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Algebraic formalism of differential one-forms for nonlinear control systems on time scales

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Abstract. The paper develops algebraic formalism of differential one-forms, associated with the nonlinear control system defined on homogeneous time scales. This formalism unifies the existing theories for continuous- and discrete-time systems. A field of meromorphic functions, corresponding to a control system, is introduced. It is equipped with two operators whose properties are studied. An inversive closure of this field is constructed with the aid of one-forms.

Key words: time scale, nonlinear system, differential field, one-form, inversive closure.

1. INTRODUCTION

A time scale is a model of time. Both the continuous- and discrete-time cases in a time-scale formalism are considered and merged into a general framework. Besides unification, extension is another main feature of the time scale calculus based on the so-called delta derivative. The latter is a generalization of both the standard time-derivative and that of the difference operator but accommodates much more possibilities. There is actually a whole spectrum of different time scales which serve as models of time; continuous and discrete time are just the two most important cases. The theory of dynamical systems on time scales is an active and new research area initiated in 1988 by Aulbach and Hilger in [[1,2]]. Recently, the

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first monograph on this topic was published [3]. However, less than ten papers concerning control systems on time scales are available [4–12].

Many results concerning continuous-time control systems carry over quite easily to the corresponding results for discrete-time systems, while other results seem to be completely different in nature from their continuous-time counterparts. The study of control systems on time scales will help to reveal and explain such discrepancies. Besides, the time scales formalism has a tremendous potential for non-traditional application areas such as biology, economics, and medicine, where the system dynamics are described on the time scale partly continuous and partly discrete. Moreover, time scale formalism accommodates easily the non-uniformly sampled system.

The aim of this paper was to develop the mathematical formalism that allows later study of nonlinear control systems on time scales. We restrict ourselves to homogeneous time scales, which are models of continuous or uniformly sampled time (discrete time).

As a starting point for developing a unified framework for nonlinear control systems on time scales we take the universal algebraic formalism developed in [13,14], which is based on the classification of differential one-forms related to the control system and can be applied to solve different modelling, analysis, and synthesis problems [13,15]. The key tasks are to construct the σ-differential field, associated with the control system on a time scale, and to *find its inversive closure*. However, later the algebraic formalism of differential one-forms can be applied to study many different problems like, for example, input-output and transfer equivalence of systems, reduction and feedback linearization, accessibility and realization of the input-output model in the state space form.

Actually, if one works only on homogeneous time scales, a purely algebraic approach using the tools of pseudo-linear algebra [16] is all that is necessary to extend the formalism of one-forms into the unified framework of discrete and continuous time; see, for example, [17]. However, a pseudo-linear algebra is unable to accommodate the systems defined on non-homogeneous time scales. Though in this paper we consider only the homogeneous case, our future goal is to build a framework that allows the study of the non-homogeneous case, and this paper has to be seen as the first step towards this goal.

2. TIME SCALE CALCULUS

The calculus on time scales was initiated in order to create a theory that can unify and extend discrete and continuous analysis. For a general introduction to the calculus on time scales, see [3]. Here we give only those notions and facts that we need in our paper and most of them were taken from [3]. The main task is to introduce the concept of derivative for real functions defined on a time scale.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, but also $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k \mid k \in \mathbb{Z}\} \cup \{0\}$, for $q > 1$, is a time scale. We assume that $\mathbb{T}$ is a topological space with the topology induced by $\mathbb{R}$. In the definition of derivative, the so-called forward and backward jump operators play an important role.

Definition 2.1. *For $t \in \mathbb{T}$ the* forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} |\ s > t\},$$

while the backward jump operator $\rho(t) : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) = \sup\{s \in \mathbb{T} |\ s < t\}.$$

In this definition we set in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \min T$ and $\rho(t) = t$, then t is called *left-dense*. The points that are right-scattered and left-scattered at the same time are called *isolated*.

Finally, the *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t$$

for all $t \in \mathbb{T}$.

Example 2.2.

- If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$, and the graininess function $\mu(t) \equiv 0$.
- If $\mathbb{T} = h\mathbb{Z}$, for $h > 0$, then for every $t \in h\mathbb{Z}$, $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = h$.
- If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, for $q > 1$, then for every $t \in \mathbb{T}$, $\sigma(t) = qt$, $\rho(t) = t/q$, and $\mu(t) = (q-1)t$.

Let $\mathbb{T}^\kappa$ denote a truncated set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point.

Definition 2.3. *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The* delta derivative *of f at t, denoted by $f^\Delta(t)$ (or by $\frac{\Delta}{\Delta t} f(t)$), is the real number (provided it exists) with the property that given any ε, there is a neighbourhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that*

$$|(f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leqslant \varepsilon |\sigma(t) - s| \tag{1}$$

for all $s \in U$. Moreover, we say that f is delta differentiable *on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.*

Remark 2.4. If $t \in \mathbb{T} \backslash \mathbb{T}^\kappa$, then $f^\Delta(t)$ is not uniquely defined, since for such a point t, small neighbourhoods $\mathcal{U}$ of t consist only of t and, besides, we have $\sigma(t) = t$. Therefore (1) holds for an arbitrary number $f^\Delta(t)$. This is a reason why we omit a maximal left-scattered point.

Example 2.5.

- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if $f^\Delta(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s} = f'(t)$, i.e. if and only if f is differentiable in the ordinary sense at t.
- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)} = f(t+1) - f(t) = \triangle f(t)$, where $\triangle$ is the usual forward difference operator defined by the equation above.
- If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, for $q > 1$, then $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$ for all $t \in \mathbb{T} \setminus \{0\}$.

For $f : \mathbb{R} \to \mathbb{R}$ define $f^\sigma := f \circ \sigma$.

Proposition 2.6. *Let $f : \mathbb{T} \to \mathbb{R}$, $g : \mathbb{T} \to \mathbb{R}$ be two delta differentiable functions defined on $\mathbb{T}$ and let $t \in \mathbb{T}$. The delta derivative satisfies the following properties*:

(i) $f^\sigma = f + \mu f^\Delta$,
(ii) $[\alpha f + \beta g]^\Delta = \alpha f^\Delta + \beta g^\Delta$, for any constants α and β,
(iii) $(fg)^\Delta = f^\sigma g^\Delta + f^\Delta g$,
(iv) if $gg^\sigma \neq 0$, then $(f/g)^\Delta = (f^\Delta g - fg^\Delta)/(gg^\sigma)$.

If $f, g : \mathbb{R} \to \mathbb{R}$, then the chain rule from the calculus states that if g is differentiable at t and if f is differentiable at $g(t)$, then $(f \circ g)'(t) = f'(g(t))g'(t)$. In general, this rule does not hold for time scales.

