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NONLINEAR ACCELERATION WAVES IN APPROXIMATELY CONSTRAINED ELASTIC BODIES

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Abstract. We give a general treatment of the propagation of nonlinear acceleration waves in approximately constrained elastic materials exhibiting constitutive equations with a second-order pole. The theory is applied to transversely isotropic bodies around an approximately inextensible fibre, such as fibre-reinforced materials.

Key words: acceleration waves, approximately constrained elastic materials, nonlinearity, constitutive equations, second-order pole.

1. INTRODUCTION

A formal perturbation method for solving a class of traction boundary-value problems of finite elasticity was introduced by Signorini in 1930 [1]. A detailed account of this method, its applications, and related questions can be found in [2] and references quoted therein.

In 1985 Marzano and Podio-Guidugli [3] resumed a Signorini-type perturbative scheme in order to construct a theory of approximately constrained elastic materials. In our opinion, this theory appears the most natural one to describe, for instance, the behaviour of elastic bodies which at the first order of approximation are incompressible, or rigid, or inextensible in some direction. In [3] Marzano and Podio-Guidugli confine their attention to a static problem, that is the traction problem in finite elasticity for an approximately constrained material. However, as previously noted by Capriz and Podio-Guidugli in [2], Signorini's perturbation method can be employed with considerable advantages also in problems of elastodynamics.

In particular, in [4] we apply the theory of approximately constrained elastic materials proposed by Marzano and Podio-Guidugli in [3] to the study of the propagation of acceleration waves in approximately constrained materials exhibiting constitutive equations with a first-order pole. By means of a suitable Laurent expansion for the square of the speed of propagation and a series expansion for the amplitude, we obtain in [4] successive approximations of the propagation condition from the corresponding approximations of the balance equations. In [4] we also show that at each step the propagation condition provides one, and only one, term in the Laurent expansion for the square of the speed of propagation and the corresponding term for the amplitude. In [4] the general method is also applied to St. Venant–Kirchhoff materials, namely isotropic materials which can be used to approximate rigid or incompressible bodies. Qualitative, interesting results are obtained and compared with those of other authors; in particular, the results concerning wave propagation in approximately rigid bodies are compared with those of Grioli [5], while the results concerning approximately incompressible bodies are compared with those of Rogerson and Scott [6] and Scott [7].

Afterwards, in [8] we apply the same technique to the study of the propagation of acceleration waves in approximately constrained elastic materials characterized by constitutive equations with a second-order pole, as approximately inextensible bodies, with the same advantages. The main results concerning wave propagation in such materials are exposed in a very concise form; moreover, the analysis of [8] is confined to the first-order approximation of the propagation condition.

The aim of this paper is to present a more complete study of the propagation of acceleration waves in approximately constrained elastic materials with a second-order pole.

Following [4], we obtain from the first approximation of the balance equations the corresponding approximation of the propagation condition; by means of an ordinary eigenvalue problem, we obtain the first term for the square of the speed of propagation and for the amplitude. Moreover, we complete the analysis of [8] by discussing the second-order approximation of the propagation condition: this approximation allows us to find the second term for the square of the speed of propagation and for the amplitude.

Finally, we study the propagation of acceleration waves in approximately inextensible bodies, as an example of approximately constrained elastic materials with a second-order pole. There are many real materials which exhibit such a behaviour, characterized by a strong anisotropy and mechanical properties which are highly dependent on a preferred direction in the material. For a complete survey on these materials, as well as their applications in engineering, we refer to [9].

For approximately inextensible bodies, we completely solve the first-order and second-order approximations of the propagation condition, obtaining results which are qualitatively in agreement with those of other authors; in particular, our results are compared with those of Green [10].

2. A PERTURBATIVE SCHEME FOR ELASTIC MATERIALS WITH A SECOND-ORDER POLE

In this section we sketch the basic features of Signorini's perturbation technique for an elastic material described by a constitutive equation exhibiting a second-order pole. We identify a continuous body with a fixed regular region B, called also reference configuration, of a three-dimensional Euclidean space; we denote by $\mathring{B}$ the interior part of B and by ∂B the boundary of B, with outward unit normal $\mathbf{m}$. We let $\mathbf{u}$ denote the displacement field, so that $\mathcal{H} = \nabla\mathbf{u}$ is the displacement gradient and $\mathcal{J} = \mathcal{I} + \mathcal{H}$ is the deformation gradient, where $\mathcal{I}$ is the unit tensor. Moreover, we denote by $\mathbf{b}$ the external body force density, $\mathbf{s}$ the traction vector, ρ the mass density, and $\mathcal{S}$ the first Piola–Kirchhoff stress tensor.

