Optimality of theoretical error estimates for spline collocation methods for linear weakly singular Volterra integro-differential equations

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Abstract. Two spline collocation methods for solving linear weakly singular Volterra integrodifferential equations are considered. A result on the superconvergence at the collocation points is proved and optimality of several theoretical error estimates is demonstrated by extensive numerical experiments. Based on numerical results, a conjecture about the theoretical error estimates at the collocation points is stated for the cases not covered by known theorems.

Key words: weakly singular, Volterra, integro-differential equations, spline collocation, superconvergence.

1. INTRODUCTION

Consider the linear Volterra integro-differential equation (VIDE)

$$y'(t) = p(t)y(t) + q(t) + \int_{0}^{t} K(t,s)y(s)ds, \quad t \in [0,T], \ T > 0,$$
(1)

with a given initial condition

$$y(0) = y_0, y_0 \in \mathbb{R} = (-\infty, \infty).$$
 (2)

It will be assumed that

$$p, q \in C^{m,\nu}(0,T], K \in \mathcal{W}^{m,\nu}(\Delta_T), m \in \mathbb{N} = \{1, 2, \ldots\}, \nu \in \mathbb{R}, \nu < 1.$$
 (3)

Here $C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $\nu < 1$, is the set of all m times continuously differentiable functions $x: (0,T] \to \mathbb{R}$ such that the estimates

$$|x^{(j)}(t)| \le c \begin{cases} 1 & \text{if } j < 1 - \nu, \\ 1 + |\log t| & \text{if } j = 1 - \nu, \\ t^{1-\nu-j} & \text{if } j > 1 - \nu \end{cases}$$

hold with a constant c = c(x) for all $t \in (0,T]$ and j = 0, 1, ..., m. The set $\mathcal{W}^{m,\nu}(\Delta_T)$, with $m \in \mathbb{N}, \nu < 1, \Delta_T = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le T, 0 \le s < t\}$ consists of m times continuously differentiable functions $K : \Delta_T \to \mathbb{R}$ satisfying

$$\left| \left(\frac{\partial}{\partial t} \right)^{i} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right)^{j} K(t,s) \right| \leq c \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log(t-s)| & \text{if } \nu + i = 0, \\ (t-s)^{-\nu-i} & \text{if } \nu + i > 0, \end{cases}$$
(4)

with a constant c = c(K) for all $(t, s) \in \Delta_T$ and all integers $i, j \ge 0$ such that $i + j \le m$.

It is well known (see [¹]) that under the assumptions (3) Eq. (1) has a unique solution $y \in C^{m+1,\nu-1}(0,T]$.

Spline collocation methods for weakly singular VIDEs have been examined by many authors (see, for example, [1-7]). There are two different approaches to solving weakly singular VIDEs with piecewise polynomial spline collocation methods:

- 1. Consider Eq. (1) as an integral equation with respect to y'. The spline collocation method is applied to Eq. (1) for finding an approximate solution for y' first. Then, the approximation for y can be constructed by integration.
- 2. The second reformulation is based on integrating both sides of Eq. (1) over (0, t), which gives us a linear Volterra integral equation with respect to y. The spline collocation method is then applied to that equation for finding an approximate solution for y.

The most general theoretical results about the attainable orders of convergence of the proposed methods (Method 1 and Method 2) are proved in papers $[^{1,3}]$. A result about superconvergence in the maximum norm for Method 1 is proved in $[^4]$.

The purpose of this paper is to perform numerical experiments in order to verify the optimality of available theoretical results, to state a conjecture for the cases not covered by existing theorems, and to provide data for comparison with other methods.

For Method 2, we prove the superconvergence at the collocation points for sufficiently large values of r, which improves an analogous theorem of $[^1]$. For other values of r, we present a conjecture about the theoretical error estimate, based on our numerical results.

2. TEST PROBLEMS

For numerical verification of theoretical results we consider the following integro-differential equation

$$y'(t) = -y(t) + q_{\nu}(t) + \int_{0}^{t} K_{\nu}(t,s)y(s)ds, \quad 0 \le t \le 1,$$
(5)

where

$$K_{\nu}(t,s) = \begin{cases} -(t-s)^{-\nu} & \text{if } \nu \neq 0\\ -\log(t-s) & \text{if } \nu = 0 \end{cases}$$

and

$$q_{\nu}(t) = \begin{cases} (2-\nu)t^{1-\nu} + t^{2-\nu} + t^{3-2\nu}\gamma_1 & \text{if } \nu \neq 0, \\ \frac{1}{3}t^3(\log t)^2 - (\frac{13}{18}t^3 - t^2 - 2t)\log t + t + t^3\gamma_2 & \text{if } \nu = 0, \\ \gamma_1 = \int_0^1 (1-x)^{-\nu}x^{2-\nu}\mathrm{d}x, \ \gamma_2 = \int_0^1 x^2\log x\log(1-x)\mathrm{d}x, \end{cases}$$

with the initial condition y(0) = 0. The choice of the function q_{ν} corresponds to the exact solution $y(t) = t^{2-\nu}$ in the case $\nu \neq 0$ and $y(t) = t^2 \log t$ in the case $\nu = 0$. Equation (5) is an equation of type (1) with p(t) = -1, and for any $\nu \in (-\infty, 1)$ the assumptions (3) hold with arbitrary m.

3. GRID, SPLINE SPACE

For solving problem $\{(1), (2)\}$ we use piecewise polynomial collocation methods on graded grids. Fix $r \in \mathbb{R}$, $r \geq 1$. For $N \in \mathbb{N}$ define a graded grid Π_N^r on the interval [0, T] by

$$\Pi_{N}^{r} = \left\{ t_{0}, t_{1}, \dots, t_{N} : t_{j} = T\left(\frac{j}{N}\right)^{r}, \ j = 0, \dots, N \right\}.$$

Here r is a parameter describing the nonuniformity of the grid Π_N^r . If r = 1, we get the uniform grid and if r increases, the density of the gridpoints (near 0) also increases.

