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CONTINUOUS AND DISCRETE ELASTIC STRUCTURES

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Abstract. A theory of elastic trusses is developed on the model of the elasticity theory of continuous bodies. The analogies and differences between the continuous and discrete representatives of elasticity are discussed. Using the concept of material uniformity, configurational nodal forces in a truss are introduced, which correspond to the configurational stresses in a continuous elastic body.

Key words: elasticity, truss, configurational forces.

1. INTRODUCTION

Elasticity theory is usually regarded as a branch of continuum mechanics in which the material behaviour is characterized by a constitutive equation relating the local stress to the local deformation of the material. The study of the mapping from the undeformed body to its deformed state is closely related to differential geometry, and, from the historical point of view, it was the same people that created both elasticity theory and differential geometry.

Three-dimensional elasticity theory is taught as an "advanced" subject, since its mere formulation needs differential-geometric prerequisites, such as vector and tensor analysis, integral theorems, etc. One-dimensional models of elastic bodies are simpler in the sense that only one space variable enters the theory. Therefore the bending of elastic beams, for instance, is taught prior to the general elasticity theory. But is it really "simpler"? Once the notions of multivariate and tensor analysis are absorbed and accepted as a standard, three-dimensional elasticity appears easier and more reasonable than the Cosserat-type theory of beams! Although the

Euler–Bernoulli beam theory provides only an approximation to the exact three-dimensional behaviour of a beam, it leads to equations of the fourth order and is, from this point of view, more complicated than the three-dimensional theory of elasticity.

Still more elementary is the static theory of elastic trusses, which leads to algebraic rather than differential equations. No prerequisites from calculus are needed to formulate the theory, and in this sense it is the simplest elastic structure. But, again, the simplicity is questionable. In a continuous elastic body the stress depends only on the *local* deformation. In a truss, however, there is no locality. Forces can be communicated directly between distant nodes if these nodes are connected by a member of the truss.

Another aspect is related to the formulation of the theory: In continuous elasticity all the relevant quantities, such as stress, strain, dislocation density, stress functions, etc., are fields defined in every interior point of the elastic body. In a truss we have to attach displacements or external forces to the nodes, elongation, stress, and strain-energy density to the members of the truss, and it is not clear from the outset, where and how a stress function or the strength of a dislocation has to be defined for a truss. In this respect a truss is far more complicated than a continuous elastic body, where all these quantities are fields depending on the material position $\boldsymbol{X}$ and, in the dynamical case, on time t.

Further insight into both the continuous and the discrete version of elasticity may be obtained by exploiting the analogy between continuous elastic bodies and trusses as far as this is possible. The present paper is but the first attempt in this direction. A theory of trusses is developed to some extent by carrying over the well-known concepts of continuum mechanics, as presented in [1,2], to this problem of discrete elasticity. The analogies and also the differences between continuous and discrete elastic systems are made evident.

In Section 2, following this introduction, the deformation of continuous elastic bodies and trusses are contrasted with each other. The relevant quantities for the material behaviour are the right stretch tensor in the continuous case and the stretches of the members in the case of a truss. Section 3 is devoted to strain energy, which is obtained by integrating its density in the continuous case and by collecting the contributions of all the individual members in the case of a truss. The equilibrium conditions of statics are presented in Section 4.

Epstein and Maugin [3] have used the concept of material uniformity and inhomogeneity to introduce the Eshelby stress tensor in continuous elasticity. Translating this idea to trusses leads to internal configurational forces, or Eshelby forces, acting along the members of a truss. Configurational forces in a truss are studied in Section 5.

The last section takes up J. Engelbrecht's motif of "complexity and simplicity" [4]. In the present context the discussion is centred around the question "Is the theory of trusses really simpler than that of continuous bodies?"

2. DEFORMATION

The time-dependent deformation of an elastic continuum is characterized by a mapping

$$\boldsymbol{X} \mapsto \boldsymbol{x}(\boldsymbol{X}, t), \tag{2.1}$$

which describes the position $\boldsymbol{x}$ at the time t of any material point that has been located at $\boldsymbol{X}$ in a certain reference placement (Fig. 1). A neighbourhood of a material point $\boldsymbol{X}$ is mapped to the corresponding neighbourhood of $\boldsymbol{x}(\boldsymbol{X}, t)$ by the linear transformation

$$\mathrm{d}\boldsymbol{X} \mapsto \mathrm{d}\boldsymbol{x} = \boldsymbol{F}\,\mathrm{d}\boldsymbol{X}. \tag{2.2}$$

The deformation gradient $\boldsymbol{F} = \partial\boldsymbol{x}/\partial\boldsymbol{X}$ assigns to any material line element $\mathrm{d}\boldsymbol{X}$ the corresponding line element $\mathrm{d}\boldsymbol{x}$ in the actual placement of the body (Fig. 2).