Then the field equations and boundary conditions are

$$\begin{aligned} \operatorname{Div}\mathcal{S} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}} \quad &\text{in} \quad \mathring{B} \times (0,T), \\ \mathcal{S}\mathbf{m} = \mathbf{s} \quad &\text{in} \quad \partial B \times (0,T), \end{aligned} \tag{2.1}$$

where Div is the divergence operator and the superposed dots denote time differentiation.

The central idea of Signorini's method is to replace (2.1) with a special one-parameter class of similar problems. Then, according to [3], we formally expand the displacement field $\mathbf{u}$ and the fields $\mathcal{H}$, $\mathbf{b}$, and $\mathbf{s}$ in terms of a real parameter z, which can be usefully identified with the inverse of some constitutive modulus:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + z\mathbf{u}_1 + z^2\mathbf{u}_2 + \cdots, \\ \mathcal{H} &= \mathcal{H}_0 + z\mathcal{H}_1 + z^2\mathcal{H}_2 + \cdots \quad (\mathcal{H}_i = \nabla\mathbf{u}_i, i = 0,1,\ldots), \\ \mathbf{b} &= \mathbf{b}_0 + z\mathbf{b}_1 + z^2\mathbf{b}_2 + \cdots, \\ \mathbf{s} &= \mathbf{s}_0 + z\mathbf{s}_1 + z^2\mathbf{s}_2 + \cdots. \end{aligned} \tag{2.2}$$

Similar expansions have been successfully used in many problems. We refer, for instance, to [2], where Signorini's method is applied to the initial-value problem of dead traction in finite elastodynamics, to [11], where it is employed in the case when the starting equilibrium placement is under stress, and finally to [12], where the method of [11] is generalized to the case when the loads depend on the solution.

The constitutive equation which specifies $\mathcal{S}$ is

$$\mathcal{S} = \mathcal{F}(\mathcal{J}, z). \tag{2.3}$$

The theory of approximately constrained elastic materials proposed by Marzano and Podio-Guidugli in [3] is based on constitutive equations (2.3) exhibiting a pole of some order, in such a way that the right-hand side of (2.3) can be written in terms of a suitable Laurent expansion. Examples of materials characterized by constitutive equations with a first-order pole are the so-called St. Venant–Kirchhoff materials;

they are hyperelastic, isotropic continua, whose constitutive equation has a first-order pole, according to two different choices of the constitutive parameter z (see [3], Sec. 5).

In the following sections we give a general treatment of the propagation of acceleration waves in approximately constrained materials characterized by constitutive equations with a second-order pole. Then, we write for $\mathcal{F}$ the following Laurent expansion in a suitable neighbourhood $\mathcal{N}$ of the origin $\mathcal{I}$:

$$\mathcal{F}(\mathcal{J}, z) = \frac{1}{z^2}\mathcal{F}_{-2}(\mathcal{J}) + \frac{1}{z}\mathcal{F}_{-1}(\mathcal{J}) + \mathcal{F}_0(\mathcal{J}) + z\mathcal{F}_1(\mathcal{J}) + \cdots, \tag{2.4}$$

where each one of the operators $\mathcal{F}_{-2}, \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, \ldots$ is analytic within $\mathcal{N}$.

We seek a solution $\mathbf{u}$ of type $(2.2)_1$ of the problem (2.1) with finite stress $\mathcal{S}$ for $z = 0$, corresponding to a load system $(\mathbf{b}, \mathbf{s})$ given by $(2.2)_{3,4}$; then we must require

$$\mathcal{F}_{-2}(\mathcal{I}) = \mathcal{O}, \quad \mathcal{F}_{-1}(\mathcal{I}) = \mathcal{O}, \tag{2.5}$$

$$\hat{\mathcal{S}}_{-2}\mathcal{H}_1 = \mathcal{O}, \tag{2.6}$$

where $\hat{\mathcal{S}}_{-2} = \nabla\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}$. Condition (2.6) means that the first approximation of the displacement gradient $\mathcal{H}$ belongs to the kernel of the operator $\hat{\mathcal{S}}_{-2}$.