Example 2.7. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, and $g : \mathbb{T} \to \mathbb{R}$. Then

$$(f \circ g)^\Delta (t) = \left(g^2\right)^\Delta (t) = (g^\sigma(t) + g(t))\, g^\Delta(t).$$

For $\mathbb{T} = \mathbb{Z}$ this is different from

$$f'(g(t))\, g^\Delta(t) = 2g(t) \cdot g^\Delta(t).$$

Theorem 2.8. [Chain rule]. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))\mathrm{d}h \right\} g^\Delta(t) \tag{2}$$

holds.

Definition 2.9. *A time scale $\mathbb{T}$ is called* homogeneous *if* $\mu \equiv \text{const}$.

Definition 2.10. [18] *A time scale* $\mathbb{T}$ *is called* regular *if the following two conditions are satisfied simultaneously*:
(i) $\sigma(\rho(t)) = t$, *for all* $t \in \mathbb{T}$,
(ii) $\rho(\sigma(t)) = t$, *for all* $t \in \mathbb{T}$.

From (i) it follows that the operator $\sigma : \mathbb{T} \to \mathbb{T}$ is "onto" while (ii) implies that σ is "one-to-one". Therefore, if $\mathbb{T}$ is regular, then σ is invertible and $\rho : \mathbb{T} \to \mathbb{T}$ is also invertible. Moreover, $\sigma^{-1} = \rho$ and $\rho^{-1} = \sigma$.

Remark 2.11. Every homogeneous time scale is regular, since in that case $\mu \equiv \mathrm{const} = h$, $\sigma(t) = t + h$, and $\rho(t) = t - h$.

Example 2.12. The time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$, $h > 0$ are both homogeneous and regular. The time scales $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $\mathbb{T} = (-\infty, 0] \cup \{\frac{1}{k} | \ k \in \mathbb{N}\} \cup \{\frac{k}{k+1} | \ k \in \mathbb{N}\} \cup [1, 2]$ are both regular, but not homogeneous.

For a function $f : \mathbb{T} \to \mathbb{R}$ we can talk about second delta derivatives $f^{[2]} := f^{\Delta\Delta}$, provided that f^{Δ} is delta differentiable on $\mathbb{T}^{\kappa^2} := (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{[2]} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly we define higher-order derivatives $f^{[n]} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$, where $\mathbb{T}^{\kappa^n} = \left(\mathbb{T}^{\kappa^{n-1}}\right)^{\kappa}$, $n \geqslant 1$.

Let us define $f^{\Delta\sigma} := \left(f^{\Delta}\right)^{\sigma}$ and $f^{\sigma\Delta} := (f^{\sigma})^{\Delta}$.

Proposition 2.13. *Let* $\mathbb{T}$ *be a homogeneous time scale and* f, f^{Δ} *be delta differentiable functions. Then we have*

$$f^{\Delta\sigma} = f^{\sigma\Delta}. \tag{3}$$

Proof. Applying condition (i) of Proposition 2.6 to functions f, f^{Δ} and using the fact that $\mathbb{T}$ is homogeneous, we obtain

$$f^{\Delta\sigma} = f^{\Delta} + \mu f^{[2]} = \left(f + \mu f^{\Delta}\right)^{\Delta} = f^{\sigma\Delta}. \qquad \square$$

3. DIFFERENTIAL FIELD

Let us consider an analytic system, defined on a homogeneous time scale $\mathbb{T}$:

$$x^{\Delta}(t) = f(x(t), u(t)), \tag{4}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, $m \leqslant n$. Assume that the map $(x, u) \mapsto \widetilde{f}(x, u) = x + \mu f(x, u)$ generically defines a submersion, i.e. generically

$$\operatorname{rank} \frac{\partial \widetilde{f}(x, u)}{\partial (x, u)} = n \tag{5}$$

holds. Assumption (5) is not restrictive, since it is a necessary condition for system accessibility [19].

Let us recall that $u_j^{[k]}$ denotes the kth delta derivative of u_j. The delta derivative can be computed recursively, $u_j^{[k+1]} = \left(u_j^{[k]}\right)^\Delta$, with $u_j^{[0]} = u_j$. For notational convenience, $(x_1, \dots, x_n)$ will simply be written as x, and $(u_1^{[k]}, \dots, u_m^{[k]})$ as $u^{[k]}$, for $k \geqslant 0$. For $i \leqslant k$ let $u^{[i\dots k]} := \left(u^{[i]}, \dots, u^{[k]}\right)$. We assume that the input applied to system (4) is infinitely many times delta differentiable, i.e. $u^{[0\dots k]}$ exists for all $k \geqslant 0$.

Let us consider the infinite set of real (independent) indeterminates

$$\mathcal{C} = \left\{x_i, i = 1, \dots, n,\ \ u_j^{[k]}, j = 1, \dots, m,\ \ k \geqslant 0\right\}. \tag{6}$$

Let $\mathcal{R}$ be a ring of analytic functions that depend on a finite number of variables from the set $\mathcal{C}$. Thus for each $\varphi \in \mathcal{R}$ there is $k \geqslant 0$ such that φ depends on $(x, u^{[0\dots k]})$. Let $\sigma : \mathcal{R} \to \mathcal{R}$ be an operator defined by

$$\sigma(\varphi)\left(x, u^{[0\dots k+1]}\right) := \varphi\left(x^\sigma, \left(u^{[0\dots k]}\right)^\sigma\right), \tag{7}$$

where $x^\sigma = x + \mu x^\Delta = x + \mu f(x, u)$, $\left(u^{[0\dots k]}\right)^\sigma = u^{[0\dots k]} + \mu u^{[1\dots k+1]}$, for $k \geqslant 0$. Such defined σ is an *endomorphism*, i.e. a map satisfying the following conditions:

(i) $\sigma(\varphi + \psi) = \sigma(\varphi) + \sigma(\psi)$, for all $\varphi, \psi \in \mathcal{R}$;

(ii) $\sigma(\varphi\psi) = \sigma(\varphi)\sigma(\psi)$, for all $\varphi, \psi \in \mathcal{R}$;

(iii) $\sigma(1_\mathcal{R}) = 1_\mathcal{R}$, where $1_\mathcal{R}$ is a unit of $\mathcal{R}$.