In general, the linear transformation involves both a rigid-body rotation and a real deformation. According to the polar decomposition theorem, the deformation gradient can be written as a product

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}. \tag{2.3}$$

The neighbourhood undergoes a deformation described by the symmetrical, positive definite stretch tensor $\boldsymbol{U}$, which is followed by a rigid-body rotation expressed by the rotation tensor $\boldsymbol{R}$. As a consequence of the principle of material frame indifference, it is only the (right) stretch tensor $\boldsymbol{U}$ which is relevant to the material response of the body.

The previous description of deformation in the continuous case is now carried over, as closely as possible, to the case of a discrete truss. A truss consists of a finite number of nodes which are connected by members. So we have to deal not with a continuum of material points $\boldsymbol{X} \in \mathcal{B}$, but with a finite collection of nodes

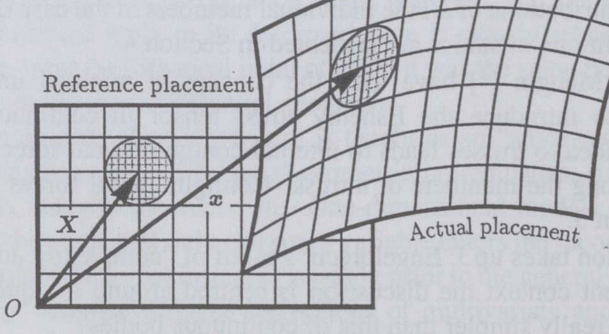

Fig. 1. Deformation of a continuous body.

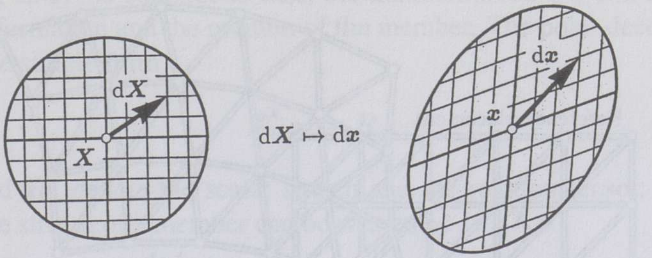

Fig. 2. Deformation of a neighbourhood.

at positions $\{\boldsymbol{X}_i\}_{i=1}^n$ in the reference placement. Each of the m members connects two different nodes of the truss. The topological structure is described by incidence numbers [5]

$$[a,i] = \begin{cases} -1 & \text{if member } a \text{ starts at node } i, \\ +1 & \text{if member } a \text{ ends at node } i, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

constituting an $m \times n$ matrix $\mathbf{C}$. Each row of $\mathbf{C}$ contains $+1$ and -1 exactly once, while all other entries vanish, and each column of $\mathbf{C}$ has at least one nonzero entry. In principle, it would not be necessary to distinguish between "start" and "end" of a member, but it is convenient to give each member an orientation. Actually, the use of signed incidence numbers endows the truss with the structure of a *directed* graph.

The deformation of the truss is characterized by the mapping

$$\boldsymbol{X}_i \mapsto \boldsymbol{x}_i = \boldsymbol{x}_i(t), \tag{2.5}$$

which assigns to every material node $\boldsymbol{X}_i$ its actual position $\boldsymbol{x}_i$ at the time t (Fig. 3). A placement or configuration of the truss, in analogy to the placement of a continuous body, is the mapping $i \mapsto \boldsymbol{x}_i$. This means that a placement of a truss is simply the collection of all its nodal positions $\{\boldsymbol{x}_i\}_{i=1}^n$. The deformation is the mapping from the reference placement with nodal positions $\boldsymbol{X}_i$ to the actual placement with nodal positions $\boldsymbol{x}_i$.

In a continuous body the neighbourhood of a material point $\boldsymbol{X}$ is the local tangent space $T_{\boldsymbol{X}}(\mathcal{B})$ at this point. In a truss the neighbourhood of a node i is formed by the members emerging from this node (Fig. 4). Instead of the line elements $\mathrm{d}\boldsymbol{X}$ and $\mathrm{d}\boldsymbol{x}$ we have a finite number of vectors

$$\Delta \boldsymbol{X}_a = \sum_{i=1}^n [a,i]\boldsymbol{X}_i \quad \text{and} \quad \Delta \boldsymbol{x}_a = \sum_{i=1}^n [a,i]\boldsymbol{x}_i, \tag{2.6}$$

which are aligned with the members of the truss in the reference placement and in the actual placement, respectively. Actually, the sums are reduced to a difference of two

