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Betweenness plane geometry and its relationship with convex, linear, and projective plane geometries

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Abstract. In a previous paper the author recapitulated betweenness geometry, developed in 1904–64 by O. Veblen, J. Sarv, J. Hashimoto, and the author. The relationship of this geometry with join geometry (by W. Prenowitz) was investigated. Now this relationship will be extended to convex and linear geometry. The achievements of the well-developed projective plane geometry are used to enrich betweenness plane geometry with coordinates, ternary operation, algebraic extension, Lenz–Barlotti classification, translation, and Moufang type. The final statement is that every Moufang-type betweenness plane is Desarguesian.

Key words: betweenness plane, ordered projective plane, ternary operation, Lenz–Barlotti classification, Moufang-type plane.

1. INTRODUCTION

In the foundation of geometry the betweenness relation has fascinated the investigators for a long time. Already C. F. Gauss, in his letter to F. Bolyai (6 March 1832; see [1], p. 222), pointed to the absence of betweenness postulates in Euclid's treatment. Elimination of this defect was started 50 years later by Pasch [2]. Further development in the 19th century (through the works of G. Peano, F. Amodeo, G. Veronese, G. Fano, F. Enriques, and M. Pieri) led to Hilbert's fundamental *Grundlagen der Geometrie* [3], where the betweenness relation is subject to the *axioms of connection and of order* (I 1–7, II 1–5 of Hilbert's list), called by Schur [4] the *projective axioms* of geometry. Here the axioms of order II 1–5 are presented as dependent on the axioms I 1–7.

In the first decade of the 20th century Hilbert's projective axioms were investigated by Moore [5] and Veblen [6]. Moore indicated some redundancy in

Hilbert's axioms of order, which Hilbert took into consideration in the following editions (e.g. in the seventh edition of [3]). But in these editions there was not considered the question asked by Henri Poincaré in 1902: "Ne setait-il pas préférable de donner aux axiomes du deuxième groupe une forme qui les affranchit de cette dépendance et les séparât complètement du premier groupe?" (see [7], Appendice. Ch. 1: *Les foundaments de la géométrie*, page 112; first published in *Journal des savants*, Mai 1902).[1]

Veblen [6] was the first to respond to this question in 1904. He gave independent axioms of betweenness relation and showed that the lines and planes can then be defined as special sets of points.

Huntington [8–11] gave an elaborated system of axioms for the betweenness relation, but only in dimension one, i.e. for the case of a line.

This standpoint was developed further in Estonia, first by Nuut [12] in 1929 (for dimension one, as a geometrical foundation of real numbers). Then Sarv [13] proposed in 1931 an axiomatics for the betweenness relation for an arbitrary dimension n, extending the Moore–Veblen approach so that all axioms of connection, including also those concerning lines, planes, etc., became consequences. This self-dependent axiomatics was simplified and then perfected by Nuut [14] and Tudeberg (from 1936 Humal) [15]. As a result, an extremely simple axiomatics was worked out for n-dimensional geometry using only two basic concepts: "point" and "between".

The author of the present paper developed a comprehensive theory of the *models of betweenness*, based on this axiomatics [16]. At the same time he established [17] that in dimension > 2 this model reduces to a convex domain in an n-dimensional linear space over an ordered skew field.[2] As a whole, the theory of these models, including also the Huntington–Nuut theory for dimension 1, can be called *betweenness geometry*. The same term was introduced independently in a similar situation by Hashimoto [19] in 1958.

In 1970–80 Rubinshtein [20–22] developed (together with Rutkovskij) a theory of *axial structures*. This theory is tightly connected with betweenness geometry and uses some of its results (with exact references to [16,17]).

Independently there evolved also another approach, independent of the axioms of connection. In 1909, Schur [23] tried to work out a part of geometry based on the basic concepts "point" and "line segment" (Ger. *Strecke*). This approach was elaborated in 1961 by Prenowitz [24] (see also [25,26]). The segment was considered as the "join" of its endpoints, and so the *join operation* was introduced in the set of points. This approach was then developed as the theory of *convexity spaces* in the 1970s by Bryant and Webster [27], Doignon [28] and others (summarized in papers [29–31], and in monographs [32,33]).

[1] "Wouldn't it be better to give the axioms in the second group a form which makes them free of this dependence and separates them from the first group?"

[2] Later Pimenov [18] (in Appendix: Local betweenness relation) called this perfected axiomatics the *Humal–Lumiste axiomatics* and its model in dimension 2, when the above result cannot be used, the *Lumiste plane*.

In a recent paper [34] the author studied the relationship of betweenness geometries with join geometries, treated in [32]. He proved there Theorems 14 and 15, according to which betweenness geometry coincides with the Pasch–Peano geometry of [28]. Due to Theorems 16 and 18 of [34], betweenness geometry coincides with convex geometry, according to the Theorem in Sec. 1.3 of [28].

Betweenness and convex geometries were developed by several authors to so-called *linear geometry* (in the sense of [32,33,35,36]).

Betweenness geometry is tightly connected also with projective geometry (and also with absolute geometry; see [37]). Already in [16], §20 and [17] it was established that in the 3-dimensional betweenness space every bundle of lines through a fixed point has the structure of a Desarguesian projective plane (see also [34], Sec. 8). In the further study of the betweenness planes (which can also be non-Desarguesian), which follows below, several constructions of the theory of projective planes will be useful. First, however, betweenness geometry must be recapitulated.

2. TOWARDS BETWEENNESS GEOMETRY

Recall that the author worked out betweenness geometry, considered here, more than 40 years ago in [16,17], being guided by [13,15]. (Independently it was initiated by Hashimoto [19].) Recently this geometry was recapitulated in [34] as follows.

Let S be a set, and let $\mathbb{B}$ be a subset in $S \times S \times S$ (i.e. a *ternary relation* for S). Further (abc) will mean that $(a,b,c) \in \mathbb{B}$, then b is said to be *between* (or *inter*) a and c. Moreover, let us denote

$$\langle abc\rangle = (abc)\vee(bca)\vee(cab); \qquad [abc] = \langle abc\rangle\vee(a=b)\vee(b=c)\vee(c=a). \quad (1)$$

The triplet (a,b,c) is said to be *correct* if $\langle abc\rangle$, and *collinear* if $[abc]$.