Let $\mathcal{K}$ be the quotient field of $\mathcal{R}$, i.e. the field of meromorphic functions in a finite number of the variables from $\mathcal{C}$. A typical element of $\mathcal{K}$ would have the form $F(\zeta) = \varphi(\zeta)/\psi(\zeta)$, where φ and ψ are elements of $\mathcal{R}$ and ψ is not the zero function, and ζ denotes the finite number of elements of the set $\mathcal{C}$. Under assumption (5), σ is *injective*, i.e. the condition $\sigma(F) = \sigma(G)$ implies $F = G$, for all $F, G \in \mathcal{K}$. Then, since the kernel of the endomorphism $\sigma : \mathcal{K} \to \mathcal{K}$ is trivial, σ is well defined on $\mathcal{K}$: $\sigma\left(F/G\right) = \sigma(F)/\sigma(G)$, for $F, G \in \mathcal{K}$ and $G \not\equiv 0$.

Remark 3.1. If $\mu = 0$, then $\sigma(F) = F$, for $F \in \mathcal{K}$ ($\sigma = \mathrm{id}$) and assumption (5) is satisfied trivially.

Remark 3.2. In the discrete-time case ($\mu = 1$) system (4) can be rewritten as

$$x^\sigma = \widetilde{f}(x, u), \tag{8}$$

where $\widetilde{f}(x, u) = x + f(x, u)$. Obviously, assumption (5) agrees with the standard submersivity assumption [19] for system (8),

$$\operatorname{rank}_\mathcal{K} \frac{\partial \widetilde{f}(x, u)}{\partial\left(x, u\right)} = n.$$

The chain rule (2) can be generalized and used to define the delta derivative of a function $F \in \mathcal{K}$ as follows:

$$F^{\Delta}\left(x, u^{[0\ldots k+1]}\right)$$
$$:= \int_0^1 \left\{ \operatorname{grad} F\left(x + h\mu f(x,u), u^{[0\ldots k]} + h\mu u^{[1\ldots k+1]}\right) \cdot \begin{bmatrix} f(x,u) \\ \left(u^{[1\ldots k+1]}\right)^{\mathrm{T}} \end{bmatrix} \right\} \mathrm{d}h, \tag{9}$$

where $f(\cdot)$ describes the dynamics of system (4).

Proposition 3.3. *For arbitrary time scale* $\mathbb{T}$ *we have*

$$F^{\Delta}\left(x, u^{[0\ldots k+1]}\right)$$
$$= \begin{cases} \dfrac{1}{\mu}\left[F\left(x + \mu f(x,u), u^{[0\ldots k]} + \mu u^{[1\ldots k+1]}\right) - F\left(x, u^{[0\ldots k]}\right)\right], & \text{if } \mu \neq 0, \\ \dfrac{\partial F}{\partial x}\left(x, u^{[0\ldots k]}\right) f(x,u) + \displaystyle\sum_{i=0}^{k} \dfrac{\partial F}{\partial u^{[i]}}\left(x, u^{[0\ldots k]}\right) u^{[i+1]}, & \text{if } \mu = 0. \end{cases}$$

Proof. Let $\mu \neq 0$. Then

$$\begin{aligned} F^{\Delta}\left(x, u^{[0\ldots k+1]}\right) &= \int_0^1 \left\{ \operatorname{grad} F\left(x + h\mu f(x,u), u^{[0\ldots k]} + h\mu u^{[1\ldots k+1]}\right) \cdot \begin{bmatrix} f(x,u) \\ \left(u^{[1\ldots k+1]}\right)^{T} \end{bmatrix} \right\} \mathrm{d}h \\ &= \frac{1}{\mu} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}h} F\left(x + h\mu f(x,u), u^{[0\ldots k]} + h\mu u^{[1\ldots k+1]}\right) \mathrm{d}h \\ &= \frac{1}{\mu} \cdot F\left(x + h\mu f(x,u), u^{[0\ldots k]} + h\mu u^{[1\ldots k+1]}\right)\Big|_0^1 \\ &= \frac{1}{\mu}\left[F\left(x + \mu f(x,u), u^{[0\ldots k]} + \mu u^{[1\ldots k+1]}\right) - F\left(x, u^{[0\ldots k]}\right)\right]. \end{aligned}$$

The proof for the case $\mu = 0$ is obvious. □

Remark 3.4. Let $F \in \mathcal{K}$. Then by Proposition 3.3 $\sigma(F) = F + \mu F^{\Delta}$.

Remark 3.5. Let $\mathbb{T}$ be a homogeneous time scale with $\mu \neq 0$. Then

$$F^{\Delta} = \frac{1}{\mu}(\sigma(F) - F). \tag{10}$$

If $F \in \mathcal{K}$, then $\sigma(F) \in \mathcal{K}$, so from (10) we get $F^{\Delta} \in \mathcal{K}$.

Remark 3.6. Let $\mathbb{T}$ be a homogeneous time scale with $\mu = 0$ and $F : \mathbb{R}^n \times \mathbb{R}^{m(k+1)} \to \mathbb{R}$. If the function F belongs to $\mathcal{K}$, then its partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial u^{[i]}}$, $i = 0, \ldots, k$, belong to $\mathcal{K}$ as well. Hence $F^{\Delta} \in \mathcal{K}$.

Both $\Delta(F)$ and F^{Δ} are used to denote the delta derivative of the meromorphic function F depending on real indeterminates from $\mathcal{C}$. Similarly, we will use both $\sigma(F)$ and F^{σ} for endomorphism σ.

Remark 3.7. Let $u(\cdot)$ be a control applied to system (4), $x(\cdot)$ be a solution of (4) corresponding to control u, and $F \in \mathcal{K}$. Then

$$\frac{\Delta}{\Delta t}(F(x(t), u^{[0\ldots k]}(t))) = F^{\Delta}(x(t), u^{[0\ldots k+1]}(t)). \tag{11}$$

Note that the left-hand side of (11) contains the total delta time derivative of F composed of functions $x(\cdot)$ and $u^{[i]}(\cdot)$, $i = 0, 1, \ldots, k$, which depend on time. It is easy to see that in the continuous-time case for the system of the form $x^{\Delta} = f(x)$ we have

$$\frac{\Delta}{\Delta t}(F(x(t)) = L_f F(x(t)),$$

where L_f denotes the Lie derivative and x satisfies $x^{\Delta} = f(x)$.

Proposition 3.8. *For system* (4) *consider the map* Δ *defined by* (9)*. Then* Δ *satisfies the following conditions*:
(i) $\Delta(F + G) = \Delta(F) + \Delta(G)$,
(ii) $\Delta(FG) = \Delta(F)G + \sigma(F)\Delta(G)$,
for all $F, G \in \mathcal{K}$.