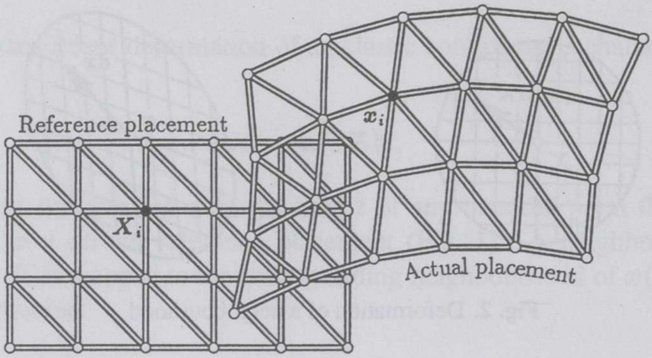

Fig. 3. Deformation of a truss.

position vectors, i. e., $\Delta \boldsymbol{X}_a = \boldsymbol{X}_{k^+} - \boldsymbol{X}_{k^-}$, where $[a, k^+] = +1$ and $[a, k^-] = -1$ are the only nonzero entries in the ath row of the incidence matrix. Depending on the signs of the incidence numbers, each member a is equipped with an orientation induced by the vector $\Delta \boldsymbol{X}_a$. The final results must not depend on the prescribed orientation. It is introduced, since we want to attach a vector to each member, and by this vector the member becomes oriented.

In a continuum the vectors $\mathrm{d}\boldsymbol{X}$ undergo a linear transformation (2.2) generated by the deformation gradient $\boldsymbol{F}$. The deformation of the whole neighbourhood of a material point $\boldsymbol{X}$ is determined by the mapping $\mathrm{d}\boldsymbol{X} \mapsto \mathrm{d}\boldsymbol{x}$ of three linearly independent line elements $\mathrm{d}\boldsymbol{X}$ attached to this material point (Fig. 2).

In a truss the situation is quite different: Each of the members emanating from a single node $\boldsymbol{X}_i$ has its own deformation. There is no linear transformation of the neighbourhood of a node (cf. Fig. 4). If a transformation is introduced in analogy to the deformation gradient $\boldsymbol{F}$, it has to be formulated independently for each member a. So we can write, for instance,

$$\Delta \boldsymbol{X}_a \mapsto \Delta \boldsymbol{x}_a = \boldsymbol{F}_a \Delta \boldsymbol{X}_a, \tag{2.7}$$

where each of the members a undergoes its own transformation $\boldsymbol{F}_a$.

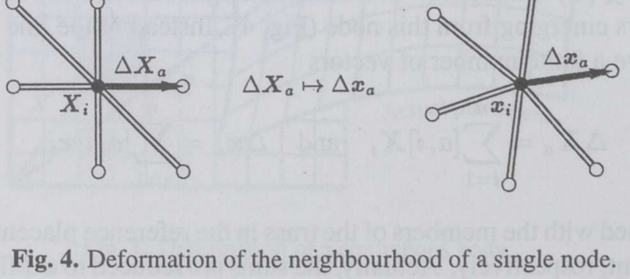

Fig. 4. Deformation of the neighbourhood of a single node.

However, as in the continuous case, the transformation $\boldsymbol{F}_a$ still contains both the actual deformation and the rotation of the member. The polar decomposition is reduced to the simple form

$$\boldsymbol{F}_a = \lambda_a \boldsymbol{R}_a, \tag{2.8}$$

where λ_a and $\boldsymbol{R}_a$ denote the scalar stretch and the rotation tensor, respectively. Explicitly, the stretch of a member can be written as

$$\lambda_a = \frac{l_a}{L_a}, \tag{2.9}$$

where

$$L_a = |\Delta \boldsymbol{X}_a| \quad \text{and} \quad l_a = |\Delta \boldsymbol{x}_a| \tag{2.10}$$

denote the lengths of the member a in the reference and the actual placement, respectively. Introducing the unit vectors

$$\boldsymbol{E}_a = \frac{1}{L_a} \Delta \boldsymbol{X}_a \quad \text{and} \quad \boldsymbol{e}_a = \frac{1}{l_a} \Delta \boldsymbol{x}_a \tag{2.11}$$

in reference and actual placements, respectively, one can express the rotation tensor explicitly as

$$\boldsymbol{R}_a = \boldsymbol{e}_a \otimes \boldsymbol{E}_a. \tag{2.12}$$

This representation does not play any further role, since the relevant part of the transformation of a truss member is its stretch λ_a.

3. STRAIN ENERGY

The strain energy stored in an elastic body $\mathcal{B}$ is

$$\Pi = \int_{\mathcal{B}} W(\boldsymbol{F}(\boldsymbol{X}, t), \boldsymbol{X}) \, \mathrm{d}V, \tag{3.1}$$

where W denotes the strain-energy density, measured per unit volume in the reference placement. Also, the integration is performed over the body in its reference placement. The strain-energy density is a function $W = W(\boldsymbol{F}, \boldsymbol{X})$ that depends on the local deformation gradient $\boldsymbol{F}$ and, in the case of an inhomogeneous material, also directly on the material position $\boldsymbol{X}$. The principle of material frame indifference states that the strain-energy density is not changed by any local rigid-body rotation. Therefore the strain-energy density may depend on the deformation gradient only via the right stretch tensor $\boldsymbol{U}$. Additional restrictions arise from the symmetry properties of the material. If the material is isotropic, the strain-energy density is a function of the eigenvalues of $\boldsymbol{U}$, i. e., the principal stretches $\lambda_1, \lambda_2, \lambda_3$. In the sequel we simply write $W = W(\boldsymbol{F}, \boldsymbol{X})$ and disregard the special form of dependence due to material frame indifference and material symmetry.