Definition 1. *The pair* $(S,\mathbb{B})$ *is called an* interimity model *and* $\mathbb{B}$ *the* interimity relation *if*

$$\mathbf{B1}\text{: } (a \neq b) \Rightarrow \exists c, (abc); \quad \mathbf{B2}\text{: } (abc) = (cba);$$

$$\mathbf{B3}\text{: } (abc) \Rightarrow \neg(acb); \quad \mathbf{B4}\text{: } \langle abc\rangle \wedge [abd] \Rightarrow [cda]; \quad \mathbf{B5}\text{: } (a \neq b) \Rightarrow \exists c, \neg[abc].$$

For any two different a, b, $a \neq b$ the subset $ab = \{x|(axb)\}$ is called an *interval* with ends a and b, and the subset $L_{ab} = \{x|[xab]\}$ is said to be a *line* through a and b.

For any non-collinear a, b, c the subset $P_{abc} = Q_a \cup Q_b \cup Q_c$, where $Q_a = L_{ab} \cup L_{ac} \bigcup_{x\in bc} L_{ax}$, is called a *plane* through a, b, c. Here Q_b and Q_c are obtained by reordering a, b, c in the definition of Q_a; hence P_{abc} does not depend on this reordering.

A one-to-one map $f : S \to S$ of an interimity model onto itself is said to be a *cointerrelation* if $(abc) \Rightarrow (f(a)f(b)f(c))$, i.e. if the interimity relation remains valid by f. Then this f is also a *collineation*, because due to (1), $[abc] \Rightarrow [f(a)f(b)f(c)]$, i.e. every collinear point-triplet maps into a collinear point-triplet, hence every line maps into a line.

The basic concept will be introduced by the following

Definition 2. *If in an interimity model, in addition,*

$$\mathbf{B6}\text{: } \neg[abc] \wedge (abd) \wedge (bec) \Rightarrow \exists f, ((afc) \wedge (def)),$$

then this model is called a betweenness model *and its relation is said to be the* betweenness relation (see [16,17]).

A subsidiary concept gives now the following

Definition 3. *If in an interimity model* $\mathbf{B6}$ *is replaced by*

$$\mathbf{B6'}\text{: } \neg[abc] \wedge (abd) \wedge (aec) \Rightarrow \exists f, ((bfc) \wedge (dfe)),$$

then this model is called a betwixtness[3] model *and its relation is said to be the* betwixtness relation.

The connecting instrument for the betweenness and betwixtness models is the so-called *Pasch postulate*

$$\mathbf{P}\text{: } \neg[abc] \wedge (bec) \wedge (d \in P_{abc}) \wedge (d \notin L_{bc}) \wedge (a \notin L_{de})$$
$$\Rightarrow \exists f, (f \in L_{de}) \wedge [(afb) \vee (afc)].$$

The above postulates $\mathbf{B6}$ and $\mathbf{B6'}$ can be considered as forms of this Pasch postulate in terms of the initial betweenness and betwixtness relation, respectively.[4]

It is natural to start with the interimity model.

Lemma 4. *In an interimity model* (abc) *implies that* a, b, c *are three distinct points.*

Proof. Indeed, $\mathbf{B3}$ excludes $b = c$, and together with $\mathbf{B2}$ excludes also $b = a$. Finally, $a = c$ is impossible as well, because if $c = a$, then $b \neq a$ and due to $\mathbf{B5}$ $\exists d, \neg[abd]$, but on the other hand $(abc) \Rightarrow \langle acb \rangle$ and $(a = c) \Rightarrow [acd]$, and these together imply due to $\mathbf{B4}$ that $[dba] = [abd]$, but this contradicts $\neg[abd]$ and finishes the proof. □

3 Here the word "betwixt" has been in mind (which, according to dictionaries, is now archaic except in the expression *betwixt and between*), as well as the word "interim".

4 They are called the *outer* and *inner Pasch axioms*, respectively, and denoted by **OP** and **IP** (see [36], where it is proved that **IP** does not imply **OP**, while **OP** does imply **IP**, according to [6]; see also [16], Theorem 13, and [34], Theorem 11; this means that every betweenness model is also a betwixtness model, but not vice versa).

If a triplet *a, b, c* is correct, i.e. $\langle abc\rangle$, then due to Lemma 4 here a, b, c are three different points, and due to $\mathbf{B2}$, $\mathbf{B3}$ only one of them is between the two others. Recall that if $[abc]$, then *a, b, c* are said to be collinear. It is obvious that correctness and collinearity of any three a, b, c does not depend on their order, i.e.

$$\langle abc\rangle = \langle bca\rangle = \langle cab\rangle, \qquad [abc] = [bca] = [cab]. \tag{2}$$

Lemma 5. *In an interimity model let* a, b, c *be collinear, i.e.* $[abc]$ *and so* (1) *holds. Here only the following four possibilities occur*:

1) $(a = b) \vee (b = c) \vee (c = a)$, 2) (abc), 3) (bca), 4) (cab).

Each of them excludes the three others.

Proof. The first possibility follows from Lemma 4. Due to $\mathbf{B2}$, $\mathbf{B3}$, $(abc) = (cba) \Rightarrow \neg(cab)$, $(abc) \Rightarrow \neg(acb) = \neg(bca)$. Due to the same Lemma 4, $(abc) \Rightarrow \neg[(a = b) \vee (b = c) \vee (c = a)]$.

Lemma 6. *In an interimity model there hold*

$$\neg[abc] \wedge \langle abd\rangle \Rightarrow \neg[acd], \tag{3}$$

$$\neg[abc] \wedge [abd] \wedge [adc] \Rightarrow (a = d), \tag{4}$$

$$(abc) \wedge (bcd) \Rightarrow \langle abd\rangle, \tag{5}$$

$$\neg[abc] \wedge (adb) \wedge (aec) \Rightarrow d \neq e. \tag{6}$$

Proof. Let us suppose for (3), by *reductio ad absurdum*, that $[acd]$. Then due to (1) and $\mathbf{B4}$, $\langle abd\rangle \wedge [acd] = \langle adb\rangle \wedge [adc] \Rightarrow [bca] = [abc]$, but this is impossible.