In virtue of the previous hypotheses, the stress tensor $\mathcal{S}$ admits the following series expansion in terms of the parameter z:

$$\mathcal{S}(z) = \mathcal{S}_0 + z\mathcal{S}_1 + z^2\mathcal{S}_2 + \cdots, \tag{2.7}$$

where the first and second approximations of $\mathcal{S}$ are given by

$$\begin{aligned}\mathcal{S}_0 &= \lim_{z\to 0}\mathcal{F}(\mathcal{J}(z), z) = \mathcal{F}_0(\mathcal{I}) + \hat{\mathcal{S}}_{-2}\mathcal{H}_2 + \hat{\mathcal{S}}_{-1}\mathcal{H}_1 \\ &\quad + \frac{1}{2}\left(\nabla^2\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_1]\right)[\mathcal{H}_1],\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{S}_1 &= \lim_{z\to 0}\frac{\mathcal{F}(\mathcal{J}(z), z) - \mathcal{S}_0}{z} \\ &= \hat{\mathcal{S}}_{-2}\mathcal{H}_3 + \hat{\mathcal{S}}_{-1}\mathcal{H}_2 + \hat{\mathcal{S}}_0\mathcal{H}_1 + \mathcal{F}_1(\mathcal{I}) + \frac{1}{2}\left(\nabla^2\mathcal{F}_{-1}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_1]\right)[\mathcal{H}_1] \\ &\quad + \sum_{r=2}^{3}\frac{1}{r!}\sum_{P_r^3}\left(\left(\cdots\left(\left(\nabla^r\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_{\alpha_1}]\right)[\mathcal{H}_{\alpha_2}]\right)\cdots\right)[\mathcal{H}_{\alpha_r}]\right).\end{aligned} \tag{2.9}$$

In (2.8) and (2.9) we have set $\hat{\mathcal{S}}_{-1} = \nabla\mathcal{F}_{-1}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}$ and $\hat{\mathcal{S}}_0 = \nabla\mathcal{F}_0(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}$, respectively, while P_r^3 denotes the set of all permutations $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of the numbers (1,2,3) taken r at a time with repetitions and such that $\sum_{s=1}^{r} \alpha_s = 3$. The last term in (2.9) is then a function of $\mathcal{H}_1$ and $\mathcal{H}_2$ only.

In the following we require the reference configuration to be at ease, that is we set $\mathcal{F}_0(\mathcal{I}) = \mathcal{O}$ on the right-hand side of (2.8). The higher-order approximations for the stress in (2.7) can be obtained by means of the same procedure (see [3], Sec. 2).

By substituting series expansions for $\mathcal{S}$, $\mathbf{u}$, $\mathbf{b}$, and $\mathbf{s}$ into (2.1), together with the use of (2.8) and (2.9), we obtain the first-order and second-order approximations of (2.1), respectively:

$$
\begin{aligned}
&\operatorname{Div}(\hat{\mathcal{S}}_{-2}\,\mathcal{H}_2)\\
&\quad = -\operatorname{Div}\left\{\hat{\mathcal{S}}_{-1}\,\mathcal{H}_1 + \frac{1}{2}\left(\nabla^2\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_1]\right)[\mathcal{H}_1]\right\} - \rho\,\mathbf{b}_0 + \rho\,\ddot{\mathbf{u}}_0,\\
&(\hat{\mathcal{S}}_{-2}\,\mathcal{H}_2)\,\mathbf{m} = -\left\{\hat{\mathcal{S}}_{-1}\,\mathcal{H}_1 + \frac{1}{2}\left(\nabla^2\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_1]\right)[\mathcal{H}_1]\right\}\mathbf{m} + \mathbf{s}_0,
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&\operatorname{Div}(\hat{\mathcal{S}}_{-2}\,\mathcal{H}_3)\\
&\quad = -\operatorname{Div}\Bigg\{\hat{\mathcal{S}}_{-1}\,\mathcal{H}_2 + \hat{\mathcal{S}}_0\,\mathcal{H}_1 + \mathcal{F}_1(\mathcal{I}) + \frac{1}{2}\left(\nabla^2\mathcal{F}_{-1}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_1]\right)[\mathcal{H}_1]\\
&\qquad + \sum_{r=2}^{3}\frac{1}{r!}\sum_{P_r^3}\left((\ldots((\nabla^r\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_{\alpha_1}])[\mathcal{H}_{\alpha_2}])\ldots)[\mathcal{H}_{\alpha_r}]\right)\Bigg\}\\
&\qquad -\rho\,\mathbf{b}_1 + \rho\,\ddot{\mathbf{u}}_1,\\
&(\hat{\mathcal{S}}_{-2}\,\mathcal{H}_3)\mathbf{m}\\
&\quad = -\Bigg\{\hat{\mathcal{S}}_{-1}\,\mathcal{H}_2 + \hat{\mathcal{S}}_0\,\mathcal{H}_1 + \mathcal{F}_1(\mathcal{I}) + \frac{1}{2}\left(\nabla^2\mathcal{F}_{-1}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_1]\right)[\mathcal{H}_1]\\
&\qquad + \sum_{r=2}^{3}\frac{1}{r!}\sum_{P_r^3}\left((\ldots((\nabla^r\mathcal{F}_{-2}(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}[\mathcal{H}_{\alpha_1}])[\mathcal{H}_{\alpha_2}])\ldots)[\mathcal{H}_{\alpha_r}]\right)\Bigg\}\,\mathbf{m} + \mathbf{s}_1.
\end{aligned}
\tag{2.11}
$$