In order to exploit the principle of virtual work, one needs the variation $\delta\Pi$ of the strain energy (3.1). This is obtained as

$$\delta\Pi = \int_{\mathcal{B}} \boldsymbol{T} \cdot \delta\boldsymbol{F} \; \mathrm{d}V, \tag{3.2}$$

where $\delta\boldsymbol{F} = \partial(\delta\boldsymbol{x})/\partial\boldsymbol{X}$ is the variation of the deformation gradient and

$$\boldsymbol{T} = \frac{\partial\Pi}{\partial\boldsymbol{F}} \tag{3.3}$$

denotes the nominal (or first Piola–Kirchhoff) stress tensor. Using Gauss' theorem, the variation of the strain energy can be written in the form

$$\delta\Pi = \int_{\partial\mathcal{B}} \delta\boldsymbol{x} \cdot \boldsymbol{T}\boldsymbol{n} \; \mathrm{d}A - \int_{\mathcal{B}} \delta\boldsymbol{x} \cdot \operatorname{Div} \boldsymbol{T} \; \mathrm{d}V, \tag{3.4}$$

with $\boldsymbol{n}$ denoting the outer normal vector of the boundary $\partial\mathcal{B}$ in the reference placement. Also, the divergence operator Div refers to the reference placement, where all the integrations have to be performed.

The strain energy of a truss is obtained by collecting the contributions of all of its members. The strain energy per unit reference length of a member is denoted by w_a. Since the strain energy is not affected by a rotation of the member, it may depend only on the stretch λ_a. Thus we have

$$\Pi = \sum_{a=1}^{m} w_a(\lambda_a(t)) \, L_a. \tag{3.5}$$

The strain energy per unit length w_a of each member is multiplied by the length L_a of the member in the reference placement; then the contributions of all members a are collected. In the continuous case we have assumed the strain-energy density W to depend on the position $\boldsymbol{X}$ not only via the deformation gradient $\boldsymbol{F}$ but also directly, to allow for inhomogeneous media. The same holds for the case of a truss: The strain energy per unit length of a member a depends on the stretch λ_a of that member, but this functional relationship may be different for each individual member, which is indicated by the subscript a in w_a. If the truss consists of uniform members, one has the same strain-energy function $w = w(\lambda_a)$ for all members.

The principle of virtual work, applied to a truss, needs the variation $\delta\Pi$ of (3.5). Following the proceeding in the continuous case, one first obtains the equation

$$\delta\Pi = \sum_{a=1}^{m} F_a(\lambda_a)\delta\lambda_a L_a, \tag{3.6}$$

in which the scalar member force

$$F_a(\lambda_a) = \frac{\mathrm{d}w_a}{\mathrm{d}\lambda_a} \tag{3.7}$$

has been introduced. As usual, we denote the force by the letter F, but this should not be mistaken for a deformation gradient, which is denoted by a bold-face $\boldsymbol{F}$. Using (2.9), $(2.10)_2$, and $(2.11)_2$, the variation of the stretch λ_a can be expressed in terms of the virtual displacements $\delta\boldsymbol{x}_i$ of the nodal positions, viz.,

$$\delta\lambda_a = \frac{1}{L_a}\,\boldsymbol{e}_a \cdot \sum_{i=1}^{n}[a,i]\delta\boldsymbol{x}_i. \tag{3.8}$$

Changing the order of summation, one obtains the variation of (3.6) in the form

$$\delta\Pi = \sum_{i=1}^{n}\left(\sum_{a=1}^{m}[a,i]F_a(\lambda_a)\boldsymbol{e}_a\right)\cdot\delta\boldsymbol{x}_i. \tag{3.9}$$

The vector $[a,i]\boldsymbol{e}_a$, for $[a,i] \neq 0$, is a unit vector aligned with the member a and pointing *towards* the node i. Thus the expression in parentheses, $\sum[a,i]F_a(\lambda_a)\boldsymbol{e}_a$, can be interpreted as the negative sum of all member forces acting on the node i.

