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ON THE "ANALYTIC CONTINUATION" OF CONTINUUM MECHANICS

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Abstract. A pictorial representation is given in order to discuss the essential generalizations of classical continuum mechanics in the presence of space and/or time nonlocality, and nonlinearity. This covers most of the recently introduced generalizations within a critical framework that emphasizes both singularity of the classical theories and the fact that real physical phenomena amenable via continuum theories in sufficiently simple analytical terms remain close, in some sense, to the ideal purely elastic and purely fluid cases. All efforts, however, must be directed at getting closer to physical reality both for intellectual satisfaction and practical endeavours, and hence at facing a reasonable complexity.

Key words: continuum mechanics, nonlocality, dispersion, nonlinearity, dissipation.

1. INTRODUCTION

Recently, some engineers have expressed a rather spurious and somewhat overenthusiastic interest typical of new converts in so-called nonlocal theories of continua, in particular in relation to problems of plasticity and damage of materials. For lack of cultural background, sufficient documentation, and a deeper knowledge of the bases of their own science, some of these authors (of whom, wantonly, we do not give the names) have discovered the Moon and in fact use concepts and redraw general schemes which have been expanded by physicists and applied mathematicians for quite a long time. Among the new fashionable expressions most commonly used at meetings on continuum physics are those of "nonlocality", "localization", "regularization", occurrence of a "characteristic length", etc. Some simple words of explanation are needed before dealing in greater detail with our subject matter. The purpose of this discursive contribution is to present a general setting for these improved complex theories that a better understanding of material behaviour by engineers, the visibly influential microstructure of some materials (composites), and the necessity of going to a finer scale due to emerging micro- and nanotechnologies require.

2. NONLOCALITY

It is commonly accepted in *continuum physics* that a theory is said to be local when the *effect* at a spatial point **x** is directly, and only, related to its *cause* at the same point. Otherwise, the theory is said to be *nonlocal.* The nonlocal mechanical theory of materials which expresses constitutive equations by means of *functionals over space* was addressed by pioneers in the field when nonlocal theories became fashionable in the 1960s–70s (see the works of D. Rogula, I. A. Kunin, E. Kröner, D. G. B. Edelen, A. C. Eringen) (see, e.g., [[1,2]]). The notions of *range of interaction* and *coherence length* enter naturally these phenomenological descriptions. In other words, one (several) *characteristic length(s)* intervene(s) in the problem (cf. themes of some conferences; [[3,4]]). Among the possible approximations to spatial functionals are representations by *gradients* of successive orders [[5]] of the *cause* and/or effects (cf. the author's contribution in [[4]]) at the point **x** where the effect is represented. The degree of *fineness* of the description depends on the order of gradients considered. In the past [[6]] we simply called *truly nonlocal theories* those theories that make use of spatial functionals and *weakly nonlocal theories* those that consider an appoximation of a certain gradient order to these functionals, it being understood that for technical (mathematical) reasons, this "order" remains small in all events, e.g., only first- and second-order gradients are contemplated as otherwise field equations become extremely stiff (with high-order derivatives). The simultaneous occurrence of a field and its gradient in the list of arguments of an energy function clearly appeals to the notion of *characteristic length*, by strict dimensional analysis (ratio of the norms of the two arguments). *Nondimensional numbers* are then introduced to characterize the "weakness" of the nonlocality of interest. In classifying "*gradient theories*" one must pay attention to the notion of *field*, i.e., the basic physico-mathematical entity of which one takes spatial gradients (cf. [[7]]). We may have gradient theories without characteristic length. Examples of these are the classical theory of elasticity, and electrostatics in which the basic fields (displacement in the former, electric potential in the latter) are excluded from the list of arguments of the potential energy on account of a *gauge condition*, and there remains only a dependence on their first-order gradient (strain in the first case, electric field in the second case) so that these are *per se* "degenerate" gradient theories. A gradient theory will exhibit a characteristic length only if two gradients of different orders are present simultaneously in the energy dependence. But this remark hints at some caution in qualifying verbally theories. For instance, the first *strain-gradient theory of elasticity* of Mindlin and Tiersten [[8]] is better called a *second-gradient theory* (of the deformation mapping). Furthermore, engineers newly converted to nonlocality often do not realize the seniority of gradient theories. For instance, Maxwell, in his celebrated treatise [[9]]

