Proc. Estonian Acad. Sci. Phys. Math., 1999, **48**, 3/4, 199–205
https://doi.org/10.3176/phys.math.1999.3/4.04

SCALING LAWS FOR TURBULENCE AT VERY LARGE REYNOLDS NUMBERS

G. I. BARENBLATT and A. J. CHORIN

Department of Mathematics, University of California, and Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA

Received 16 February 1999

Abstract. Concepts of scaling laws for turbulence at large Reynolds numbers (Re) are analyzed. Instead of classical Re-independent logarithmic dependence of the average velocity on governing parameters, a new version of the power law is proposed and verified on the basis of recent experimental data. Reasons for mistaking the logarithmic law are explained.

Key words: turbulence, scaling laws for turbulent flows, universal logarithmic law.

Turbulent fluid flow surrounds us, in the atmosphere, the oceans, in engineering, and biological systems. First examined by Leonardo, for the past century it has been intensely studied by engineers, mathematicians, and physicists, including giants such as Kolmogorov, Heisenberg, Taylor, Prandtl, and von Kármán. Every advance in a wide collection of subjects, from chaos and fractals to field theory, and every increase in the speed and parallelization of computers, is heralded as ushering in the solution of the "turbulence problem", yet turbulence remains the greatest challenge of applied mathematics as well as of classical physics. In particular, none of the main results of turbulence theory has been derived from the first principles such as the equations of fluid mechanics, and all rest on additional assumptions which must be reexamined as knowledge expands. This is exactly what we have done.

Consider the turbulent flow in a pipe – one of the most common, familiar, and useful flows. It is surely surprising that it should be poorly understood, in important respects less understood than the structure of stars. The scientific study of pipe flow began when Osborne Reynolds discovered, in the nineteenth century, that pipe flow becomes turbulent when the "Reynolds number" Re, the mean velocity across the pipe $\bar{u}$ times the pipe's diameter d divided by the fluid's viscosity ν, is sufficiently large; once turbulence sets in, the fluid's velocity and pressure fluctuate unpredictably. For engineering reasons it is important to know how the time averaged velocity u varies as the distance y from the wall increases; long ago, after much painstaking labor, engineers determined a "power" relation between u and y:

$$u = Ay^n, \tag{1}$$

where the power n and the coefficient A were known to depend slightly on Re and were determined from experiment. This law was supposed to hold everywhere except very near the wall and the centerline and was good enough for practical applications. However, at the beginning of the 1930s two famous fluid dynamicists, von Kármán and Prandtl, convinced the world [1,2] that in truth the relation between u and y had the "universal logarithmic" form:

$$u = u_* \left(\frac{1}{\kappa} \ln \frac{u_* y}{\nu} + C \right), \tag{2}$$

where u_* is a reference velocity (square root of the shear stress at the wall divided by fluid density), and κ, C should be "universal" constants, independent of the Reynolds number Re. The argument for the "universal" law was purely theoretical; it can be derived by requiring that the relation between u and y be independent of the units of measurements and that certain functions that appear in the analysis remain bounded and not vanish. Indeed, the lack of dependence on the units (dimensional analysis) gives

$$\frac{\partial u}{\partial y} = \frac{u_*}{y} \, \Phi(\eta, Re). \tag{3}$$

Here $\eta = u_* y/\nu$, Φ is an unknown dimensionless function of its dimensionless arguments.

The quantity η is large outside a very thin layer close to the wall $y = 0$. The Reynolds number Re is also very large, more than 10 000 for "developed" turbulent flows, for which the validity of asymptotic laws can be expected. Therefore it was natural to assume, as von Kármán and Prandtl did, that at large η and Re the function Φ can be replaced by its value at $\eta = \infty$, $Re = \infty$, which was denoted by $1/\kappa$; for the constant κ a special name was coined: the von Kármán constant. *There was a tacit assumption of boundedness in this argument that the constant* κ *was finite.* The relation (3) then becomes

$$\frac{\partial u}{\partial y} = \frac{u_*}{\kappa y}, \quad \frac{\partial \phi}{\partial \eta} = \frac{1}{\kappa \eta}, \tag{4}$$

where ϕ is the dimensionless velocity $\phi = u/u_*$, and the relation (2) is obtained from (4) by simple integration, however, under *another tacit assumption*: *the integration constant is* Re-*independent*. Izakson, Millikan, and von Mises (IMM) (see [3]) then provided what seemed to be a second mathematical derivation of (2) based on unassailable principles, and a famous series of experiments by Nikuradze [4] did not contradict the logarithmic law too much. This law hardened into dogma and became one of the pillars of turbulence theory and a mainstay of engineering science, widely taught and used [3,5,6].