Equations (3.4) and (3.9) are, in a sense, counterparts to each other. These equations express the variation $\delta\Pi$ of the total strain energy in terms of the virtual displacements $\delta\boldsymbol{x}(\boldsymbol{X})$ and $\delta\boldsymbol{x}_a$, respectively. Of course, in the case of a truss there is no distinction between inner points $\boldsymbol{X} \in \mathcal{B}$ and boundary points $\boldsymbol{X} \in \partial\mathcal{B}$. Therefore, instead of a volume integral and a surface integral there is only a sum extending over all nodes. Moreover, the discrete equivalent of the vector $\operatorname{Div}\boldsymbol{T}$ in (3.4) seems to be the total member force

$$-\sum_{a=1}^{m}[a,i]F_a(\lambda_a)\boldsymbol{e}_a \tag{3.10}$$

acting on the single node i.

4. EQUILIBRIUM

The equations of equilibrium can be derived from the principle of virtual work, according to which the variation of total strain energy equals the virtual work done by the external forces. This principle holds true for both the continuous and the discrete case.

A continuous elastic body is, in general, loaded by volume forces of density $\boldsymbol{f}$ and by surface tractions of density $\boldsymbol{t}$. Both the volume and the surface density are understood per unit volume and per unit surface of the body in its reference placement. Explicitly, the principle of virtual work for a continuous elastic body stipulates that

$$\delta\Pi = \int_{\mathcal{B}} \boldsymbol{f} \cdot \delta\boldsymbol{x} \; \mathrm{d}V + \int_{\partial\mathcal{B}} \boldsymbol{t} \cdot \delta\boldsymbol{x} \; \mathrm{d}A \tag{4.1}$$

holds for arbitrary virtual displacements $\delta\boldsymbol{x}$. Comparing this assertion with the expression (3.4) for the variation $\delta\Pi$, one obtains immediately the equations

$$\operatorname{Div} \boldsymbol{T} + \boldsymbol{f} = 0 \quad \text{in } \mathcal{B}, \qquad \boldsymbol{T}\boldsymbol{n} = \boldsymbol{t} \quad \text{on } \partial\mathcal{B}, \tag{4.2}$$

which constitute the corresponding boundary-value problem.

In a truss the external forces are acting on the nodes. Therefore, the principle of virtual work states that

$$\delta\Pi = \sum_{i=1}^{n} \boldsymbol{f}_i \cdot \delta\boldsymbol{x}_i \tag{4.3}$$

holds for arbitrary virtual displacements $\delta\boldsymbol{x}_i$ of the nodes. On the other hand, Eq. (3.9) represents the variation $\delta\Pi$ in terms of the virtual nodal displacements $\delta\boldsymbol{x}_i$. Comparing these two equations, one finds the condition

$$-\sum_{a=1}^{m} [a,i] F_a(\lambda_a) \boldsymbol{e}_a + \boldsymbol{f}_i = 0, \tag{4.4}$$

which expresses the balance of forces: At every node i the internal member forces (3.10) and the applied external force $\boldsymbol{f}_i$ must be in equilibrium.

5. CONFIGURATIONAL FORCES

Epstein and Maugin [3] introduce the Eshelby stress tensor by using the concept of material uniformity of an elastic body. They regard the body in its reference placement as being composed of uniform material pieces, the reference crystals, which, in general, do not fit together in their natural, stress-free form. Therefore, each of these reference crystals first has to undergo a local deformation $\boldsymbol{K}$ to assume its proper form in the reference placement (Fig. 5). The deformations $\boldsymbol{K}$ are not necessarily identical for different pieces of the body but may depend on the material position $\boldsymbol{X}$.

The strain-energy density of a *uniform* elastic body has the form

$$W(\boldsymbol{F}, \boldsymbol{X}) = \frac{1}{J_{\boldsymbol{K}(\boldsymbol{X})}} \bar{W}(\boldsymbol{F}\boldsymbol{K}(\boldsymbol{X})). \tag{5.1}$$

Here $\boldsymbol{K}(\boldsymbol{X})$ is the deformation that is necessary to insert the stress-free crystals into the reference placement, where these individual pieces are assembled, and $\bar{W}$ denotes the strain-energy density per unit volume of the stress-free crystal. The volume factor $J_{\boldsymbol{K}} = \det \boldsymbol{K}$ takes into account that the strain-energy densities W and $\bar{W}$ refer to different volumes: W is measured per unit volume in the reference placement and $\bar{W}$ per unit volume of the stress-free crystal. All material pieces obey the same constitutive equation provided by the function $\bar{W}$, but the deformation $\boldsymbol{K}$ may depend on the material position $\boldsymbol{X}$.

