von Zeipel theorem for the description of brightness distribution over the stellar surface.

In the present paper a mathematical approach to the radiative transfer equation is presented for a star with a given figure of rotation. The following investigation is deliberately confined to a physically idealized model of grey atmosphere in state of LTE and radiative equilibrium. Namely, in that case, some characteristics of outgoing radiation can be approximately found without previous knowledge of the stellar structure and compared to those of plane-parallel atmospheres. As to the whole problem of models of distorted stars, it covers a wide range of questions which are beyond the scope of the present investigation.

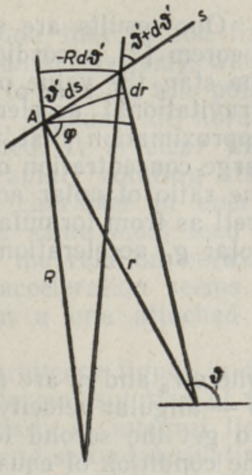
1. Integral flux for rotating stars

We examine the problem of radiative transfer in the atmosphere of a star of a given rotational figure. The equation of transfer of radiation in integrated light and for the case of pure absorption in arbitrary direction s (cf. Fig.) can be written as follows

$$\frac{dI}{ds} = \alpha (B - I). \qquad (1)$$

We choose now the direction s lying in the plane that passes through the centre of the star and along the surface normal (the plane of Fig.).

We proceed in (1) from variable s to the radius-vector r measured from the centre of the star and to the angle ϑ' between the outer normal to the surface and direction s using formulae



$$\frac{dr}{ds} = \sin(\varphi - \vartheta'), \quad \frac{d\vartheta'}{ds} = -\frac{\sin\vartheta'}{R}. \tag{2}$$

Here R is the local curvature radius, φ — the angle between the radius vector and the tangent plane to the surface at the point under consideration. Further we assume this surface to be the equipotential surface and radiative flux to propagate along the normal to it. Then we have

$$\varphi = \operatorname{arctg}\frac{r}{r'}, \quad \sin\varphi = \frac{1}{\sqrt{1+\left(\frac{r'}{r}\right)^2}}, \quad \text{where} \quad r' = \frac{dr}{d\vartheta}. \tag{3}$$

Thus, instead of (1) we can write

$$\sin\varphi\cos\vartheta'\frac{\partial I}{\partial r} - \cos\varphi\sin\vartheta'\frac{\partial I}{\partial r} - \frac{\sin\vartheta'}{R}\frac{\partial I}{\partial\vartheta'} = \alpha(B - I). \tag{4}$$

In what follows, we assume that I is independent of azimuthal angle. We may call this the approximation of locally spherical atmosphere.

Integrating equation (4) over the solid angle and taking use of the condition of radiative equilibrium, we obtain

$$\sin\varphi\frac{dF}{dr} + \frac{2F}{R} - \cos\varphi\frac{d\mathfrak{F}}{dr} = 0, \tag{5}$$

where $F = \int I\cos\vartheta' d\omega$ and $\mathfrak{F} = \int I\sin\vartheta' d\omega$. In accordance with our basic assumption about the propagation of radiative flux along the outer normal to the equipotential surface, the value $\mathfrak{F}$ is identically equal to zero, while F is the integral radiative flux. Further, the local curvature radius is

$$R = \frac{\sqrt{(dr)^2 + (r d\vartheta)^2}}{d\varphi + d\vartheta} \tag{6}$$

and consequently

$$\frac{dF}{dr} + \frac{2F}{r}\left(1 + \frac{d\varphi}{d\vartheta}\right) = 0. \tag{7}$$

Separating variables in equation (7) and using formula (3), we obtain the following solution:

$$F = \frac{C}{(r\sin\varphi)^2} \tag{8}$$

or

$$F = \frac{C}{r^2}\left[1 + \left(\frac{r'}{r}\right)^2\right], \tag{9}$$

where C is an integration constant and $r\sin\varphi$ the value of projection of radius vector upon the outer normal to the equipotential surface.