Proof. For simplicity, we assume that $F, G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ depend on x and u only. The proof of (i) comes down to the fact that $\operatorname{grad}(F + G) = \operatorname{grad} F + \operatorname{grad} G$. To prove (ii), we use Proposition 3.3. If $\mu = 0$, then $F^{\sigma} = F$ and we have (for simplicity, we omit x and u)

$$\begin{aligned}
\Delta(FG) &= \frac{\partial(FG)}{\partial x} x^{\Delta} + \frac{\partial(FG)}{\partial u} u^{\Delta} \\
&= \left[\frac{\partial F}{\partial x} G + \frac{\partial G}{\partial x} F\right] x^{\Delta} + \left[\frac{\partial F}{\partial u} G + \frac{\partial G}{\partial u} F\right] u^{\Delta} \\
&= \left[\frac{\partial F}{\partial x} x^{\Delta} + \frac{\partial F}{\partial u} u^{\Delta}\right] G + F\left[\frac{\partial G}{\partial x} x^{\Delta} + \frac{\partial G}{\partial u} u^{\Delta}\right] \\
&= \Delta(F)G + F\Delta(G) = \Delta(F)G + \sigma(F)\Delta(G).
\end{aligned}$$

Hence the condition (ii) holds for $\mu = 0$. Next, assume that $\mu \neq 0$. Then, taking also into account that $\sigma(FG) = \sigma(F)\sigma(G)$ and (7), we have

$$\begin{aligned}\Delta\left[(FG)(x,u)\right] &= \frac{1}{\mu}\left[(FG)(x^\sigma,u^\sigma) - (FG)(x,u)\right] \\ &= \frac{1}{\mu}\left[F(x^\sigma,u^\sigma)G(x^\sigma,u^\sigma) - F(x^\sigma,u^\sigma)G(x,u)\right] \\ &\quad + \frac{1}{\mu}\left[F(x^\sigma,u^\sigma)G(x,u) - F(x,u)G(x,u)\right] \\ &= \Delta\left[F\left(x,u\right)\right]G(x,u) + \sigma\left[F(x,u)\right]\Delta\left[G\left(x,u\right)\right].\end{aligned}$$

□

According to (ii) of Proposition 3.8, the delta derivative Δ satisfies a suitable generalization of the Leibniz rule:

$$\Delta(FG) = \sigma(F)\Delta(G) + \Delta(F)G. \tag{12}$$

An operator satisfying rule (12) is called a *"σ-derivation"* (see [20]).

Definition 3.9. A commutative field endowed with a σ-derivation is called a *σ-differential field.*

4. ONE-FORMS

In this section we define one-forms, extend the delta derivative operator to one-forms and prove some of its properties.

Consider the infinite set of symbols

$$\mathrm{d}\mathcal{C} = \left\{\mathrm{d}x_i,\ i = 1,\ldots,n,\ \ \mathrm{d}u_j^{[k]}, j = 1,\ldots,m,\ \ k \geqslant 0\right\} \tag{13}$$

and denote by $\mathcal{E}$ the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}\mathcal{C}$, namely

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\mathrm{d}\mathcal{C}.$$

Any element of $\mathcal{E}$ is a vector of the form

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k\geqslant 0}\sum_{j=1}^{m} B_{jk}\mathrm{d}u_j^{[k]},$$

where only a finite number of coefficients B_{jk} are nonzero elements of $\mathcal{K}$. An operator $\mathrm{d} : \mathcal{K} \to \mathcal{E}$ can be defined in the standard manner (since F is a meromorphic function in a finite number of variables from the set $\mathcal{C}$), i.e.

$$\mathrm{d}F\left(x,u^{[0\dots k]}\right):=\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}\left(x,u^{[0\dots k]}\right)\mathrm{d}x_i+\sum_{k\geqslant 0}\sum_{j=1}^{m}\frac{\partial F}{\partial u_j^{[k]}}\left(x,u^{[0\dots k]}\right)\mathrm{d}u_j^{[k]}. \tag{14}$$

The elements of $\mathcal{E}$ will be called *one-forms* and we will say that $\omega\in\mathcal{E}$ is an *exact one-form* if $\omega=\mathrm{d}F$ for some $F\in\mathcal{K}$. We will refer to $\mathrm{d}F$ as to the *total differential* (or simply the *differential*) of F.

If $\omega=\sum_i A_i\mathrm{d}\zeta_i$ is a one-form, where $A_i\in\mathcal{K}$ and $\zeta_i\in\mathcal{C}$, one can define the operators $\Delta:\mathcal{E}\to\mathcal{E}$ and $\sigma:\mathcal{E}\to\mathcal{E}$ by

$$\Delta(\omega):=\sum_i\left\{\Delta(A_i)\mathrm{d}\zeta_i+\sigma(A_i)\mathrm{d}\left[\Delta(\zeta_i)\right]\right\}, \tag{15}$$

$$\sigma(\omega):=\sum_i\sigma(A_i)\mathrm{d}\left[\sigma(\zeta_i)\right]. \tag{16}$$

Since $\sigma(A_i)=A_i+\mu\Delta(A_i)$,

$$\Delta(\omega)=\sum_i\left\{\Delta(A_i)\mathrm{d}\zeta_i+\left(A_i+\mu\Delta(A_i)\right)\mathrm{d}\left[\Delta(\zeta_i)\right]\right\}.$$

As earlier for function $F\in\mathcal{K}$, now both $\Delta(\omega)$ and ω^{Δ}, and similarly both $\sigma(\omega)$ and ω^{σ} are used to denote the delta derivative of the one-form and operator σ acting on the one-form, respectively, by choosing the one which will be more convenient.

Proposition 4.1. *For the homogeneous time scale* $\mathbb{T}$ *we have*

$$\mathrm{d}\left[F^{\Delta}\right]=\left[\mathrm{d}F\right]^{\Delta}\quad\text{and}\quad\mathrm{d}\left[F^{\sigma}\right]=\left[\mathrm{d}F\right]^{\sigma}.$$

Proof. For simplicity of presentation we assume that F depends only on x and u, hence F^{Δ} depends on x, u, $x^{\Delta}=f(x,u)$ and u^{Δ}.