insists on the *potentials* being the primary quantities, and for that very reason, using the alphabetic order, he designates by **A** the magnetic potential. What we nowadays call fields are gradients (sometimes of a special type, e.g., *curl*) of such quantities. But classical electromagnetism admits gauge conditions. Early gradient theories that exhibit length scales are the liquid theory of Korteweg (considering the gradient of density) at the end of the nineteenth century, and Einstein's general relativity of gravitation (1916) that is none other than a special second-gradient theory of the elasticity of space-time. The corresponding Lagrangian–Hamiltonian formulation involves the space-time metric (through its determinant), its first space-time gradients (through the Christoffel symbols), and its second-order space-time gradients (curvature). In condensed-matter physics it was also soon realized that *gradient theories* provide a key to opening the way to a phenomenological representation of the *ordering phenomena* prevailing in low-temperature phases. The names of E. M. Landau, I. M. Lifshitz (e.g., in ferromagnetism), and V. L. Ginzburg remain attached to this *weak nonlocality* (moderately long-range interactions) ever present in those quantum-mechanics based phenomena that have macroscopic manifestations in everyday life. This led theorists to introducing the celebrated Frank energy in terms of the spatial gradient of the so-called director field in a liquid crystal (see the modelling by Ericksen, Leslie, and de Gennes, in [10]).

The simultaneous presence of fields and their gradients in a physical theory is tantamount to saying that the corresponding field theory will exhibit *dispersion* from the point of view of *harmonic-wave propagation*, because the field equations will then present nonhomogeneous polynomials of differentiation: signals at different frequencies travel at different speeds. This phenomenon, combined with *nonlinearity* present in many physical theories that exhibit relatively high-energy levels (see below), yields by compensation the phenomenon of *wave localization* of which the most publicized example is the *solitonic structure*, a strongly localized stable wave structure that acts more or less like a particle in elastic collisions [11,12]. Thus, through some fancy verbal dialecticism, *nonlocality* yields *localization* of solutions. Domain wall structures in ferromagnets and shape-memory alloys are physical examples of such structures [13]. Applied mathematicians employ another expression for, if there were no gradients, only *jump-like* solutions of otherwise spatially uniform fields would have to be introduced in order to explain the clearly observed domain structure. Accounting for gradients smooths out those discontinuities and thus *regularizes* the solutions. The characteristic length then naturally materializes in the thickness of the smooth transition zone, i.e., the *regularization* of the solution.

In his research works, Prof. Jüri Engelbrecht has always shown a vivid interest in all manifestations of solitonic phenomena, as also in nonlocality whether in space [14] or in time [15], where by the latter the effects of *viscosity* must be understood. This naturally brings us to considering a larger framework in which space, time, and energy level will all enter in competition or will collectively contribute to a better understanding of physical reality.

3. "ANALYTIC CONTINUATION" OF CLASSICAL CONTINUUM MECHANICS

From here on we shall perform all reasonings with respect to Fig. 1 in which we propose to relate axes to the time scale, the length scale, and the energy-level scale of the studied phenomena and where, to be specific, we consider continuum mechanics as a paragon of continuum physics. Perhaps surprisingly enough to many readers, we place at the origin *not* particle physics, but the continuum mechanics of nondissipative behaviours that present neither viscosity, nor characteristic length, nor also typical energy level. These are, indeed, pure *linear elasticity* in solids and the *Eulerian fluidity* in fluids. Then one may wonder where the rest of continuum mechanics and of mechanics in a more general way stands when the continuity hypothesis becomes doubtful. The three axes of the figures are marked with *nondimensional numbers*. We remind the reader that such numbers are usually offshoots of their parent subject, *dimensional analysis*. Most such numbers consist of the ratio of two "forces", such as viscous and gravitational, or viscous and magnetic. But they can be contrived without formal analysis to practical utility. They typically permit simple but still quantitative views of complicated physical phenomena. This is the case here.