For (4), $\neg[abc] \Rightarrow (a \neq b)$, $\neg[abc] \wedge [adc] \Rightarrow (b \neq d)$; now by *reductio ad absurdum*,

$$[abd] \wedge (a \neq b) \wedge (b \neq d) \wedge (a \neq d) \Rightarrow \langle abd\rangle = \langle adb\rangle$$

and then due to $\mathbf{B4}$ $\langle adb\rangle \wedge [adc] \Rightarrow [bca] = [abc]$, but this is impossible.

For (5), due to Lemma 4, $(abc) \wedge (bcd) \Rightarrow (a \neq b) \wedge (b \neq d)$. Also $d \neq a$, because otherwise, due to $\mathbf{B2}$, $(bcd) = (bca) = (acb)$ and, now due to $\mathbf{B3}$, $\neg(abc)$, which is impossible. Further, due to (1), $(abc) \wedge (bcd) = \langle bca\rangle \wedge [bcd]$, and now due to $\mathbf{B4}$, $[adb]$, which is, due to (1), equivalent with $[abd]$, but this together with $(a \neq b) \wedge (b \neq d) \wedge (d \neq d)$ implies $\langle abd\rangle$, as needed.

For (6), $(adb) \Rightarrow [abd]$, and now, by *reductio ad absurdum*, if one supposes $d = e$, then $(aec) = (adc) \Rightarrow [adc]$, and (4) would yield $a = d$. On the other hand, due to (1), $(adb) \Rightarrow a \neq d$, which gives a contradiction.

This finishes the proof. □

For a line the following assertions can be proved, which show that in an interimity model the points a, b are not some specific points of a line L_{ab}, but can be exchanged by every two of its different points c, d. Indeed, there holds

Lemma 7. *If $c \in L_{ab}$ and $c \neq a$, then $L_{ac} = L_{ab}$.*

Proof. This is obvious if $c = b$. Otherwise $[abc] \wedge (a \neq b) \wedge (b \neq c) \wedge (c \neq a) \Rightarrow \langle abc \rangle$ and, due to $\mathbf{B4}$, $\langle abc \rangle \wedge [abx] \Rightarrow [cxa]$, thus $x \in L_{ab} \Rightarrow x \in L_{ac}$. But also $\langle acb \rangle \wedge [acy] \Rightarrow [bya]$, thus $y \in L_{ac} \Rightarrow y \in L_{ab}$. □

Using this Lemma two times, one obtains

Theorem 8. *If in an interimity model two different points c, d belong to a line L_{ab}, then $L_{cd} = L_{ab}$.*

Hence, a line is uniquely determined by any two of its different points.

Recall that in the definition of a line L_{ab} due to (1) $[xab] = (xab) \vee (abx) \vee (bxa) \vee (x = a) \vee (x = b)$ (note that here $a = b$ is excluded). Hence a and b divide the remaining part of L_{ab} into three subsets: 1) $ab = \{x|(axb)\}$ (note that, due to $\mathbf{B2}$, $ab = ba$), 2) $a/b = \{x|(xab)\}$, and 3) $b/a = \{x|(abx) = (xba)\}$.

Recall that ab is the interval with ends a and b; further, a/b will be called its *extension over an end a*.

It follows that $L_{ab} = ab \cup (a/b) \cup (b/a) \cup a \cup b$, i.e. a line L_{ab} is a union of an interval, its ends, and its extensions over both ends.

Note that up to now only $\mathbf{B1}$–$\mathbf{B4}$ are used and, in an extreme case, S can consist only of the points of one and the same line L_{ab}.

Further let also $\mathbf{B5}$ be taken along. Here $\neg[abc]$ means that a, b, c are three non-collinear points, i.e. three different points, not belonging to one line.

If a, b, c are non-collinear, then they are said to be *vertices*; the intervals bc, ca, ab are the *sides* (opposite to a, b, c, respectively) of the *triangle* $\triangle abc$, which is considered as the union of all of them.

Here a/b and b/a are the *extensions of the side ab*, and $ab \cup a \cup b$ is the *closed side*.

Note that the subset $Q_a = L_{ab} \cup L_{ac} \bigcup_{x \in bc} L_{ax}$ in the definition of a plane P_{abc} can now be interpreted as the union of points on the lines, which are determined by a vertex a of the triangle $\triangle abc$ and the points of its opposite closed side.

The plane P_{abc} itself can be interpreted as the union of the points on the lines, which are determined by any of the vertices and the points of its opposite closed side of a triangle $\triangle abc$.

An interimity model will turn into a betweenness model if one adds $\mathbf{B6}$ to $\mathbf{B1}$–$\mathbf{B5}$. The premise $\neg[abc]$ of $\mathbf{B6}$ means that there exists a triangle $\triangle abc$. The other premises $(abd) \wedge (bec)$ mean that there are $d \in b/a$ and $e \in bc$, where the side bc and extension b/a have a common endpoint b.

Note that here the premises of $\mathbf{B6'}$ differ from those of $\mathbf{B6}$ only by the fact that $e \in bc$ is replaced by $e \in ac$ and so bc is changed by the side ac, which does not have a common endpoint with the extension b/a.

The theory of the betweenness model, called *betweenness geometry*, is developed in [[34]], where the following theorems have been proved:

(1) *Every betweenness geometry is also a betwixtness geometry*, *i.e. if* $\mathbf{B1}$–$\mathbf{B6}$ *hold*, *then also* $\mathbf{B6}'$ *holds* (Theorem 11 in [34]).
(2) *The interval ab is not empty but is an infinite subset* (Theorem 12 in [34]).
(3) *For a triangle* $\triangle abc$, *the subset* $\{x|\exists y, (byc) \wedge (axy)\}$ *does not depend on the reordering of vertices* a, b, c (Theorem 13 in [34]).

It is natural to call the subset considered in the last theorem the *interior of the triangle* $\triangle abc$. Here any permutation of a, b, c is admissible.

The interpretation of the Pasch postulate $\mathbf{P}$ can be detailed as follows. Its premises mean that there is a line L_{ed}, which is determined by a point e of a side bc of the triangle $\triangle abc$ and a point d of the plane P_{abc} of this triangle, and does not contain any of its vertices. The assertion is that this line must intersect at least one of the other two sides in a point f. (Both of these cannot intersect, because this is excluded by Lemma 6 in [34].) Briefly: If a line in a plane of a triangle intersects one side and does not contain any of the vertices, then it intersects one of the other two sides (but not both of them).