Our results are somewhat different from the well-known von Zeipel theorem [5], according to which, in the frame of reference rotating with the star, the value of integral radiative flux is proportional to the local gravitational acceleration. To make comparison easier, we take an approximation that the star is ellipsoid of rotation, with an infinitely large concentration of matter towards the centre (Roche model), and find the ratio of polar and equatorial fluxes from the von Zeipel theorem as well as from formula (9). The ratio of the values of the equatorial g_e and polar g_p accelerations is

$$\frac{g_e}{g_p} = \left(\frac{r_p}{r_e}\right)^2 - \frac{\omega^2}{GM} r_e r_p^2, \tag{10}$$

where r_e and r_p are respectively the equatorial and polar radii of the star, ω — angular velocity of its axial rotation and G — gravitational constant. To get the second term of (10) in a more appropriate form, we apply the condition of equality of the gravitational potential at the pole to that at the equator:

$$\frac{GM}{r_e} + \frac{1}{2}\omega^2 r_e^2 = \frac{GM}{r_p}. \tag{11}$$

If a star is an ellipsoid of rotation with eccentricity e of its equatorial section normal to the axis of rotation, then, on the grounds of the von Zeipel theorem, we get

$$\frac{F_e}{F_p} = \frac{g_e}{g_p} = 3(1-e^2) - 2\sqrt{1-e^2}, \tag{12}$$

while according to formula (9)

$$\frac{F_e}{F_p} = 1 - e^2. \tag{13}$$

In the Table below, the values of F_e/F_p for some e are given.

e	$\left(\frac{F_e}{F_p}\right)_{vZ}$	$\left(\frac{F_e}{F_p}\right)_{\text{author}}$
0.100	0.980	0,990
0.200	0.920	0.960
0.300	0.822	0.920
0.400	0.687	0.840
0.500	0.518	0.750
0.600	0.320	0.640
0.700	0.102	0.510
0.745 $\left(\frac{1}{3}\sqrt{5}\right)$	0.000	0.444

If real stars are similar to the Roche model, which seems to be true, then for slowly rotating stars, i. e. for light distortion, our results (8) or (9) are in good agreement with the von Zeipel theorem. In the case of rapid rotation, however, the variations of radiative flux over the surface should be appreciably lower than one would expect from the von Zeipel theorem. Most probably such variations are even smaller than one would expect from formula (9), due to the evolved convection in distorted stars.

It follows from (12) that for $e = \sqrt{5}/3$ the value of F_e/F_p turns into zero ($g_e = 0$), which is physically unreasonable. Similarly, the von

Zeipel theorem leads to the unreasonable conclusion that in the first Lagrangian point in close binary (for example, in contact system), where $g=0$, the non-reflected radiative flux value equals to zero. In this point, however, our approximation cannot be used either, because the normal to the equipotential surface is not uniquely defined. It is clear that, irrespectively of the scale of deviations of real rapidly rotating stars from spherical figures as well as of the actual character of energy transfer (radiative, convective), the theory all the same must lead to physically justified results. Our results show that the relation between flux of radiation and the value of gravitational acceleration seems to realize rather in fixed coordinate system than in a one attached to the star.

Von Zeipel deduced his theorem for a star of arbitrary figure in the state of radiative equilibrium. First he obtained a formal solution of the radiative transfer equation and expressed the intensity of integral light as an algebraic sum of the Planck function, its first and second derivatives by radial optical depths. He assumed that this expansion was justified for sufficiently large optical depths. Then he rewrote both the radiative equilibrium condition and the integral radiative flux expression with the aid of this expansion. Further, he assumed the first Eddington moment as well as the joint moments of higher order to be equal to zero. Having used Poisson's equation and transformed with its aid the condition of radiative equilibrium, von Zeipel found finally