For given integers $m \ge 0$ and $-1 \le d \le m-1$, let $S_m^{(d)}(\Pi_N^r)$ be the spline space of piecewise polynomial functions on the grid Π_N^r :

$$S_m^{(d)}(\Pi_N^r) = \left\{ u : u \Big|_{[t_{j-1}, t_j]} =: u_j \in \pi_m, \quad j = 1, \dots, N; \\ u_j^{(k)}(t_j) = u_{j+1}^{(k)}(t_j), \qquad 0 \le k \le d, \ j = 1, \dots, N-1 \right\}.$$

Here π_m denotes the set of polynomials of degree not exceeding m and $u|_{[t_{j-1},t_j]}$ is the restriction of u to the subinterval $[t_{j-1},t_j]$. Note that the elements of $S_m^{(-1)}(\Pi_N^r) = \{u : u|_{[t_{j-1},t_j]} \in \pi_m, j = 1,\ldots,N\}$ may have jump discontinuities at the interior grid points t_1,\ldots,t_{N-1} .

In the following we consider spline collocation methods for two equivalent reformulations of problem $\{(1),(2)\}$. In both cases the collocation points

$$t_{jk} = t_{j-1} + \eta_k (t_j - t_{j-1}), \quad k = 1, \dots, m, \ j = 1, \dots, N,$$
 (6)

where η_1, \ldots, η_m do not depend on j and N and satisfy $0 \le \eta_1 < \ldots < \eta_m \le 1$, are used.

4. METHOD 1

The first reformulation is based on introducing a new unknown function z = y'. Using y' = z and (2), Eq. (1) may be rewritten as a linear Volterra integral equation of the second kind with respect to z:

$$z(t) = f_1(t) + p(t) \int_0^t z(s) ds + \int_0^t K(t,s) \left(\int_0^s z(\tau) d\tau \right) ds,$$
(7)

where

$$f_1(t) = q(t) + y_0 p(t) + y_0 \int_0^t K(t, s) \mathrm{d}s, \quad t \in [0, T].$$
(8)

We look for an approximation v to the solution z of Eq. (7) in $S_{m-1}^{(-1)}(\Pi_N^r)$, $m, N \in \mathbb{N}$. We determine $v = v^{(N)} \in S_{m-1}^{(-1)}(\Pi_N^r)$ $(m \ge 1)$ by the collocation method from the following conditions:

$$v_{j}(t_{jk}) = f_{1}(t_{jk}) + p(t_{jk}) \int_{0}^{t_{jk}} v(s) ds + \int_{0}^{t_{jk}} K(t_{jk}, s) \left(\int_{0}^{s} v(\tau) d\tau \right) ds, \qquad (9)$$

$$k = 1, \dots, m; \ j = 1, \dots, N.$$

Here $v_j = v|_{[t_{j-1}-t_j]}$ is the restriction of v to $[t_{j-1}, t_j]$, j = 1, ..., N, and the function f_1 and the set of collocation points $\{t_{jk}\}$ are given by (8) and (6), respectively. Having determined the approximation v for z = y', we can determine also the approximation u for y, the solution of the Cauchy problem $\{(1),(2)\}$, setting

$$u(t) = y_0 + \int_0^t v(s) \mathrm{d}s, \qquad t \in [0, T].$$
 (10)

Brunner et al. [³] have proved the following convergence result for method $\{(9),(10)\}$.

Theorem 1. Assume (3) and let the collocation points (6) be used. Then for all sufficiently large $N \in \mathbb{N}$ and for every choice of parameters $0 \leq \eta_1 < \ldots < \eta_m \leq 1$ with $\eta_1 > 0$ or $\eta_m < 1$, Eqs. (9) and (10) determine unique approximations $u \in S_m^{(0)}(\Pi_N^r)$ and $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ to the solution y of the Cauchy problem $\{(1), (2)\}$ and its derivative y', respectively.

The following error estimates hold for k = 0 and k = 1: 1) if $m < 2 - \nu - k$, then

$$||u^{(k)} - y^{(k)}||_{\infty} \le cN^{-m} \quad for \quad r \ge 1;$$

2) if $m = 2 - \nu - k$, then

$$||u^{(k)} - y^{(k)}||_{\infty} \le c \begin{cases} N^{-m}(1 + |\log N|) & for \quad r = 1, \\ N^{-m} & for \quad r > 1; \end{cases}$$

3) if $m > 2 - \nu - k$, then

$$||u^{(k)} - y^{(k)}||_{\infty} \le c \begin{cases} N^{-r(2-\nu-k)} & for \quad 1 \le r < \frac{m}{2-\nu-k}, \\ N^{-m}(1+|\log N|)^{1-k} & for \quad r = \frac{m}{2-\nu-k}, \\ N^{-m} & for \quad r > \frac{m}{2-\nu-k}. \end{cases}$$
(11)

Here c is a positive constant which is independent of N.

To illustrate the theoretical results, tables with numerical experiments in the case m = 2 for $\nu = -\frac{1}{4}$, 0, $\frac{1}{2}$, $\frac{9}{10}$ are presented. In order to estimate the errors $||u - y||_{\infty}$ and $||u' - y'||_{\infty}$, the points

$$\tau_{jk} = t_{j-1} + k \frac{t_j - t_{j-1}}{10}, \ k = 1, \dots, 9, \ j = 1, \dots, N$$

are used. The corresponding error estimates are denoted by

$$\varepsilon_N = \{\max |u(\tau_{jk}) - y(\tau_{jk})| : k = 1, \dots, 9; j = 1, \dots, N\}$$

and

$$\varepsilon'_N = \{ \max |u'(\tau_{jk}) - y'(\tau_{jk})| : k = 1, \dots, 9; j = 1, \dots, N \}.$$

The ratios of the actual errors $\rho_N = \frac{\varepsilon_{N/2}}{\varepsilon_N}$ and $\rho'_N = \frac{\varepsilon'_{N/2}}{\varepsilon'_N}$ are presented in the columns with headings in the form $\rho(x)$ and $\rho'(x)$, where x is a real number corresponding to the ratios of the error estimates. In order to save space, we have presented numerical results only for N = 4, 32, 256, 1024 although the computations were performed for all values $N = 2^j$, $j = 1, \ldots, 10$.