Theorems 14 and 15 of [34] state the following:
In betweenness geometry the Pasch postulate is valid. In an interimity geometry, *the Pasch postulate* $\mathbf{P}$ *yields* $\mathbf{B6}$, *i.e. one can obtain a betweenness geometry adding* $\mathbf{P}$, *instead of* $\mathbf{B6}$, *to* $\mathbf{B1}$–$\mathbf{B5}$.

3. COORDINATES AND TERNARY OPERATION

Coordinates and the ternary operation, introduced for the projective plane by Hall [38] (see also [39,40]), can also be defined for a betweenness plane.

Let O, X, Y be some three non-collinear points of the betweenness plane and E be a point in the interior of the triangle $\triangle OXY$, i.e. $\exists F, (OEF) \wedge (XFY)$. The line L_{OX} with the point X excluded will be called the *x-axis*, the line L_{OY} with Y excluded the *y-axis*, and the line L_{XY} with Y excluded the $[\xi]$-*axis*.

To the points of the line L_{OE} the coordinates (a, a) will be assigned, where these a are symbols, different for different points. In particular, to O and E the symbols 0 and 1 will be assigned, respectively; thus $O = (0, 0)$ and $E = (1, 1)$. If for a point p the line L_{Yp} intersects L_{OE} in (a, a) and the line L_{Xp} intersects L_{OE} in (b, b), then to p the coordinates (a, b) will be assigned. Then the intersection points of these lines with the x-axis and y-axis, respectively, are $(a, 0)$ and $(0, b)$.

On the $[\xi]$-axis the symbol $[m]$ will be assigned to the point, which is collinear with $(0, 0)$ and $(1, m)$. For instance, [1] is collinear with $O = (0, 0)$ and $E = (1, 1)$ (and also with every (a, a)).

The ternary operation, which was originally defined for the projective plane [38–40], can be applied in the case of the betweenness plane. Namely, for the sides OX and OY of the triangle $\triangle OXY$ let this operation be defined by the following construction.

Let $(x, 0)$, $[m]$, $(0, b)$ be points of the appropriate sides. Due to $\mathbf{B6}'$ there exists a point q such that $((x, 0)qY) \wedge ((0, b)qX)$, and then p such that $((0, b)p[m])$.

Further, $\neg[(0,0)(x,0)Y] \wedge ((0,0)(x,0)X) \wedge ((x,0)pY)$; hence, due to **B6**, there exists such a point $(0,y)$ that $((0,0)(0,y)Y)$. Here this y will be defined as the result of the ternary operation:

$$y = x \circ m \div b; \tag{7}$$

this definition can be extended naturally also to $x = 0$, $m = 0$, or $b = 0$.

This ternary operation has the following properties:

$$0 \circ m \div b = x \circ 0 \div b = b, \tag{8}$$

$$1 \circ m \div 0 = m \circ 1 \div 0 = m, \tag{9}$$

$$x \circ m \div z = y \;\text{ can have only unique solution with respect to } z, \tag{10}$$

$$\begin{gathered} x \circ m_1 \div b_1 = x \circ m_2 \div b_2 \,(\text{with}\, m_1 \neq m_2)\,\text{can have} \\ \text{only unique solution with respect to } x, \end{gathered} \tag{11}$$

$$\begin{gathered} \text{system} \quad a_1 \circ m \div b = c_1, \;\; a_2 \circ m \div b = c_2 \,(\text{with}\, (a_1, c_1) \neq (a_2, c_2)) \\ \text{can have only unique solution with respect to the pair } \; m, b. \end{gathered} \tag{12}$$

The properties (8), (9), and (10) follow immediately from the definition (7). For (11) the construction of x can be seen if one makes a suitable figure; uniqueness can be proved easily. (If $m_1 = m_2$, then no solution exists, in general, unless also $b_1 = b_2$, when the solution x becomes arbitrary.) To verify (12), let (7) be considered as the equation of the line $L_{[m](0,b)}$, and let us state that only one line can pass through two different points.

As a consequence, the following properties can be obtained:

$$x \circ m \div b = c \,(m \neq 0) \;\text{ can have only a unique solution with respect to } x, \tag{13}$$

$$a \circ \xi \div b = c \,(a \neq 0) \;\text{ can have only a unique solution with respect to } \xi. \tag{14}$$

Indeed, due to (11), the equation $x \circ m \div b = x \circ 0 \div c$ (with $m \neq 0$) can have only a unique solution x. But due to (8), $x \circ 0 \div c = c$. This verifies (13). Further, due to (12), the system $a \circ \xi \div y = c$, $0 \circ \xi \div y = b$ (with $a \neq 0$) can have only a unique solution with respect to the pair ξ, y, but due to (8), here $y = b$. This verifies (14).

Following Hall [38,40], *natural operations* + and $\cdot$ can be defined by this ternary operation, now for the betweenness plane:

$$a + b = (a \circ 1) \div b, \tag{15}$$

$$a \cdot m = (a \circ m) \div 0. \tag{16}$$

These operations have the following properties:

$$a + 0 = 0 + a = a, \tag{17}$$

$$m \cdot 0 = 0 \cdot m = 0, \tag{18}$$

$$m \cdot 1 = 1 \cdot m = m, \tag{19}$$

$$a + x = c \ \text{ can have only a unique solution with respect to } x, \tag{20}$$

$$x \cdot m = c \ \ (m \neq 0) \ \text{ can have only a unique solution with respect to } x, \tag{21}$$

$$a \cdot \xi = c \ \ (a \neq 0) \ \text{ can have only a unique solution with respect to } \xi. \tag{22}$$

The properties (17)–(19) follow immediately from (8) and (9). The properties (20)–(22) follow, respectively, from (12)–(14).

4. ALGEBRAIC EXTENSION OF A GIVEN BETWEENNESS PLANE

For a betweenness plane there exists a linear order on every line. This order on the lines through $O = (0, 0)$ can be chosen so that E follows O; then on the x-axis $(1, 0)$ follows $O = (0, 0)$, and on the y-axis $(0, 1)$ follows $O = (0, 0)$. For all points $(a, 0)$ of the side OX on the x-axis then $a > 0$; similarly, for all points $(0, a)$ of the side OY on the y-axis $a > 0$.