We originally became suspicious of the logarithmic law on mathematical grounds: A more detailed analysis cast a doubt on the boundedness assumptions in the original von Kármán–Prandtl argument and also on the *Re*-independence of the integration constant. Then a careful processing of Nikuradze data [4], based on new mathematical tools such as incomplete similarity and intermediate asymptotics (closely related in fact to a concept of the renormalization group, well known in theoretical physics), as well as vanishing-viscosity asymptotics, showed that they were compatible with the law

$$u/u_* = \left(\frac{1}{\sqrt{3}}\ln Re + \frac{5}{2}\right)\eta^{3/2\ln Re}, \tag{5}$$

which has the power-law form (1). Indeed, in Fig. 1 the complete set of Nikuradze data is presented in "universal" coordinates ψ, $\ln\eta$, where

$$\psi = \frac{1}{\alpha}\ln\frac{2\alpha\phi}{\sqrt{3}+5\alpha}\,, \qquad \phi = \frac{u}{u_*}\,, \qquad \eta = \frac{u_* y}{\nu}\,. \tag{6}$$

If (5) is correct, then at large η the experimental points should lie on the bisectrix of the first quadrant. The agreement is instructive. There is a big difference between the power law (5) and the logarithmic law; in particular, according to the logarithmic law the relation between u and y is independent of *Re*, while according to the proposal (5), it does depend on *Re*.

Could the logarithmic law be wrong, could most people have been mistaken for seventy years, and could the engineering community not have noticed that a conclusion of such practical significance was in error? The answer is yes, and the reasons are subtle: The family of curves produced by the power law (5) has an envelope, which is nearly identical to the graph of the von Kármán–Prandtl law with the usual values of its constants. If one plots points for many values of *Re* on a single graph, as is natural if one believes with von Kármán and Prandtl that *Re* does not play a role, then one produces a cloud of points whose boundary, the envelope, acquires a spurious prominence; this envelope can thus easily be mistaken for the curves themselves. In addition, as the viscosity decreases, the slopes of the family of power law curves (5) in the intermediate, middle part of the curves which only has the physical interest tend to a finite, constant, limit. However, it does not mean that the integration constant also tends to a finite *Re*-independent limit! This is enough for the power law to be permitted by the IMM argument, a possibility that had been overlooked. In fact, there is a fixed angle between the envelope and the asymptotic slope in the power law, which should be observable in good data. By a happy coincidence, new experimental data have recently become available, and they verify the proposed law (5); indeed, an anomaly that could have slightly altered the constants in (5) (but not its form) allowed us to identify a flaw in the *experimental* procedure. Figure 2 shows a schematic of the power law curves in $(\ln\eta, u/u_*)$

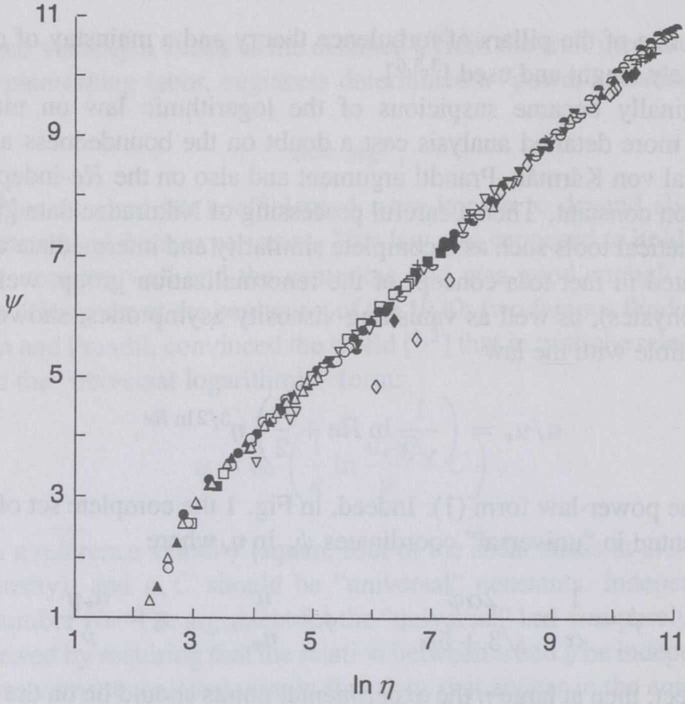

Fig. 1. The experimental points in the coordinates ($\ln \eta, \psi$) at $\eta > 30$ lie close to the bisectrix of the first quadrant, confirming the scaling law.

△ $Re = 4 \times 10^3$	□ $Re = 2.33 \times 10^4$	◡ $Re = 3.96 \times 10^5$	+ $Re = 1.959 \times 10^6$
▲ $Re = 6.1 \times 10^3$	■ $Re = 4.34 \times 10^4$	◒ $Re = 7.25 \times 10^5$	× $Re = 2.35 \times 10^6$
○ $Re = 9.2 \times 10^3$	▽ $Re = 1.05 \times 10^5$	◇ $Re = 1.11 \times 10^6$	⌓ $Re = 2.79 \times 10^6$
● $Re = 1.67 \times 10^4$	▼ $Re = 2.05 \times 10^5$	◆ $Re = 1.536 \times 10^6$	▬ $Re = 3.24 \times 10^6$