If $\mu\neq 0$, then $F^{\Delta}(x,u,u^{\Delta})=\dfrac{1}{\mu}\left[F(x+\mu f(x,u),u+\mu u^{\Delta})-F(x,u)\right]$, $F^{\sigma}(x,u,u^{\Delta})=F(x+\mu f(x,u),u+\mu u^{\Delta})$, and hence

$$
\begin{aligned}
\mathrm{d}\left[F^{\Delta}(x,u,u^{\Delta})\right] &= \frac{1}{\mu}\left[\frac{\partial F}{\partial x}(x+\mu f(x,u),u+\mu u^{\Delta})\left(1+\mu\frac{\partial f}{\partial x}(x,u)\right)\right.\\
&\quad \left.-\frac{\partial F}{\partial x}(x,u)\right]\mathrm{d}x+\frac{1}{\mu}\left[\frac{\partial F}{\partial x}(x+\mu f(x,u),u+\mu u^{\Delta})\mu\frac{\partial f}{\partial u}(x,u)\right.\\
&\quad \left.+\frac{\partial F}{\partial u}(x+\mu f(x,u),u+\mu u^{\Delta})-\frac{\partial F}{\partial u}(x,u)\right]\mathrm{d}u\\
&\quad +\frac{\partial F}{\partial u}(x+\mu f(x,u),u+\mu u^{\Delta})\mathrm{d}u^{\Delta}\\
&= \frac{1}{\mu}\left[\frac{\partial F}{\partial x}(x+\mu f(x,u),u+\mu u^{\Delta})-\frac{\partial F}{\partial x}(x,u)\right]\mathrm{d}x\\
&\quad +\frac{1}{\mu}\left[\frac{\partial F}{\partial u}(x+\mu f(x,u),u+\mu u^{\Delta})-\frac{\partial F}{\partial u}(x,u)\right]\mathrm{d}u\\
&\quad +\frac{\partial F}{\partial x}(x+\mu f(x,u),u+\mu u^{\Delta})\left[\frac{\partial f}{\partial x}(x,u)\mathrm{d}x+\frac{\partial f}{\partial u}(x,u)\mathrm{d}u\right]\\
&\quad +\frac{\partial F}{\partial u}(x+\mu f(x,u),u+\mu u^{\Delta})\mathrm{d}u^{\Delta}\\
&= \left(\frac{\partial F}{\partial x}\right)^{\Delta}\left(x,u,u^{\Delta}\right)\mathrm{d}x+\left(\frac{\partial F}{\partial u}\right)^{\Delta}\left(x,u,u^{\Delta}\right)\mathrm{d}u\\
&\quad +\left(\frac{\partial F}{\partial x}\right)^{\sigma}(x,u,u^{\Delta})\mathrm{d}f(x,u)+\left(\frac{\partial F}{\partial u}\right)^{\sigma}\left(x,u,u^{\Delta}\right)\mathrm{d}u^{\Delta}\\
&= \left[\frac{\partial F}{\partial x}(x,u)\mathrm{d}x+\frac{\partial F}{\partial u}(x,u)\mathrm{d}u\right]^{\Delta}=\left[\mathrm{d}F(x,u)\right]^{\Delta},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\left[F^{\sigma}(x,u,u^{\Delta})\right] &= \frac{\partial F}{\partial x}(x+\mu f(x,u),u+\mu u^{\Delta})\left(1+\mu\frac{\partial f}{\partial x}(x,u)\right)\mathrm{d}x\\
&\quad +\left[\frac{\partial F}{\partial x}(x+\mu f(x,u),u+\mu u^{\Delta})\mu\frac{\partial f}{\partial u}(x,u)\right.\\
&\quad \left.+\frac{\partial F}{\partial u}(x+\mu f(x,u),u+\mu u^{\Delta})\right]\mathrm{d}u\\
&\quad +\mu\frac{\partial F}{\partial u}(x+\mu f(x,u),u+\mu u^{\Delta})\mathrm{d}u^{\Delta}\\
&= \left(\frac{\partial F}{\partial x}\right)^{\sigma}\left(x,u,u^{\Delta}\right)\mathrm{d}\left[x+\mu f(x,u)\right]
\end{aligned}
$$

$$
\begin{aligned}
& + \left(\frac{\partial F}{\partial u}\right)^{\sigma} \left(x, u, u^{\Delta}\right) \mathrm{d}\left[u + \mu u^{\Delta}\right] \\
= \; & \left(\frac{\partial F}{\partial x}\right)^{\sigma} \left(x, u, u^{\Delta}\right) \mathrm{d}\left[x^{\sigma}\right] + \left(\frac{\partial F}{\partial u}\right)^{\sigma} \left(x, u, u^{\Delta}\right) \mathrm{d}\left[u^{\sigma}\right] \\
= \; & [\mathrm{d}F(x,u)]^{\sigma} .
\end{aligned}
$$

If $\mu = 0$, then these commutation rules come from properties of standard derivatives and from the fact that $\sigma = \mathrm{id}$. □

Example 4.2. Let $F(x) = x^2$ and $x^{\Delta} = xu$. Then by the 2nd formula of Table 1 (see Appendix) we get $F^{\Delta}(x,u) = (x + x^{\sigma})\, x^{\Delta} = 2x^2u + \mu x^2u^2$ and

$$
\mathrm{d}\left[F^{\Delta}(x,u)\right] = \left(4xu + 2\mu x u^2\right) \mathrm{d}x + \left(2x^2 + 2\mu x^2 u\right) \mathrm{d}u.
$$

Since $\mathrm{d}F(x) = 2x\mathrm{d}x$, by definition (15) we obtain

$$
\begin{aligned}
[\mathrm{d}F(x)]^{\Delta} &= 2x^{\Delta}\mathrm{d}x + 2x^{\sigma}\mathrm{d}\left[x^{\Delta}\right] = 2xu\mathrm{d}x + 2\left(x + \mu x u\right)\left[u\mathrm{d}x + x\mathrm{d}u\right] \\
&= \left(4xu + 2\mu x u^2\right) \mathrm{d}x + \left(2x^2 + 2\mu x^2 u\right) \mathrm{d}u = \mathrm{d}\left[F^{\Delta}(x,u)\right] .
\end{aligned}
$$

Additionally, since $F^{\sigma}(x,u) = (x + \mu x u)^2$,

$$
\begin{aligned}
[\mathrm{d}F(x)]^{\sigma} &= 2x^{\sigma}\mathrm{d}\left[x^{\sigma}\right] = 2\left(x + \mu x u\right)\mathrm{d}\left[x + \mu x u\right] \\
&= 2\left(x + \mu x u\right)\left[(1 + \mu u)\mathrm{d}x + \mu x \mathrm{d}u\right] = \mathrm{d}\left[F^{\sigma}(x,u)\right] .
\end{aligned}
$$

Example 4.3. Let $F(x,u) = \dfrac{x}{u}$ and $x^{\Delta} = xu$. Then, by the 3rd formula of Table 1 and condition (ii) of Proposition 3.8, $F^{\Delta}(x,u,u^{\Delta}) = \dfrac{ux^{\Delta} - xu^{\Delta}}{u^{\sigma}u} = \dfrac{xu^2 - xu^{\Delta}}{\left(u + \mu u^{\Delta}\right) u}$ and

$$
\begin{aligned}
\mathrm{d}\left[F^{\Delta}(x,u,u^{\Delta})\right] = & \frac{u^2 - u^{\Delta}}{\left(u + \mu u^{\Delta}\right) u} \mathrm{d}x \\
& + \frac{\mu x u^2 u^{\Delta} + 2xuu^{\Delta} + \mu x \left(u^{\Delta}\right)^2}{\left(u + \mu u^{\Delta}\right)^2 u^2} \mathrm{d}u - \frac{x + \mu x u}{\left(u + \mu u^{\Delta}\right)^2} \mathrm{d}\left[u^{\Delta}\right] .
\end{aligned}
$$