Thus the strain-energy density W depends on both the local deformation $\boldsymbol{K}$ generating the reference placement from the disconnected reference crystals and the deformation gradient $\boldsymbol{F}$ that maps the reference placement to the actual placement (Fig. 5). The derivative of W with respect to the deformation gradient $\boldsymbol{F}$ yields the nominal stress tensor (3.3), which, on account of (5.1), can be written as

$$\boldsymbol{T} = \frac{\partial W}{\partial \boldsymbol{F}} = \frac{1}{J_{\boldsymbol{K}}} \frac{\partial \bar{W}}{\partial(\boldsymbol{F}\boldsymbol{K})} \boldsymbol{K}^{\mathsf{T}} . \tag{5.2}$$

The derivative of W with respect to $\boldsymbol{K}$ is obtained as

$$\frac{\partial W}{\partial \boldsymbol{K}} = -\frac{1}{J_{\boldsymbol{K}}} \bar{W}(\boldsymbol{F}\boldsymbol{K})\, \boldsymbol{K}^{-\mathsf{T}} + \frac{1}{J_{\boldsymbol{K}}} \boldsymbol{F}^{\mathsf{T}} \frac{\partial \bar{W}}{\partial(\boldsymbol{F}\boldsymbol{K})} . \tag{5.3}$$

Using (5.1) and (5.2), the function $\bar{W}(\boldsymbol{F}\boldsymbol{K})$ and its derivative are substituted back and expressed in terms of the original strain-energy density function W and its derivative. The derivative (5.3) can then be written as

$$\frac{\partial W}{\partial \boldsymbol{K}} = -\left(W\boldsymbol{I} - \boldsymbol{F}^{\mathsf{T}}\boldsymbol{T}\right) \boldsymbol{K}^{-\mathsf{T}}, \tag{5.4}$$

where $\boldsymbol{I}$ denotes the unity tensor. The tensor in parentheses coincides with the Eshelby stress tensor or, strictly speaking, a modification of Eshelby's original

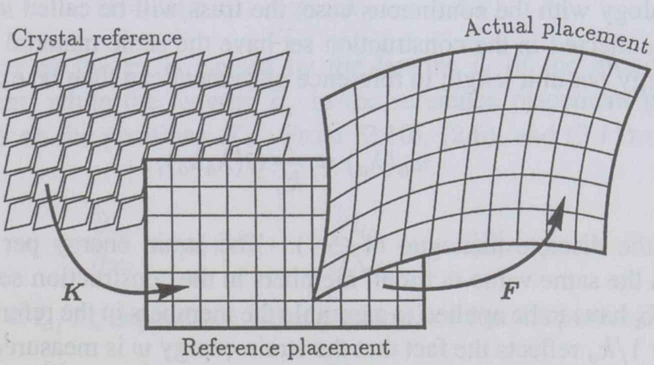

Fig. 5. Various placements of a continuous body.

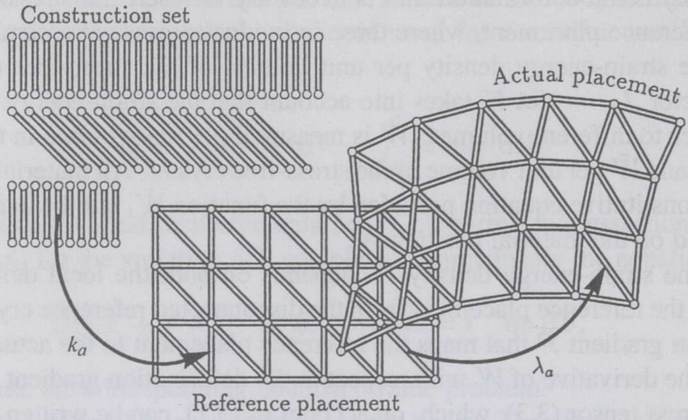

Fig. 6. Various placements of a truss.

tensor adjusted to nonlinear elasticity. Epstein and Maugin [3] use this equation to introduce the Eshelby stress tensor (or energy-momentum tensor)

$$\mathfrak{E} = -\frac{\partial W}{\partial \boldsymbol{K}} \boldsymbol{K}^{\mathsf{T}} = W\boldsymbol{I} - \boldsymbol{F}^{\mathsf{T}}\boldsymbol{T}. \tag{5.5}$$

A thorough discussion of this approach can be found in [6].

The concept presented for a continuous elastic body can be transferred to the case of the discrete elastic truss. Also here the reference placement of the truss is not the real starting point. The truss is assembled from members that are collected in a construction set. To generate the reference placement, each member a has to be rotated into its proper attitude and, if it does not fit as it is, stretched by an amount k_a to its reference length L_a. Afterwards, by application of appropriate loads, the truss is deformed from its reference placement into its actual placement, stretching the member a further by the amount λ_a (Fig. 6).

In analogy with the continuous case, the truss will be called *uniform* if all the members collected in the construction set have the same material properties. The strain-energy per unit length in reference placement can then be expressed as

$$w_a(\lambda_a) = \frac{1}{k_a}\,\bar{w}(\lambda_a k_a), \tag{5.6}$$

which is the discrete analogue of (5.1). The strain energy per unit stress-free length has the same value $\bar{w}$ for all members in the construction set, but individual stretches k_a have to be applied to assemble the members in the reference placement. The factor $1/k_a$ reflects the fact that the strain energy w is measured per unit length in reference placement, while $\bar{w}$ refers to the stress-free length. The product $\lambda_a k_a$ describes the total stretch from original to actual length of the member a.