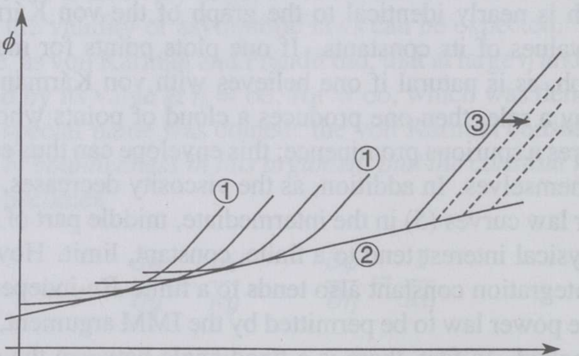

Fig. 2. A schematic of the power law curves in a pipe, their envelope and their asymptotic slope. The apparent motion of the curves to the right is due to the changes in the Reynolds number. (1) The velocity as a function of the distance to the wall (in appropriate units), (2) the envelope of the power laws (formerly mistaken for the curves themselves), (3) the asymptotic slope.

coordinates, and Fig. 3 exhibits the corresponding experimental data for a pipe. A detailed comparison is available; the logarithmic law must be abandoned (see the details in [7–10]).

Why are these results important? Engineers have long known that conclusions drawn from the logarithmic law, for example friction coefficients and other quantities of practical interest, are unreliable, and they use instead empirical functions that fit the data. They will presumably be happy to hear that now the empirical fits can be derived from a better law. Laws such as (2), (5) and their generalizations to other geometries enter into various computer models of turbulence, which now will presumably be upgraded. However, the main significance of this work lies in its impact on theory; it corrects but also solidifies our understanding of key issues in turbulence and has broad implications.

Consider the second pillar of turbulence theory, the Kolmogorov–Obukhov law [3,11,12], which applies to turbulence far from a wall. Turbulent flow has many scales of motion, just as a detailed weather map contains patterns that encompass continents and others that affect mere neighborhoods. The larger scales are determined by what stirs the fluid, while the smallest simply dissipate energy by friction. It is the range of scales in between, the "inertial range of local structure", that is the proper locus of a general theory, and that controls effects such as the diffusion of pollutants in the atmosphere or of fuel in a turbine. The great mathematician Kolmogorov and his student at that time, Obukhov, proposed a theory for that range of scales, which has been challenged over the years on various grounds. The tools we used in pipe flow can be used here too, and affirm the basic correctness of the classical version of the Kolmogorov–Obukhov theory.

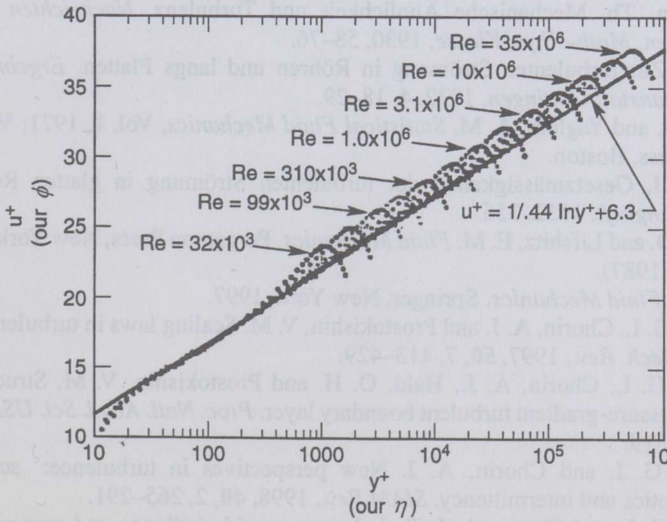

Fig. 3. The corresponding experimental data (reproduced with permission from Zagarola et al. [13]). The curves turn down at the center of the pipe, where the theory above is invalid.

More generally, the solutions of the equations of motion in fluid mechanics are chaotic when the Reynolds number Re is large, and the minutest perturbations change them. What is of interest is not specific solutions, which may never be observed, but the properties of collections of solutions with attendant probability measures. Such "random solutions" are also more amenable to analysis; we have made progress towards describing them, and indeed this progress contributed to the analyses above. We have also been able to calculate numerically, from the statistical theory, some of the properties of the inertial range, and found that they are consistent with the new understanding. The general statistical theory is thus validated by the data in special cases, and it is our contention that this happy agreement constitutes a step towards a fundamental theory of turbulence.

ACKNOWLEDGMENTS

Professors N. Goldenfeld (Illinois), O. Hald (Berkeley), and Dr. V. M. Prostokishin (Moscow) participated in several aspects of this work. This research was supported in part by the Applied Mathematical Science subprogram of the Office of Energy Research, U.S. Department of Energy, under contract DE–AC03–76–SF00098, and by National Science Foundation grant DMS–97–32710.

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TURBULENTSI SKALEERUMISE SEADUSED VÄGA SUURTE REYNOLDSI ARVUDE KORRAL

G. I. BARENBLATT ja A. J. CHORIN

Essee vormis on analüüsitud turbulentsi struktuuri, sh. keskmise kiiruse kirjeldamise kontseptsioone suurte Reynoldsi arvude puhul. Uute eksperimentaalandmete põhjal on esitatud klassikalise Reynoldsi arvu sõltumatu logaritmilise seose asemel erikujuline astmefunktsioon. On näidatud, et logaritmilist sõltuvust kirjeldav graafik on samastatav nimetatud astmefunktsioonide parve mähisjoonega.