As we can see in Table 1, the observed errors of u' behave exactly according to the right-hand side of the estimate (11) starting from N = 32. We may conclude that in the case of the test equations, the estimate (11) corresponds to the leading term of the error $||u' - y'||_{\infty}$, which is dominating even for small values of N.

$\nu = -\frac{1}{4}$	r = 1	r = 1.2	r = 1.4	r = 1.6
4	$\varepsilon_N = \rho(4.0)$	$\varepsilon_N \rho(4.0)$	$\varepsilon_N \rho(4.0)$	$\varepsilon_N \rho(4.0)$
4	7.9E-4 4.1	6.8E-4 4.6	7.5E-4 4.4	8.6E-4 4.3
32	1.3E-5 3.9	8.9E-6 4.1	9.4E-6 4.2	1.1E-5 4.2
256	2.2E-7 3.9	1.4E-7 4.0	1.4E-7 4.0	1.6E-7 4.0
1024	1.4E-8 3.9	8.5E-9 4.0	8.8E-9 4.0	1.0E-8 4.0
N	$\varepsilon'_N \rho'(2.4)$	$\varepsilon'_N \rho'(2.8)$	$\varepsilon'_N \rho'(3.4)$	$\varepsilon'_N \rho'(4.0)$
4	1.3E-2 2.2	9.6E-3 2.6	6.9E-3 3.2	5.5E-3 3.5
32	1.0E-3 2.4	4.4E-4 2.8	1.9E-4 3.4	9.2E-5 4.0
256	7.9E-5 2.4	2.0E-5 2.8	4.9E-6 3.4	1.4E-6 4.0
1024	1.4E-5 2.4	2.5E-6 2.8	4.3E-7 3.4	9.0E-8 4.0
$\nu = 0$	r = 1	r = 1.333	r = 1.667	r = 2.0
N	$\varepsilon_N \qquad \varrho(4.0)$	$\varepsilon_N \varrho(4.0)$	$\varepsilon_N \varrho(4.0)$	$\varepsilon_N \varrho(4.0)$
4	4.7E-3 3.5	3.7E-3 4.0	3.9E-3 4.1	4.6E-3 4.0
32	1.1E-4 3.6	5.9E-5 4.0	5.9E-5 4.0	6.9E-5 4.0
256	2.2E-6 3.7	9.2E-7 4.0	9.2E-7 4.0	1.1E-6 4.0
1024	1.6E-7 3.7	5.8E-8 4.0	5.7E-8 4.0	6.6E-8 4.0
N	$\varepsilon'_N \qquad \varrho'(2.0)$	$\varepsilon'_N \qquad \varrho'(2.5)$	$\varepsilon'_N \qquad \varrho'(3.2)$	$\varepsilon'_N \qquad \varrho'(4.0)$
4	7.3E-2 1.9	4.7E-2 2.4	3.0E-2 3.0	2.6E-2 3.6
32	9.6E-3 2.0	3.1E-3 2.5	9.6E-4 3.2	4.2E-4 4.0
256	1.2E-3 2.0	1.9E-4 2.5	3.0E-5 3.2	6.6E-6 4.0
1024	3.0E-4 2.0	3.0E-5 2.5	3.0E-6 3.2	4.1E-7 4.0
$\nu = \frac{1}{2}$	r = 1	r = 1.2	r = 1.4	r = 4
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\varepsilon_N \varrho (2.8)$	$r = 1.2$ $\varepsilon_N \varrho(3.5)$	$r = 1.4$ $\varepsilon_N \varrho(4.0)$	$r = 4$ $\varepsilon_N \varrho(4.0)$
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\varepsilon_N \varrho \text{ (2.8)}$ 3.2E-3 2.6	$r = 1.2$ $\varepsilon_N \varrho(3.5)$ 2.2E-3 3.2	r = 1.4 $\varepsilon_N \varrho(4.0)$ 1.4E-3 3.9	$r = 4$ $\varepsilon_N \varrho(4.0)$ 3.0E-3 3.8
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{1}{4} $ $ 32 $	r = 1 $\varepsilon_N \varrho (2.8)$ 3.2E-3 2.6 1.5E-4 2.8	$r = 1.2 \\ \varepsilon_N \varrho(3.5) \\ 2.2E-3 3.2 \\ 6.0E-5 3.2 \\ \end{cases}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4\text{E-3} & 3.9 \\ \hline 3.0\text{E-5} & 3.6 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ 2.9E-5 & 4.6 \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ 256 $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ \hline 3.2E-3 & 2.6 \\ \hline 1.5E-4 & 2.8 \\ \hline 7.0E-6 & 2.7 \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2\text{E-3} & 3.2 \\ \hline 6.0\text{E-5} & 3.2 \\ \hline 1.7\text{E-6} & 3.3 \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4\text{E-3} & 3.9 \\ \hline 3.0\text{E-5} & 3.6 \\ \hline 5.9\text{E-7} & 3.7 \end{array}$	$r = 4$ $\varepsilon_N \varrho(4.0)$ 3.0E-3 3.8 2.9E-5 4.6 3.8E-7 4.1
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $	$r = 1$ $\varepsilon_{N} \varrho (2.8)$ 3.2E-3 2.6 1.5E-4 2.8 7.0E-6 2.7 9.1E-7 2.8	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2\text{E-3} & 3.2 \\ \hline 6.0\text{E-5} & 3.2 \\ \hline 1.7\text{E-6} & 3.3 \\ \hline 1.6\text{E-7} & 3.3 \\ \hline \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4\text{E-3} & 3.9 \\ \hline 3.0\text{E-5} & 3.6 \\ \hline 5.9\text{E-7} & 3.7 \\ \hline 4.2\text{E-8} & 3.8 \end{array}$	$r = 4$ $\varepsilon_N \varrho(4.0)$ 3.0E-3 3.8 2.9E-5 4.6 3.8E-7 4.1 2.3E-8 4.0
$\nu = \frac{1}{2}$ N 4 32 256 1024 N	$r = 1$ $\varepsilon_{N} \varrho (2.8)$ 3.2E-3 2.6 1.5E-4 2.8 7.0E-6 2.7 9.1E-7 2.8 $\varepsilon'_{N} \varrho'(1.4)$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon'_N & \varrho'(1.5) \end{array}$	$\begin{array}{c c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4\text{E-3} & 3.9 \\ \hline 3.0\text{E-5} & 3.6 \\ \hline 5.9\text{E-7} & 3.7 \\ \hline 4.2\text{E-8} & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \end{array}$	$r = 4$ $\varepsilon_{N} \varrho(4.0)$ 3.0E-3 3.8 2.9E-5 4.6 3.8E-7 4.1 2.3E-8 4.0 $\varepsilon'_{N} \varrho'(4.0)$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1024}{N} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ \hline \end{array}$	$\begin{array}{c c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon'_N & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 4.5E-2 & 1.4 \\ \hline \end{array}$	$\begin{array}{c c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4\text{E-3} & 3.9 \\ \hline 3.0\text{E-5} & 3.6 \\ \hline 5.9\text{E-7} & 3.7 \\ \hline 4.2\text{E-8} & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0\text{E-2} & 1.5 \\ \hline 0.0\text{E-1} & 1.5 \\ \hline 0.0\text{E-2} & 1.5 \\ \hline 0.0\text{E-2}$	$r = 4$ $\varepsilon_{N} \varrho(4.0)$ 3.0E-3 3.8 2.9E-5 4.6 3.8E-7 4.1 2.3E-8 4.0 $\varepsilon'_{N} \varrho'(4.0)$ 1.9E-2 2.6 $\varepsilon_{N} \varrho(4.0)$