Using (5.6), the scalar member force (3.7) can be written as

$$F_a(\lambda_a) = \frac{\partial w_a}{\partial \lambda_a} = \frac{\mathrm{d}\bar{w}_a}{\mathrm{d}(\lambda_a k_a)}. \tag{5.7}$$

Differentiating w_a with respect to the stretch k_a yields

$$\frac{\partial w_a}{\partial k_a} = -\frac{1}{k_a^2}\,\bar{w}_a(\lambda_a k_a) + \frac{1}{k_a}\,\frac{\mathrm{d}\bar{w}_a}{\mathrm{d}(\lambda_a k_a)}\,\lambda_a. \tag{5.8}$$

The function $\bar{w}_a(\lambda_a k_a)$ and its derivative can now be expressed in terms of the original strain-energy function w_a by using (5.6) and (5.7), respectively. Thus the derivative (5.8) becomes

$$\frac{\partial w_a}{\partial k_a} = -\frac{1}{k_a}(w_a - \lambda_a F_a). \tag{5.9}$$

In analogy with (5.5), we define the scalar Eshelby force within a member a as

$$\mathfrak{F}_a = -\frac{\partial w_a}{\partial k_a}\,k_a = w_a - \lambda_a F_a(\lambda_a). \tag{5.10}$$

This corresponds to a one-dimensional version of the Eshelby stress tensor $\mathfrak{E}$.

There is one difference in our treatment of the continuous and the discrete case: The strain-energy function of a truss member is assumed as a function $w_a(\lambda_a)$ depending on the stretch λ_a of a member but not on its rotation. In the continuous case the corresponding property has not been exploited. The strain-energy density is expressed as a function $W(\boldsymbol{F}(\boldsymbol{X}), \boldsymbol{X})$, although the dependence on $\boldsymbol{F}$ is restricted to take place only via the right stretch tensor $\boldsymbol{U}$.

There is also a more direct way to introduce configurational or Eshelby forces in a discrete elastic structure [7]. The total strain energy (3.5) of a truss depends on the nodal positions in both the actual and the reference placement. It can be regarded as a function

$$\Pi = \Pi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n; \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n). \tag{5.11}$$

The dependence on the $\boldsymbol{x}_i$ is caused by the lengths l_a of the members in the actual placement, while the lengths L_a in the reference placement give rise to the dependence on the positions $\boldsymbol{X}_i$. From (2.10), (2.6), and (2.11) one obtains immediately

$$\frac{\partial l_a}{\partial \boldsymbol{x}_i} = [a,i]\boldsymbol{e}_i, \qquad \frac{\partial L_a}{\partial \boldsymbol{X}_i} = [a,i]\boldsymbol{E}_i. \tag{5.12}$$

The stretch $\lambda_a = l_a/L_a$ depends on either position and has the partial derivatives

$$\frac{\partial \lambda_a}{\partial \boldsymbol{x}_i} = \frac{1}{L_a}[a,i]\boldsymbol{e}_i, \qquad \frac{\partial \lambda_a}{\partial \boldsymbol{X}_i} = -\frac{\lambda_a}{L_a}[a,i]\boldsymbol{E}_i. \tag{5.13}$$

With these formulas everything is prepared to write down the partial derivatives of the total strain energy Π with respect to the nodal positions $\boldsymbol{x}_i$ and $\boldsymbol{X}_i$. From the explicit expression (3.5) one obtains

$$\frac{\partial \Pi}{\partial \boldsymbol{x}_i} = \sum_{a=1}^{m} [a, i] F_a(\lambda_a) \boldsymbol{e}_a. \tag{5.14}$$

In the balance of forces this expression has already been interpreted as the external nodal force $\boldsymbol{f}_i$ acting on the node i.

The partial derivative with respect to the material positions is

$$\frac{\partial \Pi}{\partial \boldsymbol{X}_i} = \sum_{a=1}^{m} [a, i] \left\{ w_a(\lambda_a) - \lambda_a F_a(\lambda_a) \right\} \boldsymbol{E}_a. \tag{5.15}$$

Using the definition (5.10) of the Eshelby force in a member, one can write this derivative in the form

$$\frac{\partial \Pi}{\partial \boldsymbol{X}_i} = \sum_{a=1}^{m} [a, i] \mathfrak{F}_a(\lambda_a) \boldsymbol{E}_a, \tag{5.16}$$

which has the same appearance as (5.14).

The partial derivatives of the total strain energy of a truss with respect to the nodal positions $\boldsymbol{x}_i$ and $\boldsymbol{X}_i$ have been presented here without recourse to a corresponding counterpart in the continuous case. It seems that this approach is closely related to the "variational formulation using two variations" as described by Maugin [6].