If a completely analogous procedure is applied to the higher-order approximations of the stress $\mathcal{S}$, we can replace the nonlinear differential problem (2.1) by an infinite sequence of linear differential problems for the successive approximations of the solution. Equations (2.10) and (2.11) show the central role of the linear operator $\hat{\mathcal{S}}_{-2}$ for the analytic properties of this succession of problems; we refer to [11] for a detailed account of this subject.

3. INTERNAL CONSTRAINTS: MATHEMATICAL IDEALIZATION AND BEHAVIOUR OF REAL MATERIALS. FIBRE-REINFORCED MATERIALS

A consistent part of the mathematical theory of elasticity is devoted to the study of elastic materials subject to internal constraints. Such constraints arise when the deformation gradient $\mathcal{J}$ must satisfy some form of local constraint, which reduces the number of independent deformation components.

For example, the material may be inextensible in some direction, or incompressible. These constraints, of course, are an idealization as far as real materials are concerned; however, in many cases they provide a good approximation to actual material responses.

A more realistic approach can be attained by using constitutive equations which take into account the effects of small changes in the material response. As shown in [3], a constitutive equation for the stress $\mathcal{S}$ exhibiting a pole of some order is the starting-point to describe an approximately constrained material. For instance, we can require the body to be not incompressible, but incompressible at the first order of approximation; rubber-like materials are characterized by such behaviour. To describe this situation, a constitutive equation for the stress with a first-order pole, as for a St. Venant–Kirchhoff material, can be used; for more details we refer to [3], Sec. 5.

Another interesting case arises when an elastic material is approximately inextensible at the first order of approximation. In the following we consider an example of such materials, that is a transversely isotropic body around an approximately inextensible fibre, such as a fibre-reinforced material.

In the following example, we will show that an approximately inextensible material is described by a constitutive equation with a second-order pole. To this aim, according to Marzano and Podio-Guidugli [3], we give the following definition: an elastic material with a second-order pole, for which (2.3)–(2.7) hold, is an approximately constrained material at the first order if

$$\ker \hat{\mathcal{S}}_{-2} = \mathcal{M}, \tag{3.1}$$

where $\mathcal{M}$ is a subspace of the space of all second-order tensors and $\dim \mathcal{M} \leq 8$.

Example. We consider a linear elastic material which is reinforced with a single family of fibres. Such a material has strong directional properties and it can be regarded as an anisotropic material. In particular, its constitutive equation is the constitutive equation of a linear elastic solid, transversely isotropic in the direction of a unit vector **e**, that is

$$\begin{aligned}\mathcal{S} = \{2\mu\, Sym + \lambda\,[\mathcal{I}\otimes\mathcal{I} + \delta\,(\mathcal{I}\otimes(\mathbf{e}\otimes\mathbf{e}) + (\mathbf{e}\otimes\mathbf{e})\otimes\mathcal{I}) \\ + \delta^2(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e})]\}\,\mathcal{H}\,,\end{aligned} \tag{3.2}$$

where λ, μ, δ are elastic moduli and Sym denotes the identical transformation of the vector space of the symmetric tensors into itself. The modulus δ is responsible for the anisotropy: in fact δ goes to zero in the limit as the material becomes isotropic. The preferred direction, that is the fibre direction, is characterized by $\mathbf{e}$.