Since $\mathrm{d}F(x,u)=\dfrac{1}{u}\mathrm{d}x-\dfrac{x}{u^2}\mathrm{d}u$, using (15) we obtain

$$\begin{aligned}
\left[\mathrm{d}F(x,u,u^{\Delta})\right]^{\Delta}=&\left(\frac{1}{u}\right)^{\Delta}\mathrm{d}x-\left(\frac{x}{u^2}\right)^{\Delta}\mathrm{d}u+\frac{1}{u^{\sigma}}\mathrm{d}\left[x^{\Delta}\right]-\frac{x^{\sigma}}{\left(u^{\sigma}\right)^2}\mathrm{d}\left[u^{\Delta}\right]\\
&=-\frac{u^{\Delta}}{\left(u+\mu u^{\Delta}\right)u}\mathrm{d}x+\frac{-u^2x^{\Delta}+2xuu^{\Delta}+\mu x\left(u^{\Delta}\right)^2}{\left(u+\mu u^{\Delta}\right)^2u^2}\mathrm{d}u\\
&\quad+\frac{1}{u+\mu u^{\Delta}}\mathrm{d}\left[xu\right]-\frac{x+\mu x^{\Delta}}{\left(u+\mu u^{\Delta}\right)^2}\mathrm{d}\left[u^{\Delta}\right]\\
&=\frac{u^2-u^{\Delta}}{\left(u+\mu u^{\Delta}\right)u}\mathrm{d}x+\frac{\mu xu^2u^{\Delta}+2xuu^{\Delta}+\mu x\left(u^{\Delta}\right)^2}{\left(u+\mu u^{\Delta}\right)^2u^2}\mathrm{d}u\\
&\quad-\frac{x+\mu xu}{\left(u+\mu u^{\Delta}\right)^2}\mathrm{d}\left[u^{\Delta}\right]=\mathrm{d}\left[F^{\Delta}(x,u,u^{\Delta})\right].
\end{aligned}$$

Additionally, since $F^{\sigma}(x,u,u^{\Delta})=\dfrac{x+\mu xu}{u+\mu u^{\Delta}}$,

$$\begin{aligned}
\left[\mathrm{d}F(x,u)\right]^{\sigma}=&\left[\frac{1}{u}\mathrm{d}x-\frac{x}{u^2}\mathrm{d}u\right]^{\sigma}=\frac{1}{u^{\sigma}}\mathrm{d}x^{\sigma}-\frac{x^{\sigma}}{(u^{\sigma})^2}\mathrm{d}u^{\sigma}\\
=&\frac{1}{u^{\sigma}}\mathrm{d}x+\frac{\mu}{u^{\sigma}}\mathrm{d}\left(xu\right)-\frac{x^{\sigma}}{(u^{\sigma})^2}\mathrm{d}u^{\sigma}\\
=&\frac{1}{u^{\sigma}}\mathrm{d}x+\frac{\mu}{u^{\sigma}}\left(u\mathrm{d}x+x\mathrm{d}u\right)-\frac{x^{\sigma}}{(u^{\sigma})^2}\left(\mathrm{d}u+\mu\mathrm{d}\left[u^{\Delta}\right]\right)\\
=&\left(\frac{1}{u^{\sigma}}+\frac{\mu u}{u^{\sigma}}\right)\mathrm{d}x+\left(\frac{\mu x}{u^{\sigma}}-\frac{x^{\sigma}}{(u^{\sigma})^2}\right)\mathrm{d}u-\frac{\mu x^{\sigma}}{(u^{\sigma})^2}\mathrm{d}\left[u^{\Delta}\right]\\
=&\frac{1+\mu u}{u+\mu u^{\Delta}}\mathrm{d}x+\frac{\mu^2xu^{\Delta}-x}{(u+\mu u^{\Delta})^2}\mathrm{d}u-\frac{\mu(x+\mu xu)}{(u+\mu u^{\Delta})^2}\mathrm{d}\left[u^{\Delta}\right]\\
=&\mathrm{d}\left[F^{\sigma}(x,u,u^{\Delta})\right].
\end{aligned}$$

Note that for homogeneous time scales the following relation holds:

Proposition 4.4. *Let $\omega\in\mathcal{E}$. Then for a homogeneous time scale*

$$\omega^{\sigma}=\omega+\mu\omega^{\Delta}. \tag{17}$$

Proof. For simplicity of presentation let us assume that $n=m=1$. Then $\omega=A\mathrm{d}x+\sum_{k\geqslant 0}B_k\mathrm{d}u^{[k]}\in\mathcal{E}$ and

$$\begin{aligned}\omega^\sigma &= A^\sigma \mathrm{d}x^\sigma + \sum_{k\geqslant 0} B_k^\sigma \mathrm{d}\left(u^{[k]}\right)^\sigma = A^\sigma \mathrm{d}x + \mu A^\sigma \mathrm{d}f(x,u) + \sum_{k\geqslant 0} B_k^\sigma \mathrm{d}u^{[k]} \\ &\quad + \mu \sum_{k\geqslant 0} B_k^\sigma \mathrm{d}u^{[k+1]} = A\mathrm{d}x + \mu A^\Delta \mathrm{d}x + \sum_{k\geqslant 0} B_k \mathrm{d}u^{[k]} + \mu \sum_{k\geqslant 0} B_k^\Delta \mathrm{d}u^{[k]} \\ &\quad + \mu A^\sigma \mathrm{d}f(x,u) + \mu \sum_{k\geqslant 0} B_k^\sigma \mathrm{d}u^{[k+1]} = \omega + \mu\omega^\Delta.\end{aligned}$$

□

5. THE CONSTRUCTION OF THE INVERSIVE CLOSURE OF $\mathcal{K}$

From condition (i) of Proposition 3.8, Δ is an endomorphism of the *Abelian group* $\mathcal{K}$.