Despite the close resemblance of the two expressions (5.14) and (5.16), there is a marked difference in their interpretation: According to the balance of forces or, equivalently, the principle of virtual work, the partial derivative $\partial \Pi / \partial \boldsymbol{x}_i$ equals the external force applied to the node i. If the node is free of external loads, the derivative has to vanish. No such law holds for the derivative $\partial \Pi / \partial \boldsymbol{X}_i$. Of course, one can *define* an external configurational force $\mathfrak{f}_i$ impressed on the node i and being in equilibrium with the internal configurational forces (5.16) acting on this node, but the physical relevance of such an external configurational force is questionable. In designing a truss we are completely free to choose the geometry at will. There is no rule stating that a truss should be constructed such that the total strain energy attains a stationary value, i. e., that the Eshelby forces acting on a node are in equilibrium. The reason for this lack of symmetry is the following: The geometrical shape of a truss in its actual placement, described by the nodal positions $\boldsymbol{x}_i$, is determined by the load applied to the truss, according to physical laws. The original shape, however, given by the nodal positions $\boldsymbol{X}_i$, is at our disposal and, in general, free of any restrictions.

6. COMPLEXITY AND SIMPLICITY

Engelbrecht [4] has raised the question of complexity and simplicity in science. With respect to the present problem, one could ask: What is simpler, the theory of continuous elastic deformations or the theory of elastic trusses?

Certainly, the theory of trusses has some aspects that make it rather simple: There is no need for fields depending on space coordinates, the resulting equations are purely algebraic and, if the deformations are assumed to be small, one obtains a linear system of equations determining deformation and stresses. The truss problem consists in establishing and solving this linear algebraic system of equations. The deformation of a continuous body is governed by partial differential equations. Even if linearity is assumed, there are but a few special cases in which the resulting boundary-value problems can be solved in closed form. This seems to confirm that the discrete truss problem is simpler than the corresponding problem for a continuous elastic body.

However, from a different point of view, the theory of trusses is more complicated than continuous elasticity theory. This starts with the geometrical description of the object: A continuous body is determined by describing its shape in the reference placement. The body itself is a collection of an infinite number of material points; it is endowed with the natural metric topology. A truss is not sufficiently described by fixing only its nodes in the reference placement. To complete the description, one has to know which nodes are connected by members. The connectivity of the nodes represents the topological structure of a truss.

There are (at least) two basic elements of a truss, namely nodes and members. In a continuous body one has to deal only with material points. Whatever field quantities are introduced in continuous elasticity, be it displacement, strain, stress, dislocation density, stress function, they all can be regarded as functions of the material position $\boldsymbol{X}$. In the case of a truss, however, one has to select the proper support for a mechanical quantity: Displacements and external forces are defined in the nodes and only there. Strains (elongations), stresses, and strain-energy densities refer always to the members of the truss. One could also find counterparts to dislocation densities and stress functions, but first one has to identify the geometrical element at which these quantities have to be anchored. These considerations suggest that the theory of trusses might be more complex than the theory of continuous elastic bodies.

Whether a theory appears simple or complex depends to a great extent on the prerequisites that are allowed to formulate the theory. For instance, the problem of the catenary is quite simple if the tools for establishing and solving differential equations are available. It is simpler than the corresponding discrete problem, with mass points attached to a massless string, where a closed-form solution cannot be given as easily. On the other hand, in an elementary course one can present the discrete problem, which needs only some geometry and the equilibrium of forces as prerequisites, and in this sense the discrete problem is simpler.

Returning to the comparison of a truss versus a continuous body: It is certainly favourable to work out both theories and study their interrelations. Continuous elasticity theory is well developed and can serve as a paradigm for a corresponding discrete theory. Insights gained by the discrete theory could also be valuable for the continuous theory. Discrete conceptions are sometimes easier to understand than their continuous limits. An example is the concept of the crystal reference: The picture given in Fig. 5 (or any other picture taken from the relevant literature) represents a discrete situation, although it is meant to describe the continuous limit. But one cannot draw an appropriate picture representing a continuously dissected body. This shows that ideas of discrete systems appear also in theories for continua. Certainly, either theory can contribute to the other one, shedding some light on hidden connections and unfolding apparent complexity. The main goal of any theory is to make complex things appear simple.

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PIDEVAD JA DISKREETSED ELASTSED KONSTRUKTSIOONID

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On edasi arendatud elastselt deformeeruvate sõrestike teooriat lähtudes pideva elastse keha modelleerimisel kasutatavatest terminitest. On välja toodud sarnasused ja erinevused, mis tekivad elastsete deformatsioonide kirjeldamisel pideva ja diskreetse lähenemisviisi alusel. Eeldusel, et materjal on homogeenne, on sisse toodud sõrestiku sõlmedesse rakendatud kujujõu mõiste, mis on analoogne pideva elastse keha teoorias kasutatava kujupinge mõistega.