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Interpolation classes, operator and matrix monotone functions

Yacin Ameur[a], Sten Kaijser[b], and Sergei Silvestrov[c]

[a] Department of Chemistry and Biomedical Sciences, University of Kalmar, SE-391 82 Kalmar, Sweden; Yacin.Ameur@hik.se
[b] Department of Mathematics, Uppsala University, Box 480, SE-751 06, Uppsala, Sweden; sten@math.uu.se
[c] Centre for Mathematical Sciences, Lund University, Box 118, SE-22100 Lund, Sweden; sergei.silvestrov@math.lth.se

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Abstract. Connections between interpolation spaces, Pick functions, and matrix monotone functions are investigated. Characterizations, inclusion results, open problems, and conjectures on these function classes and their interrelations are presented.

Key words: interpolation spaces, matrix monotone functions.

1. PICK FUNCTIONS AND INTERPOLATION SPACES

A (positive) *Pick function* is a function of the form

$$h(\lambda) = \int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda} \mathrm{d}\varrho(t), \quad \lambda > 0, \tag{1}$$

where ϱ is some positive Radon measure on $[0, \infty]$. The convex cone of functions having such a representation is denoted by the letter $\mathcal{P}$. A suitable reference on the class $\mathcal{P}$ is [1]. If $S \subseteq \mathbb{R}_+ = (0, \infty)$ is an arbitrary subset, we denote by $\mathcal{P}|S$ the set of restrictions to S of $\mathcal{P}$-functions.

A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ having the property that for any positive definite $n \times n$-matrices A, B, the condition "$A \leq B$ implies $h(A) \leq h(B)$" holds, is called a (positive) *matrix monotone function of order* n. The class of all matrix monotone functions of order n is denoted by the symbol $\mathcal{P}_n$. Evidently, $\mathcal{P}_1$ consists of all positive increasing functions on $\mathbb{R}_+ = (0, \infty)$ and $\mathcal{P}_{i+1} \subseteq \mathcal{P}_i$ for all $i \in \mathbb{N}$. A well-known theorem due to Löwner [2] (cf. [1]) states that $\cap_{n=1}^{\infty} \mathcal{P}_n = \mathcal{P}$.

A *regular Hilbert couple* $\overline{H} = (H_0, H_1)$ is a pair of Hilbert spaces, which are embedded into some Hausdorff topological vector space, and such that $H_0 \cap H_1$ is dense in H_0 and in H_1. If $\overline{H}$ is a regular Hilbert couple, there exists a densely defined, positive, possibly unbounded operator A in H_0 such that $\|x\|_{H_1}^2 = (Ax, x)_{H_0}$ for all $x \in H_0 \cap H_1$. Recall that a Banach space X is called *intermediate* with respect to the couple $\overline{H}$ if $H_0 \cap H_1 \subseteq X \subseteq H_0 + H_1$ where the inclusions are continuous. An intermediate space X is called an *exact interpolation space* with respect to $\overline{H}$ if $\|T\|_{B(X)} \leq \max\{\|T\|_{B(H_0)}, \|T\|_{B(H_1)}\}$ for all $T \in B(H_0) \cap B(H_1)$. If H_* is a Hilbert space which is intermediate with respect to $\overline{H}$, we may express the norm in H_* as $\|x\|_{H_*}^2 = (Bx, x)_{H_0}$ for all $x \in H_0 \cap H_*$ for some other densely defined positive operator B in H_0. The condition that H_* be an exact interpolation space with respect to A is then equivalent to

$$\|B^{1/2} T B^{-1/2}\|_{B(H_0)} \leq \max\{\|T\|_{B(H_0)}, \|A^{1/2} T A^{-1/2}\|_{B(H_0)}, \}, \qquad (2)$$

being true for all $T \in B(H_0)$. By a lemma of Donoghue [3], Lemma 1 and Lemma 2, cf. [4], if the interpolation inequality (2) is satisfied, then the operator B is affiliated with A and $B = h(A)$ for a (unique) continuous function h defined on $\sigma(A)$.

Interpolation of Hilbert spaces has been studied by several authors, notably Foiaş and Lions [5], Donoghue [3] (see also [1] and [6]), Peetre [7], Foiaş et al. [8], and one of the present authors (Ameur [4,9]). This has also led to a new proof of Löwner's theorem by Sparr [10], and to discovery of a new class of orthogonal polynomials [11].

2. INTERPOLATION FUNCTIONS

Let A be a densely defined positive operator in a Hilbert space H. A positive continuous function h defined on $\sigma(A)$ is an *interpolation function with respect to* A if

$$\|h(A)^{1/2} T h(A)^{-1/2}\| \leq \max(\|T\|, \|A^{1/2} T A^{-1/2}\|) \qquad (3)$$

for every bounded operator T on H. The set of interpolation functions with respect to A form a convex cone, which is denoted by C_A. Fix $n \in \mathbb{N}$, and assume that $H = \ell_2^n$ is an n-dimensional Hilbert space. We shall say that a function h defined everywhere on $\mathbb{R}_+$ is an *interpolation function of order* n and write $h \in C_n$ if h satisfies (3) for *every* positive definite operator $A \in B(\ell_2^n)$.