$$\varrho d\Phi = C \frac{4\pi}{3c} \frac{dB}{\varkappa}, \tag{14}$$

where Φ is gravitational potential in rotating frame of reference, B denotes the integrated Planck function, $\varkappa$ is the absorption coefficient, c — velocity of light, C is constant. Precisely from formula (14) and the Eddington approximation follows the von Zeipel theorem. The equation (14) is identically satisfied if one assumes that the temperature is constant over the equipotential surface. But on the other hand the radiative flux varies from point to point on the same surface proportionally to the value of local gravitational acceleration. The reason of this disagreement apparently lies in the fact that the second derivative of the integrated Planck's function by radial optical depth equals to zero. If one takes into account the integrated Planck's function and its first derivative only, one will easily find that the condition of radiative equilibrium transforms identically into zero; neither equation (14) (as long as constant C is zero) nor the von Zeipel theorem are valid any longer. In outer layers of a star, derivatives of the integrated Planck's function of a higher order are likely to be taken into account, but some other assumptions made by von Zeipel are invalid here. Particularly, the first and the joint Eddington moments of higher order are not neglectingly small.

2. The source function; the law of darkening; luminosity of the star

We turn now back to equation (4) and examine local-plane-parallel atmosphere. More exactly, still dealing with the atmosphere of a distorted star, we assume the curvature of the atmospheric layers to be constant along the path of beam. Then the equation of radiative transfer reduces, as follows:

$$\sin\varphi\cos\vartheta' \frac{\partial I}{\partial r} - \cos\varphi\sin\vartheta' \frac{\partial I}{\partial r} = \alpha(B - I). \tag{15}$$

Multiplying both parts of equation (15) first by $\sin\vartheta' d\omega/4\pi$ and then by $\cos\vartheta' d\omega/4\pi$ and integrating each time over the solid angle, we get

$$\left\{\begin{aligned} &\sin\varphi\frac{d\varkappa}{dr}-\cos\varphi\frac{d\overline{I}}{dr}+\cos\varphi\frac{dK}{dr}=0,\\ &\sin\varphi\frac{dK}{dr}-\cos\varphi\frac{d\varkappa}{dr}=-\alpha H,\end{aligned}\right. \tag{16}$$

where

$$K=\frac{1}{4\pi}\int I\cos^2\vartheta'\,d\omega,$$

$$\varkappa=\frac{1}{4\pi}\int I\sin\vartheta'\cos\vartheta'\,d\omega$$

and $H=\frac{1}{4}F$. Combining equations (16), one has

$$\frac{d}{d\tau}\left(K-\overline{I}\cos^2\varphi\right)=H\sin\varphi. \tag{17}$$

Here τ, as usual is the radial optical depth. It can be easily checked from equation (15) that in the plane-parallel case integral radiative flux is independent of optical depth, but varies from point to point over the surface according to formula (9). At first sight, equation (17) is quite different from the one we have used to deal with in the plane-parallel case. But once we rotate our coordinate system by angle $90°-\varphi$, we get the usual differential equation connecting K-integral with integral radiative flux H.

Integrating equation (17) over τ, we obtain

$$K=H\tau\sin\varphi+\overline{I}\cos^2\varphi+C, \tag{18}$$

where C denotes the integration constant.

It has usually been assumed in the case of plane-parallel atmospheres that $K=\frac{1}{3}\overline{I}$. We apply here the approximation

$$K=\frac{1}{3}\overline{I}+\frac{2}{3}H, \tag{19}$$

which has been previously used in the theory of an extended spherically symmetric atmosphere [6]. Then equation (18) may be rewritten with the aid of (19), radiative equilibrium condition for grey atmosphere $\overline{I}=B$ and boundary condition $\overline{I}_0=2H$ (for $\tau=0$), as follows:

$$B=\frac{1}{2}F\left(\frac{3}{2}\tau f(\varphi)+1\right), \tag{20}$$

where $f(\varphi)=\dfrac{\sin\varphi}{1-3\cos^2\varphi}$ and B stands for the source function, while integral radiative flux F varies according to (8) or (9). Having thus obtained the expression of the source function, we integrate then the equation of radiative transfer (15) over τ and get the so-called law of darkening in form