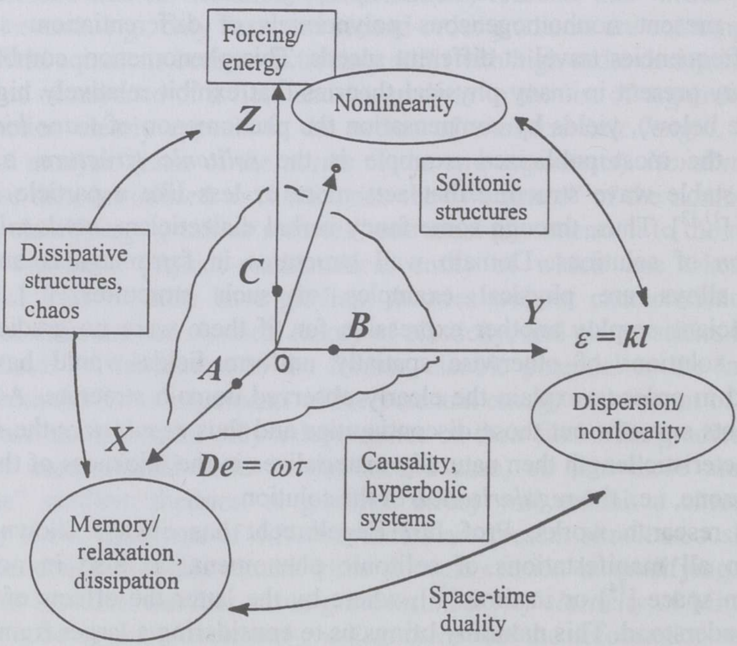

Fig. 1. Dissipation vs. nonlocality vs. nonlinearity diagram.

In Fig. 1 the *X*-axis accounts for the time scale inherent in viscous or relaxation effects. A characteristic nondimensional number then is the *Deborah number* [16] *De* which compares a typical relaxation time τ inherent in the system and a time scale attached to an external source, e.g., a frequency of excitation ω, through the obvious definition $De = \omega\tau$. A representative point such as *A* along the *X*-axis but in the vicinity of the origin, clearly represents a weakly viscous material, such as a material exhibiting *Newtonian viscosity*. A point farther out of the origin would indicate a possible *non-Newtonian* – albeit still linear – viscous behaviour, and a point far out but still along the *X*-axis a material with long-range *memory* (naturally represented by a constitutive functional over time in the manner of Boltzmann and Volterra). A truly nonlinear viscoelastic behaviour, such as that of a Bingham fluid or in electro- or magnetorheological fluids, requires a coupling with the *Z*-axis, hence an off *X*-axis situation.

The *Y*-axis relates to length-scale effects. The nondimensional number $\varepsilon = l/\lambda$ or kl, where l is an intrinsic length scale and $\lambda \equiv k^{-1}$ is a typical excitation length scale – k is a wave number – measures the degree of *nonlocality* or *dispersion*. The symbol used for that nondimensional number carries a connotation of the "*infinitesimally small*", but this being set apart, it does measure in which way one deviates from the *continuum hypothesis* as, parodying the philosopher G. Bachelard ([17], p. 136), we may say that "it is the wavelength which, by itself, creates the phenomenon. Confusion would reign if the wavelength was not large enough to overlap the discontinuities of the punctiform distribution." On the contrary, in a weakly nonlocal theory we focus attention on the possible *resonances* between structure and excitation. A nondissipative but *weakly nonlocal* behaviour has a representative point such as *B* in the vicinity of the origin along the *Y*-axis. A representative point far away from the origin but still along the *Y*-axis would then correspond to *true nonlocality*, where the notion of *contact force* dear to continuum mechanics disappears altogether to the benefit of *action at a distance* dear to Newton, Boscovich, Laplace, and others. We then touch two domains of apparently disjoint interest, the microcosm – the "infinitesimally small" –, and the "infinitesimally large", as a large ε may correspond either to a large l or an extremely small λ. The notion of *representative-volume element*, so useful in *homogenized* continuum theories [18], is essential in deciding whether an inhomogeneous material can be reasonably represented by a classical continuum or a weakly nonlocal one. Standard asymptotic periodic homogenization does not involve a characteristic length and therefore misses the *dispersion* typical of such systems. A different type of homogenization technique must be exploited which accounts for this characteristic size which enters in competition with the wavelength of an excitation (cf. Bachelard's remark). The *Bloch expansion* issued from quantum-mechanical considerations is a useful tool in that context [19].