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ (.5E-2) & 1.4 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ 2.2E-3 & 3.2 \\ 6.0E-5 & 3.2 \\ 1.7E-6 & 3.3 \\ 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ 4.5E-2 & 1.4 \\ 1.4E-2 & 1.5 \\ \hline 0.25 & 0.5 \\ \hline \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ 1.4E-3 & 3.9 \\ 3.0E-5 & 3.6 \\ 5.9E-7 & 3.7 \\ 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ 4.0E-2 & 1.5 \\ 9.8E-3 & 1.6 \\ \hline \end{array}$	$r = 4$ $\varepsilon_{N} \varrho(4.0)$ 3.0E-3 3.8 2.9E-5 4.6 3.8E-7 4.1 2.3E-8 4.0 $\varepsilon'_{N} \varrho'(4.0)$ 1.9E-2 2.6 3.3E-4 4.0 $\varepsilon'_{N} \varphi'(4.0)$
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 0.5E-2 & 1.4 \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ 2.2E-3 & 3.2 \\ 6.0E-5 & 3.2 \\ 1.7E-6 & 3.3 \\ 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ 1.4E-2 & 1.5 \\ 4.0E-3 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ 1.4E-3 & 3.9 \\ 3.0E-5 & 3.6 \\ 5.9E-7 & 3.7 \\ 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ 4.0E-2 & 1.5 \\ 9.8E-3 & 1.6 \\ 2.3E-3 & 1.6 \\ 2.3E-3 & 1.6 \end{array}$	$r = 4$ $\varepsilon_N \varrho(4.0)$ 3.0E-3 3.8 2.9E-5 4.6 3.8E-7 4.1 2.3E-8 4.0 $\varepsilon'_N \varrho'(4.0)$ 1.9E-2 2.6 3.3E-4 4.0 5.2E-6 4.0
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4\text{E-3} & 3.9 \\ \hline 3.0\text{E-5} & 3.6 \\ \hline 5.9\text{E-7} & 3.7 \\ \hline 4.2\text{E-8} & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0\text{E-2} & 1.5 \\ \hline 9.8\text{E-3} & 1.6 \\ \hline 2.3\text{E-3} & 1.6 \\ \hline 8.7\text{E-4} & 1.6 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \end{array}$
$\nu = \frac{1}{2}$ N 4 32 256 1024 N 4 32 256 1024 $\nu = \frac{9}{10}$	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ \hline 3.2E-3 & 2.6 \\ \hline 1.5E-4 & 2.8 \\ \hline 7.0E-6 & 2.7 \\ \hline 9.1E-7 & 2.8 \\ \hline \varepsilon'_N & \varrho' \ (1.4) \\ \hline 5.0E-2 & 1.3 \\ \hline 1.9E-2 & 1.4 \\ \hline 6.9E-3 & 1.4 \\ \hline 3.5E-3 & 1.4 \\ \hline r = 1 \end{array}$	$\begin{array}{c c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon'_N & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon'_N & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline r = 20 \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \end{array} $	$\begin{array}{c c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ \hline 3.2E-3 & 2.6 \\ \hline 1.5E-4 & 2.8 \\ \hline 7.0E-6 & 2.7 \\ \hline 9.1E-7 & 2.8 \\ \hline \varepsilon'_N & \varrho' \ (1.4) \\ \hline 5.0E-2 & 1.3 \\ \hline 1.9E-2 & 1.4 \\ \hline 6.9E-3 & 1.4 \\ \hline 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ \hline \end{array}$	$\begin{array}{c c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon'_N & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline \end{array}$	$\begin{array}{c c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon'_N & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline \end{array}$	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ 2.3E-8 & 4.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon'_N & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ 2.1E-3 & 1.7 \\ \hline 0 & 0 \\ \end{array}$	$\begin{array}{c c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon'_N & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline \end{array}$	$\begin{array}{c c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon'_N & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ 2.1E-3 & 1.7 \\ 3.1E-4 & 2.0 \\ 2.1E-5 & 1.7 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4Ec & 2.0 \\ \hline \end{array}$	$\begin{array}{c c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline 1.6E-5 & 3.6 \\ \hline 2.4E-7 & 2.5 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 0.4E-4 & 7 \\ \hline 0.4E-4 &$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \overline{N} \\ 4 \\ 32 \\ 256 \\ 1024 \\ \overline{N} \\ 4 \\ 32 \\ 256 \\ 1024 \\ \overline{N} \\ 4 \\ 32 \\ 256 \\ 1024 \\ \overline{N} \\ 4 \\ 32 \\ 256 \\ 1024 \\ \overline{N} \\ $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ \hline 3.2E-3 & 2.6 \\ \hline 1.5E-4 & 2.8 \\ \hline 7.0E-6 & 2.7 \\ \hline 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ \hline 5.0E-2 & 1.3 \\ \hline 1.9E-2 & 1.4 \\ \hline 6.9E-3 & 1.4 \\ \hline 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ \hline 2.1E-3 & 1.7 \\ \hline 3.1E-4 & 2.0 \\ \hline 3.3E-5 & 2.1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline \end{array}$	$\begin{array}{c c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline 1.6E-5 & 3.6 \\ \hline 3.4E-7 & 3.6 \\ \hline 2.5 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 9.3E-7 & 5.3 \\ \hline 4.6 \\ \hline r = 20 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ r = 20 \\ \hline 0 \\ \hline 0 \\ r = 20 \\ r = 20 \\ \hline 0 \\ r = 20 $