The operator Δ depends on the operator σ. In the continuous-time case $\mu = 0$ and $\sigma = \mathrm{id}$, so $\sigma^{-1} = \mathrm{id}$. In the discrete-time case, under assumption (5), σ is an injective endomorphism (see page 6) of $\mathcal{K}$, but not necessarily surjective.[1] Recall that endomorphism σ is *surjective* if for every $F \in \mathcal{K}$ there exists $G \in \mathcal{K}$ such that $\sigma(G) = F$. If σ is both injective and surjective, then it is called *bijective*. In the definitions and algorithms that follow we also need pre-images of elements from $\mathcal{K}$ with respect to σ. If σ is not surjective, not every element in $\mathcal{K}$ has a pre-image with respect to σ, i.e. $\sigma^{-1}(F)$ may not exist. However, it is always possible to embed $\mathcal{K}$ into its inversive closure $\mathcal{K}^*$ [20]. Then $\sigma^{-1}(F) \in \mathcal{K}^*$, for all $F \in \mathcal{K}$ and σ can be extended to $\mathcal{K}^*$ in such a way that $\sigma : \mathcal{K}^* \to \mathcal{K}^*$ becomes an *automorphism*, i.e. a bijective endomorphism.

We give an explicit construction of $\mathcal{K}^*$ in the general case. Since for $\mu = 0$, $\mathcal{K}^* = \mathcal{K}$, we need the construction only for $\mu \neq 0$.

Let $\mathcal{Z}$ be a complementary subspace to $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x^\sigma\}$, i.e.

$$\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x^\sigma\} \oplus \mathcal{Z}. \tag{18}$$

By assumption (5) we get $\dim \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x^\sigma\} = n$. Then $\dim \mathcal{Z} = m$ and

$$\mathcal{Z} = \mathrm{span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_m\} =: \mathrm{span}_{\mathcal{K}}\{\omega\},$$

where $\omega_k \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}u\}$, $k = 1, \ldots, m$. It is always possible to choose $\omega_k = \mathrm{d}z_k$ in such a way that $\mathrm{d}z_k$ are elements of the set $\{\mathrm{d}x_i, \mathrm{d}u_j,\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$. Therefore there exists a vector-valued function $z = \varphi(x,u) \in \mathbb{R}^m$ such that $\omega = \mathrm{d}z$.

[1] Note that we use the same symbol σ to denote the shift operator on a time scale and on $\mathcal{K}$. Although σ is certainly surjective on a homogeneous time scale, it may not be so on $\mathcal{K}$.

Let us recall that $x^\sigma = \widetilde{f}(x,u) = x + \mu f(x,u) \in \mathbb{R}^n$, where $f(\cdot)$ describes the dynamics of system (4) and $z = \varphi(x,u) \in \mathbb{R}^m$ is a function whose differential generates a complementary subspace $\mathcal{Z}$. Condition (18) means that the map $(x,u) \mapsto (x^\sigma, z) = (\widetilde{f}(x,u), \varphi(x,u))$ is a (local) diffeomorphism. This implies that (locally) there exists a vector-valued function ψ such that $(x,u) = \psi(x^\sigma, z)$. Finally, let $\mathcal{K}^*$ be a field extension of $\mathcal{K}$ consisting of meromorphic functions in a finite number of independent variables

$$\{x, u^{[k]}, z^{\langle -\ell \rangle},\ k \geqslant 0, \ell \geqslant 1\}.$$

Let $\sigma^{-1}(z) = z^{\langle -1 \rangle}$ and $\sigma^{-1}\left(z^{\langle -i+1 \rangle}\right) := z^{\langle -i \rangle}$. Since

$$(x,u) = \psi\left(x^\sigma, z\right) = \psi\left(\sigma(x), \sigma\left(z^{\langle -1 \rangle}\right)\right) = \sigma\left(\psi\left(x, z^{\langle -1 \rangle}\right)\right),$$

we have

$$\sigma^{-1}(x,u) = \psi\left(x, z^{\langle -1 \rangle}\right).$$

Therefore there exist functions ψ_s, $s = 1, \ldots, n+m$, such that

$$\sigma^{-1}(x_i) = \psi_i\left(x, z^{\langle -1 \rangle}\right),$$

$$\sigma^{-1}(u_j) = \psi_{n+j}\left(x, z^{\langle -1 \rangle}\right),$$

and

$$\sigma^{-1}(u_j^\Delta) = \frac{1}{\mu}\left[u_j - \sigma^{-1}(u_j)\right] = \mu^{-1} u_j - \mu^{-1}\psi_{n+j}(x, z^{\langle -1 \rangle}).$$

So, using the induction principle and

$$\sigma^{-1}\left(u_j^{[i]}\right) = \mu^{-1}\left[u_j^{[i-1]} - \sigma^{-1}\left(u_j^{[i-1]}\right)\right], \quad i \geqslant 1,$$

one can show that

$$\sigma^{-1}\left(u_j^{[k]}\right) = \sum_{i=0}^{k-1} (-1)^i \mu^{-i-1} u_j^{[k-i-1]} + (-1)^k \mu^{-k} \psi_{n+j}\left(x, z^{\langle -1 \rangle}\right),$$

for $j = 1, \ldots, m$, $k \geqslant 1$. Hence σ can be extended to $\mathcal{K}^*$ and it is an automorphism of $\mathcal{K}^*$. Although the choice of variables $z = \varphi(x,u)$ is not unique, each possible choice brings up a field extension of $\mathcal{K}$ which is isomorphic to $\mathcal{K}^*$.

Let $\rho = \sigma^{-1}$, where $\sigma : \mathcal{K}^* \to \mathcal{K}^*$, and let F^ρ denote the pre-image of the element $F \in \mathcal{K}^*$ with respect to σ, i.e $F^\rho := \rho(F)$.

We extend the operator Δ to variables $z^{\langle -\ell \rangle}$, $\ell \geqslant 1$, by using

$$\Delta(z^{\langle -\ell \rangle}) := \frac{z^{\langle -\ell+1 \rangle} - z^{\langle -\ell \rangle}}{\mu}.$$

The extension of operator Δ to $\mathcal{K}^*$ can be made in analogy to (9). Such an operator Δ is now σ-derivation of $\mathcal{K}^*$.

Now we demonstrate the construction of $\mathcal{K}^*$ on a simple example.