Let us consider Eq. (20) with $c = 0$ and $a > 0$; i.e. the equation $a + x = 0$ with $a > 0$. It is natural to denote the unique solution of this equation, if it exists, by $x = -a$, so that $a + (-a) = 0$; here naturally $-a < 0$. The point $(0, -a)$, if it exists, lies in the remaining part of the y-axis, except Y.

Considering the tetragon with vertices $(a, 0)$, (a, a), [1], X, one can see that one of its diagonals intersects the y-axis L_{OY} at the point $(0, a)$, the other at the point $(0, -a)$. Hence the pair of points $(0, a)$, $(0, -a)$ is harmonic with respect to the pair (0,0), Y. But there is the possibility that this intersection point $(0, -a)$ does not really exist in the considered betweenness plane. Then it will be added as an *ideal* point, following the procedure which is given for projective planes by Moufang [41] and Sperner [42] to expand a limited plane part by adding the ideal points.

For the points $(0, -y)$ of the y-axis, including the ideal ones, each of which forms with $(0, y) \in OY$, $y > 0$, a harmonic pair with respect to the pair O, Y, as above, there exists $-y < 0$. The same procedure can be performed on the x-axis: the points $(0, -x)$, harmonic with respect to O, X, including the ideal ones, can be added to the points $(x, 0) \in OX$.

The ternary operation can be extended as well, carrying the ideal points not only on the x- and y-axes, but also on the $[\xi]$-axis. After that this operation works on a linearly ordered set (called in [39] the *ternar*).

Introducing the ideal points (x, y) as the intersection points of $L_{Y(x,0)}$ and $L_{X(0,y)}$, where one of $(x, 0)$, $(0, y)$ is ideal (or both of them are ideal), one extends the given betweenness plane to an *ordered projective plane* with a ternary operation. In the properties (10)–(14), (20)–(22) the words *can have only* may be then replaced by *has*, and one has the so-called *complete linearly ordered ternar*.

Theorem 9. *By means of a complete linearly ordered ternar with elements* $a, b, ..., x, y, ...$ *a betweenness plane can be constructed, taking the pairs* (a, b) *as the points and considering* (a_2, b_2) *being between* (a_1, b_1) *and* (a_3, b_3) *if either*
(1) *there exist* m, b *so that* $b_i = a_i \circ m \div b$ *for* $i \in \{1, 2, 3\}$ *and* a_2 *is between* a_1, a_3, *like* b_2 *is between* b_1, b_3 *in this ordered ternar, or*
(2) a_2 *is between* a_1, a_3, *and* $b_2 = b_1 = b_3$, *or*
(3) $a_2 = a_1 = a_3$ *and* b_2 *is between* b_1, b_3.

Proof. For the set of these points (a, b), with this relation "between", the axioms $\mathbf{B1}$–$\mathbf{B4}$ are satisfied, because they hold for the set of the ternar's elements, as the linearly ordered set. Here the lines are the subsets $\{(x, x \circ m \div b)\}$, $\{(x, b)\}$, and $\{(a, y)\}$. For each of them there exist points not belonging to this line. Hence $\mathbf{B5}$ is also satisfied.

For a line $\{(x, y)|y = x \circ m \div c\}$ the two complementary half-planes, whose edge is this line, are $\{(x, y)|x \circ m \div c < y\}$ and $\{(x, y)|x \circ m \div c > y\}$. The validity of the Pasch postulate can be established in the following way.

On the betweenness plane just constructed, let the triangle with vertices (a_i, b_i), $i \in \{1, 2, 3\}$ be such that its vertices (a_1, b_1) and (a_2, b_2) belong to the two different half-planes above and the third vertex (a_3, b_3) is not on the line above. Then $a_1 \circ m \div c < b_1$ and $a_2 \circ m \div c > b_2$, but $a_3 \circ m \div c \neq b_3$. Here $a_3 \circ m \div c < b_3$ or $a_3 \circ m \div c > b_3$, but not both together. In the first case the vertices (a_2, b_2) and (a_3, b_3) are in the two different half-planes, i.e. the line above intersects the side between them. In the first case the same can be said for the vertices (a_1, b_1) and (a_3, b_3). So the Pasch postulate is valid, and thus the proof is finished. □

If the complete linearly ordered ternar in this theorem is obtained by the above extension of the ternar of a given betweenness plane, then the new betweenness plane, constructed according to this theorem, can be seen as an *algebraic extension* of the given plane. The latter is then a convex subset of the extended betweenness plane. Here *convex* means that with any two different points a, b it contains also the interval ab.

5. COLLINEATIONS AND LENZ–BARLOTTI CLASSIFICATION

A one-to-one map $f : S \to S$ of a betweenness plane onto itself is said to be a *collineation* if $(abc) \Rightarrow (f(a)f(b)f(c))$, i.e. if the betweenness relation remains valid under f. Then, due to (1), also $[abc] \Rightarrow [f(a)f(b)f(c)]$, i.e. every collinear point-triplet maps into a collinear point-triplet, hence every line maps into a line; from this stems the term "collineation". If considering the projective plane obtained by algebraic extension, one has here a collineation in the sense of the theory of projective geometry. The only difference is now that the linear order on every line of the betweenness plane remains invariable.

If the collineation has a fixed point z and a fixed line A, so that all lines through z and all points of A also remain fixed, then this collineation is called (z, A)-*collineation*, with *centre* z and *axis* A.

If there is a fixed point z and a fixed line A on the plane, so that the group of all (z, A)-collineations acts transitively on this plane, then the plane is said to be (z, A)-*transitive*. There exist the algebraic properties of the betweenness plane, which are equivalent to the (z, A)-transitivity.

Let the coordinates on the (z, A)-transitive betweenness plane be taken so that $L_{XY} = A$ and $Y = z$, and let us consider the (Y, L_{XY})-collineation f such that $f(O) = (0, b)$. Then all lines through Y remain invariant by f, like all points of the line L_{XY}, in particular X, [1], and $[m]$.

According to the construction, which led above to the result (1), there is a point p with coordinates $(x, x \circ m \div b)$, and a point u which has, due to (10), the coordinates $(x, x \cdot m)$. There is a point w which belongs to the line $L_{O[1]}$ and thus has the coordinates $(x \cdot m, x \cdot m)$. Due to (9), there is a point r with coordinates $(x \cdot m, x \cdot m + b)$.