With this notation, the Foiaş–Lions theorem, essentially contained in [5], states that $\cap_{n=1}^{\infty} C_n = \mathcal{P}$. A stronger variant of the Foiaş–Lions theorem states that $C_A = \mathcal{P}|\sigma(A)$ for every positive operator A in Hilbert space. This was proved by Donoghue in [3] (cf. [9] for another proof). It is important to note the following consequence of Donoghue's theorem: a function f belongs to C_n if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a function $h \in \mathcal{P}$ such that $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \dots, n$. (Of course, the function h depends on f and the set $\{\lambda_i\}_{i=1}^n$ and is in general not unique.)

The class of functions C_n admits also the following useful characterization.

Theorem 1. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$. Then h belongs to C_n if and only if for all subsets $\{a_i\}_{i=1}^n \in \mathbb{R}^n$ and $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ one has that if $\sum_1^n a_i \frac{\lambda_i}{\lambda_i+t} \geq 0$ for all $t > 0$, then $\sum_1^n a_i h(\lambda_i) \geq 0$.*

The characterization of classes C_n given by Theorem 1 reminds of the definition of the function classes used in [10] to give a new proof of Löwner's theorem. These classes, however, are somewhat different.

3. GAPS BETWEEN THE FUNCTION CLASSES

In Donoghue's book [1] it was asserted that $\mathcal{P}_{n+1} \subsetneqq \mathcal{P}_n$ for all n. In [12] a rigorous proof of this assertion was given. This was done by exhibiting an explicit function in the gap for each n. A way to construct many such functions via solutions to the truncated moment problem was recently obtained in [13]. Monotone operator functions on arbitrary C^*-algebras are considered in [12,14]. Also, connections between operator monotone functions, operator convex functions, Löwner's theorem and Jensen's type inequalities for operators are considered in [15].

The functions in the intersection $\cap_{n=1}^{\infty} \mathcal{P}_n$ have several complete characterizations, for instance, by integral representations via Löwner's theorems or by analytical continuation properties. Obtaining satisfactory characterizations and insight into the structure of the gaps $\mathcal{P}_n \setminus \mathcal{P}_{n+1}$ is a deep and important open problem and, as mentioned above, so far only some ways to construct specific functions in the gaps have been described.

The following inclusion theorem shows that the interpolation classes C_n provide further insight into the structure of the gaps $\mathcal{P}_n \setminus \mathcal{P}_{n+1}$.

Theorem 2. *For all $n \in \mathbb{N}$ it holds that $\mathcal{P}_{n+1} \subseteq C_{2n+1} \subseteq C_{2n} \subseteq \mathcal{P}_n$.*

Due to restricted size of this paper, we refer for the proof of Theorem 2 to [16].

Whereas Theorem 1 yields a necessary and sufficient condition for $h \in C_n$ to hold, we have also the following, perhaps more transparent conditions for small values of n:

(i) C_1 consists of all positive functions on $\mathbb{R}_+$,
(ii) C_2 consists of all quasi-concave functions ($h(s) \leq \max(1, t/s)h(t)$ for all $s, t > 0$),
(iii) C_3 consists of all concave functions h on $\mathbb{R}_+$ such that $(t + c)^2 h(t)/t$ is convex for all $c > 0$,
(iv) a C_4 function is either affine or strictly concave on $\mathbb{R}_+$.

Analysis of the gaps for these inclusions is of interest as a contribution to the in-depth analysis of the structure of the gaps $\mathcal{P}_n \setminus \mathcal{P}_{n+1}$ between classes of matrix monotone functions, as well as for further study of the interpolation classes C_n themselves. We have been able to prove so far only that

$$C_4 \subsetneqq \mathcal{P}_2 \subsetneqq C_3 \subsetneqq C_2 \subsetneqq \mathcal{P}_1 \subsetneqq C_1,$$

but believe that the following general result should hold.

Conjecture 1. All inclusions in Theorem 2 are proper, for all n.

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Interpolatsiooni klassid, operaatorid ja maatriksmonotoonsed funktsioonid

Yacin Ameur, Sten Kaijser ja Sergei Silvestrov

On uuritud seost interpolatsiooni ruumide, Picki funktsioonide ja maatriksmonotoonsete funktsioonide vahel. On esitatud nende funktsioonide klasside karakteriseerimisi, sisestamistulemusi, lahtisi probleeme ning hüpoteese ja samade klasside vastastikuseid seoseid.