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 32 $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ \hline 3.2E-3 & 2.6 \\ \hline 1.5E-4 & 2.8 \\ \hline 7.0E-6 & 2.7 \\ \hline 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ \hline 5.0E-2 & 1.3 \\ \hline 1.9E-2 & 1.4 \\ \hline 6.9E-3 & 1.4 \\ \hline 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ \hline 2.1E-3 & 1.7 \\ \hline 3.1E-4 & 2.0 \\ \hline 3.3E-5 & 2.1 \\ \hline 7.1E-6 & 2.1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.8E-7 & 2.9 \\ \hline \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline 1.6E-5 & 3.6 \\ \hline 3.4E-7 & 3.6 \\ \hline 2.5E-8 & 3.7 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 9.3E-7 & 5.3 \\ \hline 4.3E-8 & 4.4 \\ \hline r = 0 \\ \hline r$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ N \\ \overline{N} \\ N \\ \overline{N} \\ \overline{N} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ 2.1E-3 & 1.7 \\ 3.1E-4 & 2.0 \\ 3.3E-5 & 2.1 \\ 7.1E-6 & 2.1 \\ \hline \varepsilon_N' & \varrho' \ (1.1) \\ $	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.4E-7 & 2.9 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline \varepsilon_N' & \varrho'(1.$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline 1.6E-5 & 3.6 \\ \hline 3.4E-7 & 3.6 \\ \hline 2.5E-8 & 3.7 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline \varepsilon_N' & \varrho''(1.1) \\ \hline \varepsilon_N' & \varrho'''(1.1) \\ \hline \varepsilon_N' & \varrho'''(1.1) \\ \hline \varepsilon_N' & \varrho''' \\ \hline \varepsilon_N' & \varrho'''' \\ \hline \varepsilon_N' & \varrho''''' \\ \hline \varepsilon_N' & \varrho'''''' \\ \hline \varepsilon_N' & \varrho'''''' \\ \hline \varepsilon_N' & \varrho''''''''' \\ \hline \varepsilon_N' & \varrho''''''''''''''''''''''''''''''''''$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 9.3E-7 & 5.3 \\ \hline 4.3E-8 & 4.4 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 32 $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ 2.1E-3 & 1.7 \\ 3.1E-4 & 2.0 \\ 3.3E-5 & 2.1 \\ 7.1E-6 & 2.1 \\ \hline \varepsilon_N' & \varrho' \ (1.1) \\ 2.3E-2 & 1.5 \\ 2.2E-2 & 1.5 \\ 2.2E-2 & 1.6 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.4E-7 & 2.9 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline 2.9E-2 & 0.9 \\ \hline 2.0E-2 & 0.9 \\ \hline 2.0$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ 1.4E-3 & 3.9 \\ 3.0E-5 & 3.6 \\ 5.9E-7 & 3.7 \\ 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ 4.0E-2 & 1.5 \\ 9.8E-3 & 1.6 \\ 2.3E-3 & 1.6 \\ 2.3E-3 & 1.6 \\ 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ 1.6E-5 & 3.6 \\ 3.4E-7 & 3.6 \\ 2.5E-8 & 3.7 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline 3.3E-2 & 0.7 \\ 2.6E-2 & 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 9.3E-7 & 5.3 \\ \hline 4.3E-8 & 4.4 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 5.7E-2 & 1.0 \\ \hline 2.0E-2 & 1.1 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ 2.1E-3 & 1.7 \\ 3.1E-4 & 2.0 \\ 3.3E-5 & 2.1 \\ 7.1E-6 & 2.1 \\ \hline \varepsilon_N' & \varrho' (1.1) \\ 2.3E-2 & 1.5 \\ 3.3E-2 & 1.0 \\ 2.9E-2 & 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 2.8E-7 & 2.9 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline 2.9E-2 & 0.9 \\ \hline 3.0E-2 & 1.1 \\ \hline 2.2E-2 & 0.1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline 1.6E-5 & 3.6 \\ \hline 3.4E-7 & 3.6 \\ \hline 2.5E-8 & 3.7 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline 3.3E-2 & 0.7 \\ \hline 2.6E-2 & 1.1 \\ \hline 1.8E-2 & 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 9.3E-7 & 5.3 \\ \hline 4.3E-8 & 4.4 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 5.7E-2 & 1.0 \\ \hline 3.0E-3 & 4.1 \\ \hline 2.5E-5 & 4.0 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho \ (2.8) \\ 3.2E-3 & 2.6 \\ 1.5E-4 & 2.8 \\ 7.0E-6 & 2.7 \\ 9.1E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho' \ (1.4) \\ 5.0E-2 & 1.3 \\ 1.9E-2 & 1.4 \\ 6.9E-3 & 1.4 \\ 3.5E-3 & 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ 2.1E-3 & 1.7 \\ 3.1E-4 & 2.0 \\ 3.3E-5 & 2.1 \\ 7.1E-6 & 2.1 \\ \hline \varepsilon_N' & \varrho' \ (1.1) \\ 2.3E-2 & 1.5 \\ 3.3E-2 & 1.0 \\ 2.9E-2 & 1.1 \\ \hline \varepsilon_N & 2 & 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.2 \\ \hline \varepsilon_N & \varrho(3.5) \\ \hline 2.2E-3 & 3.2 \\ \hline 6.0E-5 & 3.2 \\ \hline 1.7E-6 & 3.3 \\ \hline 1.6E-7 & 3.3 \\ \hline \varepsilon_N' & \varrho'(1.5) \\ \hline 4.5E-2 & 1.4 \\ \hline 1.4E-2 & 1.5 \\ \hline 4.0E-3 & 1.5 \\ \hline 1.7E-3 & 1.5 \\ \hline r = 1.455 \\ \hline \varepsilon_N & \varrho(3.0) \\ \hline 1.3E-3 & 2.2 \\ \hline 5.7E-5 & 2.9 \\ \hline 2.4E-6 & 2.9 \\ \hline 1.3E-2 & 0.9 \\ \hline 3.0E-2 & 1.1 \\ \hline 1.3E-2 & 1.1 \\ \hline 1.5E-2 & 1.1 \\ \hline 1.5E-$	$\begin{array}{c} r = 1.4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 1.4E-3 & 3.9 \\ \hline 3.0E-5 & 3.6 \\ \hline 5.9E-7 & 3.7 \\ \hline 4.2E-8 & 3.8 \\ \hline \varepsilon_N' & \varrho'(1.6) \\ \hline 4.0E-2 & 1.5 \\ \hline 9.8E-3 & 1.6 \\ \hline 2.3E-3 & 1.6 \\ \hline 8.7E-4 & 1.6 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 7.1E-4 & 3.1 \\ \hline 1.6E-5 & 3.6 \\ \hline 3.4E-7 & 3.6 \\ \hline 2.5E-8 & 3.7 \\ \hline \varepsilon_N' & \varrho'(1.1) \\ \hline 3.3E-2 & 0.7 \\ \hline 2.6E-2 & 1.1 \\ \hline 1.8E-2 & 1.1 \\ \hline 1.8E-2 & 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 3.0E-3 & 3.8 \\ \hline 2.9E-5 & 4.6 \\ \hline 3.8E-7 & 4.1 \\ \hline 2.3E-8 & 4.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.9E-2 & 2.6 \\ \hline 3.3E-4 & 4.0 \\ \hline 5.2E-6 & 4.0 \\ \hline 3.2E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(4.0) \\ \hline 5.7E-3 & 1.1 \\ \hline 1.4E-4 & 4.7 \\ \hline 9.3E-7 & 5.3 \\ \hline 4.3E-8 & 4.4 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 5.7E-2 & 1.0 \\ \hline 3.0E-3 & 4.1 \\ \hline 3.5E-5 & 4.0 \\ \hline 2.2E-6 & 4.0 \\ \hline \end{array}$