Since the points $[m]$ and [1] are fixed by the f under consideration, like the lines through Y, it holds that $f : u \to p$, $f : w \to r$. Also the point X is fixed. Since u, w are collinear, lying on a line through X, the corresponding p, r must lie on a line through X. Therefore their second coordinates must be

$$x \circ m \div b = x \cdot m + b. \tag{23}$$

Hence for a (Y, L_{XY})-transitive betweenness plane the ternary operation is composed of two binary natural operations.

Further, it can be proved that, with respect to the natural operation of addition, the considered linearly ordered ternar forms a group. It is sufficient to show that this addition is associative, having in mind the properties (17) and (20).

Let us consider the (Y, L_{XY})-collineation f above. Here $y = x$ transforms to $y = x + b$; $x = c$ to $x = c$; (c, c) to $(c, c + b)$; $y = c$ to $y = c + b$; and also $x = a$ to $x = a$.

Hence $f((a, c)) = (a, c + b)$. Let now the same be done with (Y, L_{XY})-collineation g, defined by $g(O) = (0, d)$. The result is that $g((u, v)) = (u, v + d)$. For the product gf there exist

$$(gf)(O) = g(f(O)) = g((0, b)) = (0, b + d).$$

Consequently, in general, $(gf)((a, c)) = (a, c + (b + d))$. But $g(f(a, c)) = = g((a, c + d)) = (a, (c + b) + d)$. Hence

$$c + (b + d) = (c + b) + d, \tag{24}$$

i.e. addition has, indeed, the associativity property.

Conversely, if the ternar of a betweenness plane has the properties (23) and (24), i.e. if a) the ternary operation is composed of two natural binary operations: addition and multiplication, and b) with respect to the addition one has a linearly ordered group, then this plane is (Y, L_{XY})-transitive.

Indeed, let us consider for an arbitrary b a map f, which leaves Y and $[m]$ invariant and transforms (a, c) to $(a, c+b)$, and also leaves the lines L_{XY} and $x = a$ invariant and transforms the line $y = x \cdot m + t$ to the line $y = x \cdot m + (t + b)$. This f is a collineation, because if a point (a, c) lies on the line $y = x \cdot m + t$, then $c = a \cdot m + t$, and thus

$$c + b = (a \cdot m + t) + b = a \cdot m + (t + b),$$

i.e. the point $(a, c+b)$ lies on the line $y = x \cdot m + (t+b)$. Moreover, this collineation f is a (Y, L_{XY})-collineation. Since b is arbitrary, the plane is (Y, L_{XY})-transitive. For the projective planes Lenz [43] gave in 1954 a classification considering the set **L** of all pairs (z, A) with $z \in A$ such that the plane is (z, A)-transitive. He obtained seven classes: I (this set is empty) – VII (this set contains all such pairs), where IV and VI are both subdivided into two subclasses: a and b.

Three years later Barlotti [44] perfected this classification by abandoning the condition $z \in A$ and excluding class VI, which turned out to be empty, according to a result by H. F. Gingerich in 1945 (see [45], p. 103).

In [46] the Lenz–Barlotti classification is presented as follows (the classes of [43], which turned out to be empty, are excluded).

Let **B** be the set of all pairs (z, A) with $z \notin A$ such that the plane is (z, A)-transitive. The classes are now the following.

I_1. **L**=$\emptyset$=**B**.

I_2. $\mathbf{L} = \emptyset$, $|\mathbf{B}| = 1$.

I_3. $\mathbf{L} = \emptyset$, $|\mathbf{B}| = 2$, where $|\mathbf{B}|$ is the power of the set **B** (i.e. $\mathbf{B} = \{(z_1, A_1), (z_2, A_2)\}$) and $z_1 \in A_2 \setminus A_1$, $z_2 \in A_1 \setminus A_2$.

I_4. **L**=$\emptyset$, $|\mathbf{B}| = 3$, and **B**= $\{(a, L_{bc}), (b, L_{ca}), (c, L_{ab})\}$ with non-collinear a, b, c.

I_5. **L**=$\emptyset$. There exists a point a and a line A with $a \in A$, and a bijection β of $A \setminus \{a\}$ onto the set of lines X with $a \in X, X \neq A$ such that $\mathbf{B} = \{(x, \beta(x)) | x \in A \setminus \{a\}\}$.

II_1. $|\mathbf{L}| = 1$ and **B**= $\emptyset$.

II_2. $|\mathbf{L}| = 1 = |\mathbf{B}|$ and for **B**= $\{(b, B)\}$ there is $b \in A$ and $a \in B$.

II_3. $|\mathbf{L}| = 1$ and **B** is such as in I_6, while $(a, A) \in \mathbf{L}$.

III_1. There exists only one point a and one line Z with $a \notin Z$ so that $\mathbf{L} = \{(z, L_{za}) | z \in Z\}$ and **B**= $\emptyset$.

III_2. There exists only one point a and one line Z with $a \notin Z$ so that $\mathbf{L} = \{(z, L_{za}) | z \in Z\}$ and **B**= $\{(a, Z)\}$.

IV_{a_1}. There exists only one line A with **L**= $\{(z, A) | z \in A\}$ and **B**= $\emptyset$.

IV$_{a_2}$. There exists only one line A with **L**$=\{(z,A)|z\in A\}$. There exist $a,a'\in A$, $a\neq a'$ and the lines X,Y so that **B**$=\{(a,X)|a'\in X\}\cup\{(a',Y)|a\in Y\}$.

IV$_{a_3}$. There exists only one line A with **L**$=\{(z,A)|z\in A\}$. There exists an involution $\beta: A\to A$ without fix points so that **B**$=\{(x,X)|x\in A, X\neq A, \beta(x)\in X\}$.

IV$_{b_1}$, dual to IV$_{a_1}$.

V. There exists only one point a and one line Z with $a\in Z$ so that **L** $=\{(z,Z)|z\in Z\}\cup\{(a,A)|a\in A\}$ and **B**$=\emptyset$.

VII$_1$. **L**$=\{(z,A)|z\in A\}$ and **B**$=\emptyset$.

VII$_2$. **L**$=\{(z,A)|z\in A\}$ and **B**$\neq$**L**.

The same classification is presented in [47] as a table, where also the corresponding ternar of each class is characterized. Moreover, the special features of this classification for the ordered projective planes (and thus for betweenness planes) are discussed in [47] (see also [48]).