Example 5.1. Consider the nonlinear dynamical system defined on the homogeneous time scale $\mathbb{T}$ with $\mu \neq 0$

$$\begin{aligned} x_1^{\Delta} &= x_2 u, \\ x_2^{\Delta} &= x_1, \end{aligned} \tag{19}$$

which can be rewritten in the form

$$\begin{aligned} x_1^{\sigma} &= x_1 + \mu x_2 u, \\ x_2^{\sigma} &= x_2 + \mu x_1. \end{aligned} \tag{20}$$

One can choose $z = x_1$ and define $\mathcal{K}^*$ as the field of meromorphic functions in a finite number of the variables $x_1, x_2, u^{[k]},\ \ k \geqslant 0,\ \ x_1^{\langle -i \rangle},\ \ i \geqslant 1$, where $\sigma^{-1}(x_1) := x_1^{\langle -1 \rangle}$ and $\sigma^{-1}\left(x_1^{\langle -i+1 \rangle}\right) := x_1^{\langle -i \rangle}$. Then the remaining variables x_2 and u have the pre-images in $\mathcal{K}^*$:

$$\begin{aligned} \rho(x_2) &= x_2 - \mu x_1^{\langle -1 \rangle}, \\ \rho(u) &= \frac{x_1 - x_1^{\langle -1 \rangle}}{\mu\left(x_2 - \mu x_1^{\langle -1 \rangle}\right)}. \end{aligned}$$

Alternatively, one can choose $z = x_2$ and define $\mathcal{K}^*$ as the field of meromorphic functions in a finite number of the variables x_1, x_2, $u^{[k]}$, $k \geqslant 0$, $x_2^{\langle -i \rangle}$, $i \geqslant 1$, where $\sigma^{-1}(x_2) := x_2^{\langle -1 \rangle}$ and $\sigma^{-1}\left(x_2^{\langle -i+1 \rangle}\right) := x_2^{\langle -i \rangle}$. Then x_1 and u have the pre-images in $\mathcal{K}^*$:

$$\begin{aligned} \rho(x_1) &= \frac{1}{\mu}\left[x_2 - x_2^{\langle -1 \rangle}\right], \\ \rho(u) &= \frac{\mu x_1 - x_2 + x_2^{\langle -1 \rangle}}{\mu^2 x_2^{\langle -1 \rangle}}. \end{aligned}$$

A third possibility is to choose $z = u$ and define $\mathcal{K}^*$ as the field of meromorphic functions in a finite number of independent variables x_1, x_2, $u^{[k]}$, $k \geqslant 0$, $u^{\langle -i \rangle}$, $i \geqslant 1$, where $\sigma^{-1}(u) := u^{\langle -1 \rangle}$ and $\sigma^{-1}\left(u^{\langle -i+1 \rangle}\right) := u^{\langle -i \rangle}$. Then both x_1 and x_2 have the pre-images in $\mathcal{K}^*$:

$$\begin{aligned} \rho(x_1) &= \frac{x_1 - \mu x_2 u^{\langle -1 \rangle}}{1 - \mu^2 u^{\langle -1 \rangle}}, \\ \rho(x_2) &= \frac{x_2 - \mu x_1}{1 - \mu^2 u^{\langle -1 \rangle}}. \end{aligned}$$

Additionally, one can assume that

$$\operatorname{rank}_{\mathcal{K}} \frac{\partial f}{\partial u} = m. \tag{21}$$

Assumption (21) guarantees that the controls are independent. This assumption, though natural, is not necessary for the construction of $\mathcal{K}^*$, but under it $\sigma^{-1}(u)$ can be expressed as a meromorphic function of x and $\sigma^{-1}(x)$ from (4); see the first and second constructions in Example 19.

APPENDIX

Let us consider the control system given by (4). Let $F : \mathbb{R} \to \mathbb{R}$. Using Proposition 3.3, one can check that the following formulas hold:

Table 1. Delta derivatives of some elementary functions

	$F(x)$	$F^{\Delta}(x,u)$
1	const	0
2	x^n, $n \geqslant 1$	$f(x,u) \cdot \left[\sum_{k=1}^{n} x^{n-k} \left(x + \mu f(x,u)\right)^{k-1} \right]$
3	$\frac{1}{x}$	$-\frac{f(x,u)}{\left(x + \mu f(x,u)\right)^{\sigma} x}$
4	x^{-n}, $n \geqslant 1$	$-f(x,u) \cdot \left[\sum_{k=1}^{n} x^{-k} \left(x + \mu f(x,u)\right)^{k-1-n} \right]$
5	$\sqrt{x}$	$\frac{f(x,u)}{\sqrt{x + \mu f(x,u)} + \sqrt{x}}$
6	$\sqrt[n]{x}$, $n \geqslant 2$	$\frac{f(x,u)}{\sum_{k=0}^{n-1} \left(\sqrt[n]{x + \mu f(x,u)} \right)^{n-k-1} \left(\sqrt[n]{x} \right)^{k}}$
7	$\exp(x)$	$\begin{cases} \exp(x) \frac{\exp\left(\mu f(x,u)\right) - 1}{\mu}, & \mu \neq 0 \\ \exp(x) \cdot f(x,u), & \mu = 0 \end{cases}$
8	$\sin(x)$	$\begin{cases} \frac{1}{\mu} \left[\sin\left(x + \mu f(x,u)\right) - \sin x\right], & \mu \neq 0 \\ \cos(x) \cdot f(x,u), & \mu = 0 \end{cases}$
9	$\cos(x)$	$\begin{cases} \frac{1}{\mu} \left[\cos\left(x + \mu f(x,u)\right) - \cos x\right], & \mu \neq 0 \\ -\sin(x) \cdot f(x,u), & \mu = 0 \end{cases}$

For example, let us show how the second formula can be proved by using Proposition 3.3. If $\mu = 0$, then we have $(x^n)^{\Delta} = nx^{n-1}f(x,u)$, but if $\mu \neq 0$, then

$$(x^n)^{\Delta} = \frac{(x+\mu f(x,u))^n - x^n}{\mu} = f(x,u)\cdot\left[\sum_{k=1}^{n} x^{n-k}\left(x+\mu f(x,u)\right)^{k-1}\right].$$

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Diferentsiaalsete üksvormide algebraline formalism mittelineaarsete juhtimissüsteemide jaoks ajaskaaladel

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On käsitletud mittelineaarseid juhtimissüsteeme ajaskaaladel. Ajaskaala on aja mudel. Analüüs ajaskaalal põhineb niinimetatud delta-tuletisel, mis üldistab nii tavalise tuletise kui ka diferentsoperaatori ja võimaldab ühildada nii pidevate kui ka diskreetsete juhtimissüsteemide uurimise. Artiklis on välja töötatud matemaatiline aparatuur, mis võimaldab hiljem uurida mittelineaarseid juhtimissüsteeme ajaskaaladel. Lähtepunktiks on võetud universaalne algebraline formalism, mis põhineb juhtimissüsteemiga defineeritud diferentsiaalvormide klassifitseerimisel ja võimaldab lahendada erinevaid modelleerimis-, analüüsi- ning sünteesiülesandeid. Võtmeülesandeks on seejuures delta-diferentsiaalkorpuse konstrueerimine ja selle pöördsulundi konstrueerimine.