The *Z*-axis compares the energy content (say, per unit volume) of the material and the energy level of external forcing. It is clear that a high level of energy requires considering *nonlinearities* in the system. This is classically illustrated by *plasticity* without time scale – i.e., no characteristic time – in which the nonlinear response is exhibited for a sufficiently high level of input, although some materials (extremely ductile ones, such as gold and silver) show this nonlinear behaviour very early in mechanical loading, if not from its origin. Thus classical strain-rate independent plasticity would be represented by a point C along the Z-axis, but still not far from the origin, for we know that an excessive forcing yields fracture and failure of the material. Other examples include shock-wave propagation [20], but this requires a consideration of interactions between the various "coordinates" in Fig. 1, being a *dynamical effect.* The singular nature of strain-rate independent plasticity (placed on the Z-axis), which is shared by magnetic and ferroelectric hysteresis at low frequencies [12], is enhanced by the fact that, although exhibiting no time scale, it is nonetheless *dissipative*, so that its dissipation has a peculiar nature, being homogeneous of degree one in the said strain-rate [21].

It was rightly remarked by M. Ostoja-Starjewski (Atlanta) during a meeting in Poznan (Poland, August 1998), where a sketch of Fig. 1 was presented for the first time, that an additional, fourth axis accounting for *stochasticity* should be added. Not only does the representation then become visually difficult, but I am not an expert in this technical speciality and leave it out of consideration due to my own ignorance. At this point it is then natural to envisage the *couplings* between the three properties delineated by the three axes in Fig. 1 for, in practice, we have to face behaviours that do not "*diagonalize*" simply along these "principal" axes. Most representative points will be out of the origin in the three-dimensional Euclidean space spanned by (X,Y,Z). First of all, space and time responses can hardly be disconnected, for instance, because of *causality*, and because of the importance of *dynamical* or *wave-like* phenomena. That is, most physical systems exhibit not only a characteristic length or time but also a *characteristic speed*, e.g., the velocity of light in vacuum or an elastic speed related to the linear elastic behaviour, c. The general notion of *phase* $\varphi(\mathbf{x},t)$ from which one deduces *frequency* ω and *wave-vector* $\mathbf{k}$ by (cf. the kinematic theory of waves developed by J. Lighthill and G. B. Whitham; see [20])

$$\omega = -\frac{\partial \varphi}{\partial t}, \mathbf{k} = \nabla \varphi,$$

is essential in that context. The (X,Y) plane of our symbolic representation becomes the realm of *dynamical processes*, and *causality* (in other words *hyperbolicity* – the fact that information travels at a finite speed) becomes the focus of consideration of all potential "relativists". Logically enough, space and time nonlocalities come to be considered simultaneously with constitutive equations represented by space-time functionals [22,23]. Next to, but rather different from, hyperbolicity, one has *dispersion.* While nonlinearity and causality permeate the theory of shock waves, but with a dissipation in the background as a regularizing factor (in other words, shock waves exist because the considered physical phenomenon is ultimately dissipative, but with a localized dissipation), dispersion and nonlinearity provide necessary ingredients for the propagation at finite speed of strongly localized solutions, *solitonic structures*, in the complete absence of dissipation. The (Y,Z) plane is therefore the realm of such strange dynamical phenomena that have invaded part of the physico-mathematical literature. It must also be emphasized that the necessity of mixed time-space considerations allied to invariance properties (that of the phase) led to the original introduction of *wave mechanics* and the resulting *wave-particle dualism* in the expert hands of L. de Broglie. As a matter of fact, space, time, and energy scales are basically related via a characteristic velocity c and Planck's reduced constant $\hbar$ by