We refer to [9], Sec. 2, for the method of deriving the constitutive equation (3.2), as well as for many examples of real materials exhibiting such behaviour. For brevity, we only cite a carbon fibre-epoxy resin composite.

Equation (3.2) is a constitutive equation of type (2.4), where

$$\begin{aligned} z = \frac{1}{\delta}, \quad & \hat{\mathcal{S}}_{-2} = \lambda\,(\mathbf{e}\otimes\mathbf{e})\otimes(\mathbf{e}\otimes\mathbf{e}), \\ \hat{\mathcal{S}}_{-1} = \lambda(\mathcal{I}\otimes(\mathbf{e}\otimes\mathbf{e}) + (\mathbf{e}\otimes\mathbf{e})\otimes\mathcal{I}), \quad & \hat{\mathcal{S}}_0 = 2\mu\, Sym + \lambda\,\mathcal{I}\otimes\mathcal{I}. \end{aligned} \tag{3.3}$$

Condition (2.6) yields:

$$(\mathbf{e}\otimes\mathbf{e})\,\cdot\,\mathcal{H}_1 = 0\,, \tag{3.4}$$

that is the elastic fibres in the direction of transverse isotropy are inextensible at the first order of approximation, so that we have a material approximately constrained.

4. ACCELERATION WAVES IN ELASTIC MATERIALS WITH A SECOND-ORDER POLE

This section deals with some general features of the propagation of acceleration waves in approximately constrained materials with a second-order pole, according to the results briefly presented in [8]. We use the well-known method of discontinuity surfaces, extensively exposed, for instance, by Wang and Truesdell [13] and Jeffrey [14].

An acceleration wave is a surface of discontinuity which propagates through the material in a manner such that the displacement $\mathbf{u}$ and its first derivatives are continuous, whilst the second derivatives suffer finite discontinuities at the wave surface. The compatibility conditions for acceleration waves are

$$\begin{aligned} [|\ddot{\mathbf{u}}|] &= \mathbf{A}\,V^2, \\ [|\mathbf{u}_{,ij}\,|] &= \mathbf{A}\;n^i n^j \quad (i,j=1,2,3); \end{aligned} \tag{4.1}$$

in the jump conditions (4.1) the brackets indicate the discontinuities across the singular surface, the comma denotes partial differentiation, n^i are the components of the unit normal $\mathbf{n}$ to the singular surface, $\mathbf{A}$ is the amplitude vector, and V is the speed of propagation.

We show in the following that the perturbative technique introduced in Sec. 2 leads in a natural way to successive approximations of the propagation condition, obtained by the corresponding approximations of the balance equations.

First, according to $(2.2)_1$ and (2.4), we choose appropriate expansions also for $\mathbf{A}$ and V^2; in particular we write a Laurent expansion for V^2 and a series expansion for $\mathbf{A}$:

$$V^2 = \frac{1}{z^2}V_{-2}^2 + \frac{1}{z}V_{-1}^2 + V_0^2 + zV_1^2 + \ldots \,, \tag{4.2}$$

$$\mathbf{A} = \mathbf{A}_0 + z\mathbf{A}_1 + z^2\mathbf{A}_2 + \ldots \,. \tag{4.3}$$

Since we require the product $\mathbf{A}V^2$ to tend to a finite limit when $z \to 0$, while $V^2 \to \infty$, the first two terms in (4.2) are forced to vanish:

$$\mathbf{A}_0 = \mathbf{0}, \quad \mathbf{A}_1 = \mathbf{0}; \tag{4.4}$$

then

$$\mathbf{A}V^2 = \mathbf{A}_2V_{-2}^2 + z\left(\mathbf{A}_2V_{-1}^2 + \mathbf{A}_3V_{-2}^2\right) + z^2\left(\mathbf{A}_2V_0^2 + \mathbf{A}_3V_{-1}^2 + \mathbf{A}_4V_{-2}^2\right) + \ldots \,. \tag{4.5}$$