6. TRANSLATION PLANE AND ORDERED QUASIFIELDS

If for a betweenness plane its algebraic extension is (z,A)-transitive for an arbitrary centre $z\in A$, then it is called the *translation plane* with respect to the axis A.

For this, it is sufficient that this plane is (z_1,A)-transitive and (z_2,A)-transitive for two different centres z_1,z_2, on the axis A, i.e. $z_1,z_2\in A$; $z_1\neq z_2$.

Indeed, let then z be an arbitrary point of A, $z\neq z_1,z_2$, and L a line through z, $L\neq A$. Further, let u,v be two arbitrary points of L such that the lines L_{uz_1} and L_{vz_2} intersect at a point w. There exists a (z_1,A)-collineation f_1 such that $f_1(u)=w$, and a (z_2,A)-collineation f_2 such that $f_2(w)=v$. Then $f=f_2f_1$ is a desirable (z,A)-collineation, which shows that one has the (z,A)-transitivity.

Let us consider now a betweenness plane which is a translation plane with respect to the axis L. The coordinates on such a plane can be taken so that $L_{XY}=A$. This plane is (Y,L_{XY})-transitive and also (X,L_{XY})-transitive. Due to the result of Sec. 5, the ternary operation is composed of two binary operations: addition and multiplication, and with respect to the first of them, one has a linearly ordered group. In the proof of this result there is a (Y,L_{XY})-collineation f with the property $f((a,c))=(a,c+b)$. Similarly there exists an (X,L_{XY})-collineation g with the property $g((a,c))=(a,c+d)$.

It appears that $fg=gf$. Indeed, for an arbitrary point u, $u\notin L_{XY}$, one has then collinears $Y,u,f(u)$; $X,u,g(u)$; $Y,g(u),f(g(u))$; and $X,f(u),g(f(u))$. Here $X,f(u),g(f(u))=(gf)(u)$ are also collinear, like $Y,g(u),f(g(u))=(fg)(u)$. Hence $(gf)(u)$ and $(fg)(u)$ are both the intersection points of two different lines $L_{Xf(u)}$ and $L_{Yg(u)}$, thus $gf=fg$, as asserted.

Since $(gf)(a,c) = g(a, c+b) = (a, c+b+d)$ and $(fg)(a,c) = f(a, c+d) = (a, c+d+b)$, one has

$$b + d = d + b, \tag{25}$$

i.e. the linearly ordered group is an Abelian group with respect to addition.

The complete linearly ordered ternar has the properties (19), (21), and (22) with respect to multiplication, which shows that one has here a quasigroup with unit, i.e. a *loop*.

There are some properties which connect these two binary natural operations: addition and multiplication of the complete linearly ordered ternar.

Let us consider an (X, L_{XY})-collineation h such that $h(0,0) = (b,0)$. Then $y = x$ transforms to $y = x - b$; $y = a$ to $y = a$; (a,a) to $(a+b, a)$; $x = a$ to $x = a + b$; $y = a \cdot m$ to $y = a \cdot m$; $(a, a \cdot m)$ to $(a+b, a \cdot m)$; and $y = x \cdot m$ to $y = x \cdot m - b \cdot m$.

Since the point $(a, a \cdot m)$ lies on the line $y = x \cdot m$, and the point $(a+b, a \cdot m)$ lies on the line $y = x \cdot m - b \cdot m$, there holds $a \cdot m = (a+b) \cdot m - b \cdot m$, thus

$$(a + b) \cdot m = a \cdot m + b \cdot m, \tag{26}$$

i.e. the multiplication is right-distributive with respect to the addition.

The properties (11) and (23) give together the following:

$$x \cdot m_1 = x \cdot m_2 \text{ (with } m_1 \neq m_2 \text{) has a unique solution with respect to } x. \tag{27}$$

The ternar, 1) whose ternary operation is composed of its two binary natural operations, 2) which is an Abelian group with respect to the addition and a loop with respect to the multiplication, and 3) which has the properties (26) and (27), has been considered in [49], therefore it is called a *Veblen–Wedderburn system* (also said to be a *quasifield*).

For the betweenness plane this is, of course, the linearly ordered quasifield (considered, e.g., by Jousson [50]).

7. ALTERNATIVE QUASIFIELD AND MOUFANG-TYPE PLANE

Let us consider the betweenness plane, whose algebraic extension gives the ordered projective plane (with the excluded line L_{XY}), which is a translation plane with respect to every line through the point Y as the axis. It is known that for this it is sufficient that this plane is the translation plane with respect to two lines intersecting in Y (see [40], Theorem 20.5.1).

Theorem 10. *If the betweenness plane is such that its algebraic extension is a translation plane with respect to every line going through the point* Y, *then*

(1) *all its lines, different from* L_{XY}, *are given by the linear equations* $x = c$ *and* $y = x \cdot m + b$, *and*

(2) *the coordinates are subjected to the following requirements*:

2.1) *according to addition, they form a linearly ordered Abelian group*,

2.2) *according to multiplication, they form a loop (quasigroup with unit)*,

2.3) $(a+b)\cdot m = a\cdot m + b\cdot m$,

2.4) $a\cdot(s+t) = a\cdot s + a\cdot t$,

2.5) *every element* $a\neq 0$ *has an inverse* a^{-1} *such that* $a\cdot a^{-1} = 1 = a^{-1}\cdot a$,

2.6) $a^{-1}\cdot(a\cdot b) = b$.

Proof. Among the lines through the point Y there is the line L_{XY}. Hence, due to the results of the previous Sec. 6, here (1) and 2.1), 2.2), and 2.3) are valid.