$$\omega = ck, \;\; p = \hbar k, \;\; E = \hbar\omega,$$

where p and E are characteristic momentum and energy level, respectively. In this vision, the sphere at infinity S_∞ centred on the origin in Fig. 1 stands for particle (continuum is no longer valid), high energy, and action-at-a-distance physics. But we could have inversed the space-time representation by noting that, because of the *phase* definition, the couple $(\omega, \mathbf{k})$ is dual to the couple $(t,\mathbf{x})$ in Fourier space so that, just the same as in crystal physics, we could have carried out the reasoning in the dual of the (X,Y) plane. Finally, a representative point in the (X,Z) plane will relate the time scale with the energy level due to forcing. This classically provides the realm of *dissipative structures* (including chaos, when a bit of nonlinearity is injected in the system). Active systems, e.g., nerve fibres, such as those dealt with by Engelbrecht [24] belong in this vision.

4. HOW FAR FROM THE ORIGIN?

The origin in Fig. 1 represents a physical singularity (pure elasticity or pure fluidity) in the sense of the relative scarcity of this situation in real problems while being probably the most exploited scheme by reason of its mathematical simplicity. Real physical systems clearly most often reside outside this origin, but how far? The idea underlying this contribution is that staying in the continuous domain of validity of the representation of mechanical phenomena makes that we never take excursions far away from this origin. This justifies the title of the paper in the sense that all descriptions of interest in the continuum framework involve relatively small levels of forcing, small Deborah numbers, and rather small dispersion, and consequently remain in a neighbourhood of the origin. However, artificially produced materials with a microstructure and nanotechnologies will force us to consider this slight deviation. The question remains of the regularity of the origin point in Fig. 1 in the analytical sense. Unfortunately, more than often, this regularity is not guaranteed. On the contrary, introducing a dispersion leading to high-order space derivatives in the governing system of partial differential equations renders the system *singular* as ε tends towards zero. This is probably where the most important mathematical problem arises in this "analytic continuation" of standard continuum mechanics (see very stiff systems in Christov et al. [25]). This necessarily high degree of dispersion is a characteristic trend of many recent works, especially in the dynamic framework [26,27], where the degree of nonlinearity increases simultaneously to that of dispersion allowing for the emergence of new rich, practically stable, dynamical structures (e.g., soliton complexes, etc.). The representative point of such promising systems that are closer to physical reality remains in a neighbourhood of the origin, but outside it. What we also learn from the above is the importance, before attacking the phenomenological description of any physical phenomenon, of the delineation of the domain of time, space, and energy scales that one wants to accommodate in the description; any course on phenomenological physics and thermomechanics should begin with such a study [28], something too often neglected in many textbooks. The art of modelling then consists in selecting a framework that is both physically justified in its complexity, but not further, and sufficiently simple so as to be amenable by (more sophisticated than in old times) analytical means or numerical simulations.

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"ANALÜÜTILISEST JÄTKAMISEST" PIDEVA KESKKONNA MEHAANIKAS

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Loomulike üldistuste selgitamiseks klassikalise pideva keskkonna mehaanika raamides ruumi ja/või aja mittelokaalsuse ning mittelineaarsuse olemasolul on esitatud piltlik skeem. See katab suurema osa hiljuti esitatud üldistusi, mis rõhutavad klassikaliste teooriate singulaarsust ja fakti, et reaalsed füüsikalised nähtused, mis kajastuvad pideva keskkonna teooria piisavalt lihtsates analüütilistes terminites, jääksid mingis mõttes lähedaseks ideaalse, pelgalt elastse keha või vedeliku juhule. Siiski tuleb suunata kõik pingutused füüsikalise reaalsuse võimalikult täpseks kirjeldamiseks, seades eesmärgiks nii intellektuaalse rahulduse kui ka praktilised püüdlused, s.t. arvestada mõistlikku keerukust.