Table 1. Method 1; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4}$

The observed errors of u are also in good agreement with the theoretical estimates of Theorem 1 except in the cases where r is close to the value $\frac{m}{2-\nu}$, after which the maximal theoretical convergence rate is achieved. If r is close to the

critical value, then the observed convergence rate is smaller than the one predicted by the error estimate (11) but converges slowly to the theoretical value. To get a better picture of what happens near this value of r (r = 1.333) in the case $\nu = \frac{1}{2}$, Table 2 is presented. This table shows the dependence of the convergence rate on the nonuniformity parameter r, when r is increasing by steps of 0.1.

In the proof of Theorem 1 (see $[^3]$) it is actually shown that

$$||u - y||_{\infty} \le c'' N^{-r(2-\nu)} \sum_{l=1}^{N} l^{r(2-\nu)-m-1},$$
(12)

which is asymptotically equivalent to (11). In Table 3 we see that the ratios of the right-hand side of (12) behave similarly to the observed convergence rate, which explains the slow convergence of the observed rate to the theoretical one.

By a careful choice of the collocation parameters η_j it is possible (assuming a little more regularity of functions p, q, and K) to improve the convergence rate.

	<i>r</i> =	= 1	r = 1.1		r = 1.2		r =	1.3
N	ε_N	$\varrho(2.8)$	ε_N	$\varrho(3.1)$	ε_N	$\varrho(3.5)$	ε_N	$\varrho(3.9)$
4	3.2E-3	2.6	2.6E-3	2.9	2.2E-3	3.2	1.8E-3	3.5
32	1.5E-4	2.8	9.1E-5	3.1	6.0E-5	3.2	4.1E-5	3.4
256	7.0E-6	2.7	3.4E-6	3.0	1.7E-6	3.3	9.7E-7	3.5
1024	9.1E-7	2.8	3.7E-7	3.0	1.6E-7	3.3	7.7E-8	3.6
	r =	1.4	r =	1.5	r =	1.6	r =	1.7
Ν	$r = \varepsilon_N$	1.4 <i>ϱ</i> (4.0)	$r = \varepsilon_N$	1.5 <i>ϱ</i> (4.0)	$r = \varepsilon_N$	1.6 <i>ϱ</i> (4.0)	$r = \varepsilon_N$	1.7 <i>ρ</i> (4.0)
N 4	$r = \frac{\varepsilon_N}{1.4\text{E-3}}$	1.4 <u>ρ</u> (4.0) 3.9	$r = \frac{\varepsilon_N}{1.3\text{E-3}}$	1.5 <i>ρ</i> (4.0) 3.9	$r = \frac{\varepsilon_N}{1.2\text{E-3}}$	1.6 <u>ρ</u> (4.0) 3.9	$r = \frac{\varepsilon_N}{1.1\text{E-3}}$	1.7 <i>ρ</i> (4.0) 3.9
N 4 32	$r = \frac{\varepsilon_N}{1.4\text{E-3}}$ 3.0E-5	$ \begin{array}{r} 1.4 \\ \underline{\varrho}(4.0) \\ 3.9 \\ 3.6 \end{array} $	$r = \frac{\varepsilon_N}{1.3\text{E-3}}$ 2.3E-5	$ \begin{array}{r} 1.5 \\ \varrho(4.0) \\ 3.9 \\ 3.8 \\ \end{array} $	$r = \frac{\varepsilon_N}{1.2\text{E-3}}$ 1.8E-5	$ \begin{array}{r} 1.6 \\ \underline{\varrho}(4.0) \\ 3.9 \\ 3.9 \\ 3.9 \end{array} $	$r = \frac{\varepsilon_N}{1.1\text{E-3}}$ 1.6E-5	1.7 <u>ρ</u> (4.0) 3.9 4.0
N 4 32 256	$r = \frac{\varepsilon_N}{1.4\text{E-3}}$ 3.0E-5 5.9E-7	$ \begin{array}{r} 1.4 \\ \varrho(4.0) \\ 3.9 \\ 3.6 \\ 3.7 \\ \end{array} $	$r = \frac{\varepsilon_N}{1.3\text{E-3}}$ 2.3E-5 4.0E-7	$ \begin{array}{c} 1.5 \\ \varrho(4.0) \\ 3.9 \\ 3.8 \\ 3.9 \end{array} $	$r = \frac{\varepsilon_N}{1.2\text{E-3}}$ 1.8E-5 3.0E-7	$ \begin{array}{r} 1.6 \\ \varrho(4.0) \\ 3.9 \\ 3.9 \\ 3.9 \\ 3.9 \\ 3.9 \end{array} $	$r = \frac{\varepsilon_N}{1.1\text{E-3}}$ 1.6E-5 2.5E-7	$ \begin{array}{r} 1.7 \\ \underline{\rho}(4.0) \\ 3.9 \\ 4.0 \\ 4.0 \\ \end{array} $