We substitute $(2.2)_1$ and (4.5) into $(4.1)_1$. Equating the coefficients of the same powers of z, we obtain

$$\begin{aligned} [|\ddot{\mathbf{u}}_0|] &= \mathbf{A}_2\, V_{-2}^2 \,, \\ [|\ddot{\mathbf{u}}_1|] &= \mathbf{A}_2V_{-1}^2 + \mathbf{A}_3V_{-2}^2 \,, \\ [|\ddot{\mathbf{u}}_2|] &= \mathbf{A}_2V_0^2 + \mathbf{A}_3V_{-1}^2 + \mathbf{A}_4V_{-2}^2 \;\ldots \end{aligned} \tag{4.6}$$

In the same way, with the use of $(2.2)_1$, (4.2), (4.4), the jump conditions $(4.1)_2$ become

$$\begin{aligned} [|\mathbf{u}_{0,ij}|] &= \mathbf{0} \,, \\ [|\mathbf{u}_{1,ij}|] &= \mathbf{0} \,, \\ [|\mathbf{u}_{2,ij}|] &= \mathbf{A}_2 n^i n^j \,, \\ [|\mathbf{u}_{3,ij}|] &= \mathbf{A}_3 n^i n^j \;\ldots \end{aligned} \tag{4.7}$$

Now we consider the first-order and second-order approximations (2.10) and (2.11) of the balance equation $(2.1)_1$, assuming that the body force $\mathbf{b}$ is continuous.

Taking the jumps of (2.10) and substituting $(4.6)_1$, $(4.7)_{2,3}$, we obtain the first-order approximation of the propagation condition

$$[\,{}^T\!\mathcal{S}_{-2}\,(\mathbf{n}\otimes\mathbf{n}) - \rho V_{-2}^2\mathcal{I}]\mathbf{A}_2 = \mathbf{0} \,, \tag{4.8}$$

where ${}^T\!\mathcal{S}_{-2}$ is defined in terms of $\hat{\mathcal{S}}_{-2}$ by the following relation:

$$(\mathbf{a}\otimes\mathbf{b})\cdot\, {}^T\!\mathcal{S}_{-2}\,(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{a}\otimes\mathbf{c})\cdot\, \hat{\mathcal{S}}_{-2}\,(\mathbf{b}\otimes\mathbf{d}), \qquad \forall\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}. \tag{4.8$'$}$$

The second approximation (2.11) of the balance equation $(2.1)_1$ yields the second-order approximation of the propagation condition. In fact, taking the jumps of (2.11) and using conditions $(4.6)_2$, $(4.7)_{2,3,4}$, we obtain the following form for the second-order approximation of the propagation condition:

$$[\,{}^T\!\mathcal{S}_{-2}\,(\mathbf{n}\otimes\mathbf{n}) - \rho V_{-2}^2 \mathcal{I}]\mathbf{A}_3 = \rho V_{-1}^2 \mathbf{A}_2 - \Psi\mathbf{A}_2\,. \tag{4.9}$$

In (4.9) the double tensor Ψ is given by:

$$\Psi = \left({}^T\nabla^2 \mathcal{F}_{-2}\,(\mathcal{J})\,|_{\mathcal{J}=\mathcal{I}}\,[\mathcal{H}_1] + {}^T\!\mathcal{S}_{-1}\right)(\mathbf{n}\otimes\mathbf{n}), \tag{4.10}$$

where the transpose operator is defined as in $(4.8)'$ [1]. From condition (4.8) we see that $\mathbf{A}_2$ is a proper vector of the acoustic tensor ${}^T\!\mathcal{S}_{-2}\,(\mathbf{n}\otimes\mathbf{n})$ and the corresponding proper number is ρV_{-2}^2. Then, by means of a usual eigenvalue problem, we can find the first term for the square of the speed of propagation and for the amplitude. We turn now to Eq. (4.9). We set

$$\mathcal{Q} = {}^T\!\mathcal{S}_{-2}\,(\mathbf{n}\otimes\mathbf{n}) - \rho V_{-2}^2 \mathcal{I} \tag{4.11}$$

and

$$\Phi = \rho V_{-1}^2 \mathcal{I} - \Psi. \tag{4.12}$$

By using (4.11) and (4.12), Eq. (4.9) takes the form

$$\mathcal{Q}\,\mathbf{A}_3 = \Phi\,\mathbf{A}_2. \tag{4.13}$$

In Eq. (4.13) the amplitude $\mathbf{A}_2$ is known from the first approximation of the propagation condition (4.8), while (4.12) yields Φ as a function of V_{-1}^2, but V_{-1}^2 at this order of approximation is undetermined.

