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DECAY ESTIMATES FOR PARABOLIC CONSERVATION LAWS

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Abstract. A theorem concerning optimal temporal decay estimates for solutions of the Cauchy problem for a system of parabolic conservation laws has been proved.

Key words: Cauchy problem, nonlinear flux function, conservation laws.

1. INTRODUCTION

This note presents an advance announcement of a result obtained with H. Zhao concerning optimal temporal decay estimates for solutions of the Cauchy problem for the system of parabolic conservation laws

$$u_t + \sum_{j=1}^{n} f_j(u)_{x_j} = D\Delta u, \quad x \in R^N, \quad t > 0, \tag{1}$$

subject to the conditions

$$u(t,x)\big|_{t=0} = u_0(x), \quad x \in R^N, \quad N > 1, \quad n > 1. \tag{2}$$

The detailed derivation of this result, and of others that are related, will be published elsewhere.

The Cauchy problem (1),(2) has been studied by many authors, and background details of this work and key references are to be found in [1,2]. The results of [1,2] allow us to deduce that in order to obtain optimal decay estimates, the sufficient conditions that must be imposed on the nonlinear flux functions $f_j(u)\,(j=1,2,\dots,N)$ are $f_j(u) = O(|u|^3)$ $(j=1,2,\dots,N)$ as $|u| \to 0$, which are stronger than the corresponding sufficient conditions that guarantee the global existence results. This fact was noted in [2], where it was also noted that if the system (1),(2) admits a strongly convex quadratic entropy function $\eta(u)$ which is strongly consistent with the viscous matrix D, then the optimal temporal decay estimate

$$\left\|\Delta^{k/2} u(t,x)\right\|_{L^2(R^N,R^n)} \le C(r,\tau)(1+t)^{-(N+2k)/4} \tag{3}$$

can still be obtained, but only under the conditions $f_j(u) = O(|u|^2)$ $(j=1,2,\ldots,N)$ as $|u| \to 0$. However, in [1] it was noticed that for $n>2$ the corresponding entropy equation is overdetermined, so the existence of a nontrivial entropy is only because of a fortunate coincidence. Consequently, for general systems like (1),(2), it is necessary to discover if the sufficient conditions imposed on the nonlinear flux functions $f_j(u)$ $(j=1,2,\ldots,N)$ that guarantee the global existence results are also sufficient to deduce the optimal temporal decay estimate (3). The Theorem in Section 2 answers this question in the affirmative.

2. SUFFICIENCY RESULT

The result reported here, the proof of which is to be published elsewhere, concerns Theorem 1 of [2] that for convenience is repeated below.

Theorem 1 of [2] (Global existence result). *Suppose the flux function* $f(u)$ *satisfies*

$$\frac{f(u)}{|u-\overline{u}|^2} \in L^\infty(\overline{B}_r(\overline{u}), R^n)$$

for some fixed point $\overline{u} \in R^n$ *and each fixed positive constant* r, *then, if* $u_0(x) - \overline{u} \in L^\infty \cap L^1(R,R^n)$, *with* $\|u_0 - \overline{u}\|_{L^1(R,R^n)}$ *suitably small and* $\|u_0(x) - \overline{u}\|_{L^\infty(R,R^n)} < r$, *the Cauchy problem* (1),(2) *admits a unique globally smooth solution* $u(t,x)$ *which satisfies*

$$\|u_0(x) - \overline{u}\|_{L^\infty(R,R^n)} \le r.$$

□

Theorem. *Suppose for each fixed* $r>0$ *the smooth nonlinear flux functions* $f_j(u)$ $(j=1,2,\ldots,N)$ *satisfy*

$$\frac{f_j(u)}{|u|^2} \in L^\infty(\overline{B}(0), R^n) \text{ for } j = 1, 2, \ldots, N,$$

then, for $u_0(x) \in L^1 \cap L^\infty(R^N, R^n)$, *with* $\|u_0(x)\|_{L^\infty(R^N, R^n)} < r$ *and* $\|u_0(x)\|_{L^1(R^N, R^n)}$ *sufficiently small, the unique globally smooth solution* $u(t, x)$ *satisfying Theorem* 1 *of* [2] *satisfies the optimal temporal decay estimate* (3).

□

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KUSTUMISE HINNANGUD PARABOOLSETELE JÄÄVUSSEADUSTELE

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On antud funktsioonide $f_j(u)$ jaoks tingimused selleks, et probleemi (1) ja (2) lahendi eksisteerimise piisavad tingimused oleksid piisavad ka ajas optimaalse kustumise hinnangule (3).