Table 2. Method 1; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4}$, and $\nu = \frac{1}{2}$

N	r = 1.1	r = 1.2	r = 1.3	r = 1.4	r = 1.5	r = 1.6
4	2.5	2.6	2.8	3.0	3.2	3.4
8	2.6	2.8	3.0	3.2	3.3	3.5
16	2.8	3.0	3.2	3.3	3.5	3.6
32	2.9	3.1	3.3	3.4	3.6	3.7
64	2.9	3.2	3.4	3.5	3.7	3.8
128	3.0	3.2	3.4	3.6	3.7	3.8
256	3.0	3.3	3.5	3.7	3.8	3.9
512	3.1	3.3	3.5	3.7	3.8	3.9
1024	3.1	3.3	3.6	3.7	3.9	3.9

Table 3. The ratios of the right-hand side of (12) for $\nu = \frac{1}{2}$

$\nu = -\frac{1}{4}$	r = 1	r = 1.189	r = 1.378	r = 1.6
$\frac{4}{N}$	$\varepsilon_N \varrho(4.8)$	$\varepsilon_N \varrho(6.4)$	$\varepsilon_N \varrho(8.0)$	$\varepsilon_N \varrho(8.0)$
4	7.8E-4 4.5	4.4E-4 6.0	3.4E-4 7.1	4.2E-4 7.5
32	7.7E-6 4.7	1.8E-6 6.3	7.3E-7 7.8	8.4E-7 8.0
256	7.2E-8 4.8	6.8E-9 6.4	1.5E-9 7.8	1.6E-9 8.0
1024	3.2E-9 4.8	1.7E-10 6.4	2.5E-11 7.9	2.6E-11 8.0
N	$\varepsilon'_N \varrho'(2.4)$	$\varepsilon'_N \varrho'(2.8)$	$\varepsilon'_N \qquad \varrho'(3.3)$	$\varepsilon'_N \varrho'(4.0)$
4	1.4E-2 2.2	1.1E-2 2.6	7.7E-3 3.1	6.1E-3 3.5
32	1.1E-3 2.4	5.0E-4 2.8	2.2E-4 3.3	1.0E-4 4.0
256	8.5E-5 2.4	2.3E-5 2.8	6.2E-6 3.3	1.6E-6 4.0
1024	1.5E-5 2.4	2.9E-6 2.8	5.7E-7 3.3	1.0E-7 4.0
$\nu = 0$	r = 1	r = 1.275	r = 1.550	r = 2
N	$\varepsilon_N \varrho(4.0)$	$\varepsilon_N \varrho(5.9)$	$\varepsilon_N \varrho(8.0)$	$\varepsilon_N \varrho(8.0)$
4	4.4E-3 3.8	2.1E-3 5.6	1.7E-3 7.3	2.7E-3 6.6
32	7.2E-5 4.0	1.1E-5 5.8	3.7E-6 7.7	5.8E-6 7.9
256	1.1E-6 4.0	5.4E-8 5.9	8.0E-9 7.8	1.1E-8 8.0
1024	7.1E-8 4.0	1.6E-9 5.9	1.3E-10 7.8	1.8E-10 8.0
N	$\varepsilon'_N \varrho'(2.0)$	$\varepsilon'_N \varrho'(2.4)$	$\varepsilon'_N \qquad \varrho'(2.9)$	$\varepsilon'_N \qquad \varrho'(4.0)$
4	7.8E-2 1.9	5.4E-2 2.3	3.8E-2 2.8	2.8E-2 3.7
32	1.0E-2 2.0	4.0E-3 2.4	1.5E-3 2.9	4.6E-4 4.0
256	1.3E-3 2.0	2.8E-4 2.4	6.2E-5 2.9	7.2E-6 4.0
1024	3.2E-4 2.0	4.8E-5 2.4	7.2E-6 2.9	4.5E-7 4.0
$\nu = \frac{1}{2}$	r = 1	r = 1.533	r = 2.067	r = 4
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\varepsilon_N \varrho(2.8)$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$	$r = 4$ $\varepsilon_N \varrho(8.0)$
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = 1$ $\varepsilon_N \varrho(2.8)$ 3.3E-3 2.6	r = 1.533 $\varepsilon_N \varrho(4.9)$ 1.1E-3 4.5	r = 2.067 $\varepsilon_N \varrho(8.0)$ 7.7E-4 6.0	$r = 4$ $\varepsilon_N \varrho(8.0)$ 2.6E-3 4.5
$ \frac{\nu = \frac{1}{2}}{N} $ 4 32	r = 1 $\varepsilon_N \varrho(2.8)$ 3.3E-3 2.6 1.6E-4 2.8	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ \hline 1.1\text{E-3} 4.5 \\ 9.9\text{E-6} 4.9 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7\text{E-4} & 6.0 \\ 2.0\text{E-6} & 7.5 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6\text{E-3} & 4.5 \\ \hline 6.7\text{E-6} & 7.8 \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{1}{4} $ $ \frac{32}{256} $	$r = 1$ $\varepsilon_N \varrho(2.8)$ 3.3E-3 2.6 1.6E-4 2.8 7.0E-6 2.8	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ \hline 1.1\text{E-3} 4.5 \\ 9.9\text{E-6} 4.9 \\ 8.4\text{E-8} 4.9 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7E-4 & 6.0 \\ 2.0E-6 & 7.5 \\ 4.6E-9 & 7.6 \end{array}$	$r = 4$ $\varepsilon_N \varrho(8.0)$ 2.6E-3 4.5 6.7E-6 7.8 1.3E-8 8.0
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $	$r = 1$ $\varepsilon_N \varrho(2.8)$ 3.3E-3 2.6 1.6E-4 2.8 7.0E-6 2.8 8.8E-7 2.8	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1\text{E-3} 4.5 \\ 9.9\text{E-6} 4.9 \\ 8.4\text{E-8} 4.9 \\ 3.4\text{E-9} 4.9 \end{array}$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 7.7E-4 6.0 2.0E-6 7.5 4.6E-9 7.6 7.9E-11 7.7	$r = 4$ $\varepsilon_N \varrho(8.0)$ 2.6E-3 4.5 6.7E-6 7.8 1.3E-8 8.0 2.1E-10 8.1