In order to overcome this difficulty, we can apply the same procedure followed in [4], where a similar situation occurs; we refer to [4] for more details. Then, also in the present case we can deduce that

$$\det\Phi = 0. \tag{4.14}$$

Condition (4.14), together with (4.12), means that $\rho\,V_{-1}^2$ must be an eigenvalue of the double tensor Ψ. Then we can obtain from (4.14) three values for V_{-1}^2 and, if they are real and positive, we find another term in the Laurent expansion (4.3) for V^2; alternately, if Ψ is such that all the eigenvalues are not real or not positive, V_{-1}^2 can be arbitrarily chosen (perhaps $V_{-1}^2 = 0$). Finally, we return to Eq. (4.13), in which the only unknown is $\mathbf{A}_3$. Since V_{-2}^2 are the eigenvalues determined by (4.8), we have

[1] In our previous paper [4] the explicit expression of Ψ, given by formula (26), is incorrect, as well as the same expression in [8]. In fact, the right sides of those formulas represent $\Psi\mathbf{A}_1$ or $\Psi\mathbf{A}_2$, respectively, namely a vector in both cases. The correct expression of the tensor Ψ in [8] is given by (4.10), while the other one can be easily deduced in the same way.

$$\det \mathcal{Q}\left(V_{-2}^2\right) = 0. \tag{4.15}$$

Then we can obtain from (4.13) the amplitude vector $\mathbf{A}_3$, with one or two degrees of indeterminacy, depending on the rank of $\mathcal{Q}$. Of course, a similar procedure can be applied to the higher-order approximations of the propagation condition in order to derive higher-order approximations for the square of the speed and for the amplitude.

5. ACCELERATION WAVES IN APPROXIMATELY INEXTENSIBLE BODIES

In this section we apply the general results of Sec. 4 in order to solve the first-order and second-order approximations of the propagation condition for approximately inextensible bodies; their constitutive equation is given in Sec. 3.

The first-order approximation of the propagation condition (4.8) with the use of $(3.3)_2$ becomes

$$\left[\lambda\left(\mathbf{e}\cdot\mathbf{n}\right)^2 \mathbf{e}\otimes\mathbf{e} - \rho V_{-2}^2 \mathcal{I}\right]\mathbf{A}_2 = \mathbf{0}\,. \tag{5.1}$$

Solving the eigenvalue problem, we obtain a double spurious root and a nonzero root:

$$V_{-2}^2 = 0, \quad V_{-2}^2 = \frac{\lambda}{\rho}\cos^2\theta\,, \tag{5.2}$$

where θ is the angle between $\mathbf{e}$ and $\mathbf{n}$. Condition (5.1) is unchanged by the replacements $\mathbf{n}\to -\mathbf{n}$ and $\mathbf{e}\to -\mathbf{e}$, so that no generality is lost by confining θ to the interval $[0,\pi/2]$.

With regard to the eigenvalue $(5.2)_1$, it is well known that a spurious root corresponds to a possible nonpropagating singular surface. By substituting $(5.2)_1$ in condition (5.1), we see that $\mathbf{A}_2 \in \ker\left(\mathbf{e}\otimes\mathbf{e}\right)$; since, in general, $\mathbf{n}\cdot\mathbf{e}\neq 0$, we obtain $\mathbf{A}_2\cdot\mathbf{e} = 0$. In particular, if $\mathbf{n} = \mathbf{e}$, we have a transverse wave.

With regard to the wave speed $(5.2)_2$, if we suppose $\theta \neq \pi/2$, Eq. (5.1) yields $\mathbf{A}_2$ as a vector parallel to $\mathbf{e}$. In particular, if $\theta = 0$, we have a longitudinal wave.

Many authors have studied the propagation of acceleration waves in inextensible bodies, treated as examples of constrained materials, both in the linear and in the nonlinear theory of elasticity. It is well known that the properties concerning wave propagation in such materials depend largely on the directional properties of the carrying medium. For a general survey on this subject we refer to [9], Ch. VIII, and references quoted therein.

Other authors relax the constraint of inextensibility and construct different mathematical models, for instance "nearly inextensible" bodies and "almost inextensible" bodies. A general account of the wave propagation in "nearly inextensible" bodies is given by Rogerson and Scott [6], while wave propagation in "almost inextensible" materials is analysed by Green [10], Sec. 4.