To prove the remaining three requirements, let a (Y,A)-collineation be considered, whose axis A is the line $x=0$, going through the centre Y (in such a case this collineation is called an *elation*, according to [40], §20.2), and which maps the point $X=[0]$ into the point $[m]$. Then all points $(0,b)$ are invariant, like the lines L_{XY} and $x=c$. Here successively $[0]\mapsto[m]$, $(0,b)\mapsto(0,b)$, $y=b\mapsto y=x\cdot m+b$, $x=a\mapsto x=a$, $(a,b)\mapsto(a,a\cdot m+b)$. In particular, $(1,t)\mapsto(1,m+t)$ and $(0,0)\mapsto(0,0)$, hence $y=x\cdot t\mapsto y=x\cdot(m+t)$. But $(a,a\cdot t)\mapsto(a,a\cdot m+a\cdot t)$, where $(a,a\cdot t)$ belongs to the line $y=x\cdot t$. Therefore $(a,a\cdot m+a\cdot t)$ belongs to the line $y=x\cdot(m+t)$; hence the left distributivity requirement 2.4) is satisfied:

$$a\cdot m + a\cdot t = a\cdot(m+t). \tag{28}$$

Now let us consider another elation, whose centre is $(0,0)$, axis $x=0$, and which maps $X=[0]$ into $(-1-a,0)$. For this $(0,1+a)\mapsto(0,1+a)$, thus $y=1+a\mapsto y=x+1+a$. Further, $[0]\mapsto(-1-a,0)$, $(0,b+a\cdot b)\mapsto(0,b+a\cdot b)$, hence $y=b+a\cdot b\mapsto y=x\cdot b+b+a\cdot b$. Now $y=1+a\mapsto y=x+1+a$, $y=x\cdot(1+a)\mapsto y=x\cdot(1+a)$, hence $(1,1+a)\mapsto(d,d+1+a)$, if $a\neq 0$; therefore

$$d\cdot(1+a) = d+1+a.$$

Further, $Y\mapsto Y$, $(1,1+a)\mapsto(d,d+1+a)$, thus $x=1\mapsto x=d$. Consequently,

$$y=x\cdot(b+a\cdot b)\mapsto y=x\cdot(b+a\cdot b),$$

$$y=b+a\cdot b\mapsto y=x\cdot b+b+a\cdot b,$$

hence $(1,b+a\cdot b)\mapsto(d,d\cdot(b+a\cdot b))$, where $d\cdot(b+a\cdot b) = d\cdot b+b+a\cdot b$.

Let now not only $a\neq 0$ but also $(-1-a,0)\neq(0,0)$, i.e. $a\neq -1$. For such an element there exists a d, so that $d\cdot(1+a)=d+1+a$ and $d\cdot(b+a\cdot b)=d\cdot b+b+a\cdot b$ for an arbitrary b. Taking $d=u+1$ and using the distributive property, one finds that $ua=1$ and $u\cdot(a\cdot b)=b$. For $a=-1$ these relations remain valid if $u=-1$. Since for $u\neq 0$ there exists such an element v that $vu=1$ and $v\cdot(u\cdot a)=a$, it follows that $v=a$. Consequently, $u=a^{-1}$, and the result will be 2.5): $a \cdot a^{-1} = a^{-1} \cdot a = 1$, and also 2.6): $a^{-1} \cdot (a \cdot b) = b$. This finishes the proof. $\square$

One more identity may be deduced following Bruck [51]. Let there be

$$[y^{-1} - (y + z^{-1})^{-1}] \cdot [y \cdot (z \cdot y) + y] = t,$$

where $y \neq 0$, $y \neq -z^{-1}$. Multiplying here by $y + z^{-1}$, one obtains

$$(y + z^{-1}) \cdot t = (y + z^{-1})[zy + 1 - (y + z^{-1})^{-1}(y \cdot (z \cdot y)) - (y + z^{-1})^{-1}y]$$

$$= y \cdot (z \cdot y) + y + y + z^{-1} - y \cdot (z \cdot y) - y = y + z^{-1}.$$

Hence, $t = 1$, and the elements $y^{-1} - (y + z^{-1})^{-1}$ and $y \cdot (z \cdot y) + y$ are inverse to each other. Then for an arbitrary x there exists

$$[y^{-1} - (y + z^{-1})^{-1}] \cdot [(y \cdot (z \cdot y)) \cdot x + y \cdot x] = x.$$

If one denotes

$$[y^{-1} - (y + z^{-1})^{-1}] \cdot [y \cdot (z \cdot (y \cdot x)) + y \cdot x] = w,$$

then

$$(y + z^{-1}) \cdot w = (y + z^{-1})[z \cdot (y \cdot x) + x] - y \cdot [z \cdot (y \cdot x)] - y \cdot x$$

$$= y \cdot x + z^{-1} \cdot x = (y + z^{-1}) \cdot x.$$

Consequently, $w = x$. Comparing the expressions for w and x, one obtains

$$[y \cdot (z \cdot y)] \cdot x = y \cdot [z \cdot (y \cdot x)]. \tag{29}$$

This identity is called the *Moufang identity*; it is valid also for the excluded values $y = 0$, $y = -z^{-1}$, thus for all elements.

In particular, for $z = 1$ there follows the *left alternativity*:

$$(y \cdot y) \cdot x = y \cdot (y \cdot x). \tag{30}$$

A ternar, satisfying the conditions (2.1)–(2.6), (29), and (30), is called an *alternative quasifield*. A projective plane with such a ternar is said to be a *Moufang plane*.

A betweenness plane whose algebraic extension gives a Moufang projective plane is said to be the *Moufang-type betweenness plane*. For such a plane the following algebraic result (Skornyakov [52], Bruck and Kleinfeld [53]) is important:

Every ordered alternative quasifield is a skewfield.

For betweenness planes this means:

Every Moufang-type betweenness plane is Desarguesian.

Note that for the ordered projective planes this statement is formulated in [[47]], p. 264, and in [[46]], p. 135.

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Vahelsustasandi geomeetria ja selle seos kumer-, lineaar- ning projektiivtasandi geomeetriaga

Ülo Lumiste

Ühes oma varasemas artiklis on autor käsitlenud uudselt vahelsusgeomeetriat, mis arendati O. Vebleni, J. Sarve, J. Hashimoto ja autori enda poolt välja aastail 1904–1964. Samuti on käsitletud selle geomeetria seost W. A. Prenowitzi ühenduvuse (*join*) geomeetriaga. Käesolevas artiklis on see seos laiendatud kumer- ja lineaargeomeetriani. Projektiivtasandi geomeetria rohked saavutused on rakendatud vahelsustasandi geomeetria rikastamiseks koordinaatide, ternaaroperatsiooni, algebralise laienduse, Lenzi–Barlotti klassifikatsiooni, translatsioonide ja Moufangi-tüüpi vahelsustasandi mõiste sissetoomisega. Käsitluse lõpus on nenditud, et iga viimast tüüpi vahelsustasand on desargiline.