$$I(\vartheta')=\frac{1}{2}F\left[1+\frac{3}{2}\frac{\sin\varphi}{1-3\cos^2\varphi}\sin(\varphi-\vartheta')\right] \tag{21a}$$

or

$$I(\vartheta') = I(0)\,(1 - u + u\cos\vartheta' - v\sin\vartheta'), \tag{21b}$$

where

$$u = \frac{\frac{3}{2}\frac{\sin^2\varphi}{1-3\cos^2\varphi}}{1+\frac{3}{2}\frac{\sin^2\varphi}{1-3\cos^2\varphi}}$$

is the equivalent of the usual coefficient of limb darkening, while $v = u\,\mathrm{ctg}\,\varphi$.

As formerly, we suppose that the star is an ellipsoid of rotation and put down formulae (9) and (21) in an explicit form. In what follows, we adopt the polar coordinate system with the axis of rotation serving as the polar axis. The equation of the ellipsoid is then

$$r = \frac{b}{\sqrt{1 - e^2\sin^2\vartheta}}, \tag{22}$$

where ϑ is co-latitude or the polar angle and b stands for semi-minor axis or the polar radius of the star. Then we have

$$\sin\varphi = l\lambda + m\mu + n\nu, \tag{23}$$

where l, m, n — directional cosines of outer normal are given by

$$l = \frac{\sqrt{1-e^2}\sin\vartheta\sin\Phi}{\sqrt{1-e^2\sin^2\vartheta}}, \quad m = \frac{\sqrt{1-e^2}\sin\vartheta\cos\Phi}{\sqrt{1-e^2\sin^2\vartheta}}, \quad n = \frac{\cos\vartheta}{\sqrt{1-e^2\sin^2\vartheta}}. \tag{24}$$

Similarly, the values of directional cosines of radius vector look as follows:

$$\lambda = \sin\vartheta\sin\Phi, \quad \mu = \sin\vartheta\cos\Phi, \quad \nu = \cos\vartheta. \tag{25}$$

In the last two formulae, Φ is, as usually, the azimuthal angle.

Hence

$$\sin\varphi = \frac{\sqrt{1-e^2}\sin^2\vartheta + \cos^2\vartheta}{\sqrt{1-e^2\sin^2\vartheta}}. \tag{26}$$

It can be shown with the aid of last formula that denominator in the right hand part of (20) is always a positive quantity.

In a similar way, one can find that

$$\cos\vartheta' = \frac{\sqrt{1-e^2}\sin i\sin\vartheta + \cos i\cos\vartheta}{\sqrt{1-e^2\sin^2\vartheta}}. \tag{27}$$

With (22) and (27), formula (21) takes the form

$$I(\vartheta) = \frac{1}{2}F_p\Phi(e, i, \vartheta),$$

where F_p is polar flux, i — the angle between the axis of rotation and direction towards observer and

$$\Phi(e, i, \vartheta) = \frac{1}{1-u(e,\vartheta)}\Bigg[1 - u(e,\vartheta) + u(e,\vartheta)\,\frac{\sqrt{1-e^2}\sin i\sin\vartheta + \cos\vartheta\cos i}{\sqrt{1-e^2\sin^2\vartheta}} -$$

$$-v(e,\vartheta)\sqrt{\frac{(1-e^2)\sin^2\vartheta\cos^2 i+\cos^2\vartheta\sin^2 i-\sqrt{1-e^2}\sin 2i\cos\vartheta\sin\vartheta}{1-e^2\sin^2\vartheta}}\Bigg]\times$$

$$\times\left[\frac{1-e^2\sin^2\vartheta}{\sqrt{1-e^2\sin^2\vartheta+\cos^2\vartheta}}\right]^2. \tag{28}$$

The surface of an ellipsoid projected on celestial sphere (as viewed by a distant observer) is the ellipse with eccentricity ranging from 0 to e, depending on the value of the angle i. As to the character of the distribution of brightness over the "disk", it should be mentioned that generally this is not centrally symmetric, and whether all isophotes are closed curves or not also depends on the value of i. The relative brightness of the isophotes can be defined by formula (28).