A qualitative comparison of our results with those of other papers confirms that our speeds and amplitudes are in agreement with well-known results obtained by different approaches. Here we focus attention on the comparison of our results with those of [10], Sec. 2, in which Green studies plane acceleration waves in a transversely isotropic linear elastic body, inextensible in the direction of transverse isotropy. In [10], Sec. 2, the material is considered as an exactly constrained material, so that speeds and amplitudes are different from ours, nevertheless the qualitative behaviour agrees with our results. In fact, in the general case ($\cos\theta \neq 0$) Green obtains two nonzero speeds of propagation, associated with discontinuities orthogonal to the axis of transverse isotropy. In the exceptional case ($\cos\theta = 0$) two of the wave speeds and associated discontinuities correspond to the solution of the general case, while the third wave speed is associated with a discontinuity along the axis of transverse isotropy.

We turn now to the second-order approximation of the propagation condition (4.9). By using (3.2) and $(3.3)_3$, we can see that the double tensor Ψ given by (4.10) reduces to

$$\Psi = \lambda\ (\mathbf{e}\cdot\mathbf{n})\ (\mathbf{n}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{n})\,. \tag{5.3}$$

By substituting (5.3) and $(3.3)_2$ into (4.9), we obtain

$$\left[\lambda\,(\mathbf{e}\cdot\mathbf{n})^2\ \mathbf{e}\otimes\mathbf{e} - \rho V_{-2}^2\mathcal{I}\right]\mathbf{A}_3 = \left[\rho V_{-1}^2\mathcal{I} - \lambda\,(\mathbf{e}\cdot\mathbf{n})\ (\mathbf{n}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{n})\right]\mathbf{A}_2. \tag{5.4}$$

Equation (5.4) can be written in the form (4.13) by setting

$$\mathcal{Q} = \lambda\,(\mathbf{e}\cdot\mathbf{n})^2\ \mathbf{e}\otimes\mathbf{e} - \rho V_{-2}^2\mathcal{I}, \tag{5.5}$$

and

$$\Phi = \rho V_{-1}^2\mathcal{I} - \lambda\,(\mathbf{e}\cdot\mathbf{n})\ (\mathbf{n}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{n})\,. \tag{5.6}$$

Now we apply to (5.4) the discussion of condition (4.13) exposed in Sec. 4; the squares of the speeds of propagation V_{-2}^2 in (5.4) are given by (5.2), while the corresponding amplitudes $\mathbf{A}_2$ are orthogonal or parallel to $\mathbf{e}$, respectively.

At this order of approximation V_{-1}^2 is unknown, but we have shown in Sec. 4 that ρV_{-1}^2 must be an eigenvalue of the double tensor (5.3). Since Ψ in (5.3) is symmetric, we obtain three real eigenvalues, but only one is positive:

$$\rho V_{-1}^2 = \lambda\cos\theta(\cos\theta + 1). \tag{5.7}$$

Then (5.7) yields the second term in the Laurent expansion (4.3) for V^2.

Moreover, if we return to Eq. (5.4), we can now obtain $\mathbf{A}_3$, with one or two degrees of indeterminacy, depending on the rank of $\mathcal{Q}$.

By substituting in the expression (5.5) for $\mathcal{Q}$ the values (5.2) for V_{-2}^2, we obtain

$$\operatorname{rank}\mathcal{Q}\left(V_{-2}^2 = 0\right) = 1, \tag{5.8}$$

while

$$\operatorname{rank} \mathcal{Q}\left(V_{-2}^2 = \frac{\lambda}{\rho}\cos^2\theta\right) = 2, \tag{5.9}$$

so that $\mathbf{A}_3$ can be found with two degrees and one degree of indeterminacy, respectively.

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MITTELINEAARSED KIIRENDUSLAINED LIGIKAUDSETE PINGEKITSENDUSTEGA ELASTSETES KEHADES

Franco PASTRONE ja Maria Luisa TONON

On esitatud üldistatud käsitlus mittelineaarsete kiirenduslainete levi kohta teist järku poolust sisaldava olekuvõrrandiga kirjeldatavates pingekitsendustega elastsetes materjalides. Teooriat on rakendatud ristuvates suundades isotroopsete kehade, nagu ühes suunas peaaegu mittedeformeeruvate kiududega armeeritud materjalide puhul kiudude suunas.