A detailed description of the distribution of brightness over the "visible disk" of a rapidly rotating star may appear to be useful in investigating the properties of some semi-detached Algol-type binaries. In such systems primary component (quite frequently much more massive than its companion) is the rapidly rotating star, while tidal perturbations of its body may be insignificant. With the known character of the distribution of brightness over the surface of the star, one can find its full luminosity

$$L=\int_\sigma I\,d\sigma, \tag{29}$$

where $d\sigma$ is the elemental area of the surface of rotational body

$$d\sigma=2\pi r^2\cos\vartheta'\sqrt{1+\left(\frac{r'}{r}\right)^2}\,d(\cos\vartheta). \tag{30}$$

We proceed in integral (29) to the variable $\mu=\cos\vartheta$, and, after some reduction, the expression of full luminosity of ellipsoidal star yields to

$$L=L_0\int_0^{\mu_0}\Phi(e,i,\mu)\times$$

$$\times\frac{(\sqrt{(1-e^2)(1-\mu^2)}\sin i+\mu\cos i)\sqrt{(1-e^2)^2+e^2(2-e^2)\mu^2}}{(1-e^2+e^2\mu^2)^{5/2}}\,d\mu, \tag{31}$$

where $L_0=2\pi F_p b^2$, the value of $\Phi(e,i,\mu)$ is given by (28) and

$$\mu_0=\sqrt{\frac{1-e^2}{\csc^2 i-e^2}}.$$

Thus, the influence of random orientation of an ellipsoidal star on its luminosity can be quantitatively estimated.

Finally we resume some main results of the current work. Firstly, a mathematical approach to the radiative transfer equation has been found for the case of a star of a given rotational figure. The explicit expression for the distribution of radiative flux over the surface of a star in radiative equilibrium has been found and compared to the von Zeipel theorem. Further, the source function of the local-plane-parallel atmosphere has been obtained. Finally, a law of darkening and full luminosity of an ellipsoidal star as a function of orientation of rotational axis has been under consideration.

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KIIRGUSE ÜLEKANDEST PÖÖRLEVATE TÄHTEDE ATMOSFÄÄRIDES

Artiklis tuletatakse võrrand, mis iseloomustab kiirguse ülekannet pöörleva tähe atmosfääris. Oletades, et kiirgusvoog levib ekvipotentsiaalpinna normaali suunas, saadakse valem voo jaotuse kohta kiirguslikus tasakaalus oleva tähe pinnal. Selle valemi rakendamise tulemused kinnitavad seose olemasolu kiirgusvoo ning gravitatsioonilise kiirenduse vahel, kuid mitte pöörlevates, nagu väidetakse von Zeipeli teoreemis, vaid pigem paigalseisvates koordinaatides. On leitud allikafunktsioon lokaalselt tasapinnalise atmosfääri puhul. Lähtudes oletusest, et täht kujutab endast pöördellipsoidi, seletatakse tema tumenemise efekti ääre poole ning summaarse heleduse käitumist olenevalt pöördetelje orientatsioonist ruumis.

И. ПУСТЫЛЬНИК

К ТЕОРИИ ПЕРЕНОСА ИЗЛУЧЕНИЯ В АТМОСФЕРАХ ВРАЩАЮЩИХСЯ ЗВЕЗД

В работе дается математическая формулировка уравнения переноса излучения в атмосфере вращающейся звезды. В предположении, что поток излучения распространяется перпендикулярно к эквипотенциальной поверхности, получено выражение для распределения потока по поверхности звезды, находящейся в состоянии лучистого равновесия. Сравнение полученных результатов с теоремой фон Цейпеля показывает, что связь потока излучения с гравитационным ускорением имеет скорее место в покоящейся, чем во вращающейся системе координат. Найдено выражение функции источника в приближении локально плоской атмосферы. Получен закон потемнения к краю для звезды, являющейся эллипсоидом вращения, и ее полная светимость как функция наклона оси вращения к картинной плоскости.