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ N $	$r = 1$ $\varepsilon_{N} \varrho(2.8)$ 3.3E-3 2.6 1.6E-4 2.8 7.0E-6 2.8 8.8E-7 2.8 $\varepsilon'_{N} \varrho'(1.4)$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7E-4 & 6.0 \\ 2.0E-6 & 7.5 \\ 4.6E-9 & 7.6 \\ \hline 7.9E-11 & 7.7 \\ \hline \varepsilon'_N & \varrho'(2.0) \end{array}$	$r = 4$ $\frac{\varepsilon_N}{2.6E-3} \frac{\varrho(8.0)}{4.5}$ 6.7E-6 7.8 1.3E-8 8.0 2.1E-10 8.1 $\frac{\varepsilon'_N}{2} \frac{\varrho'(4.0)}{\varrho'(4.0)}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{N}{4} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 3.3E-3 & 2.6 \\ \hline 1.6E-4 & 2.8 \\ \hline 7.0E-6 & 2.8 \\ \hline 8.8E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho'(1.4) \\ \hline 5.3E-2 & 1.3 \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7E-4 & 6.0 \\ 2.0E-6 & 7.5 \\ 4.6E-9 & 7.6 \\ \hline 7.9E-11 & 7.7 \\ \hline \varepsilon'_N & \varrho'(2.0) \\ \hline 2.7E-2 & 1.9 \end{array}$	$r = 4$ $\frac{\varepsilon_N}{2.6E-3} \frac{\varrho(8.0)}{4.5}$ 6.7E-6 7.8 1.3E-8 8.0 2.1E-10 8.1 $\frac{\varepsilon'_N}{2.0E-2} \frac{\varrho'(4.0)}{2.5}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{N}{4} $ $ \frac{32}{32} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 3.3E-3 & 2.6 \\ \hline 1.6E-4 & 2.8 \\ \hline 7.0E-6 & 2.8 \\ \hline 8.8E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho'(1.4) \\ \hline 5.3E-2 & 1.3 \\ \hline 2.0E-2 & 1.4 \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N & \varrho(4.9) \\ 1.1E-3 & 4.5 \\ 9.9E-6 & 4.9 \\ 8.4E-8 & 4.9 \\ 3.4E-9 & 4.9 \\ \hline \varepsilon_N' & \varrho'(1.7) \\ 3.9E-2 & 1.5 \\ 8.2E-3 & 1.7 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7E-4 & 6.0 \\ 2.0E-6 & 7.5 \\ 4.6E-9 & 7.6 \\ \hline 7.9E-11 & 7.7 \\ \hline \varepsilon_N' & \varrho'(2.0) \\ \hline 2.7E-2 & 1.9 \\ \hline 3.2E-3 & 2.0 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ 2.6E-3 & 4.5 \\ 6.7E-6 & 7.8 \\ 1.3E-8 & 8.0 \\ 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ 2.0E-2 & 2.5 \\ 3.6E-4 & 4.0 \end{array}$
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N & \varrho(4.9) \\ 1.1E-3 & 4.5 \\ 9.9E-6 & 4.9 \\ 8.4E-8 & 4.9 \\ 3.4E-9 & 4.9 \\ \hline \varepsilon_N' & \varrho'(1.7) \\ 3.9E-2 & 1.5 \\ 8.2E-3 & 1.7 \\ 1.7E-3 & 1.7 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ 3.8E-4 2.0 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \\ 3.6E-3 1.4 \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ 3.8E-4 2.0 \\ 9.0E-5 2.0 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_N \varrho(2.8)\\ 3.3E-3 2.6\\ 1.6E-4 2.8\\ 7.0E-6 2.8\\ 8.8E-7 2.8\\ \hline \varepsilon_N' \varrho'(1.4)\\ 5.3E-2 1.3\\ 2.0E-2 1.4\\ 7.3E-3 1.4\\ 3.6E-3 1.4\\ \hline r=1 \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ \hline 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ \hline 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ \hline 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ 3.8E-4 2.0 \\ 9.0E-5 2.0 \\ \hline r = 2.818 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline r = 20 \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ 256 $ $ 1024 $ $ \frac{N}{4} $ $ \frac{4}{32} $ $ 256 $ $ 1024 $ $ \frac{\nu = \frac{9}{10}}{N} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 3.3E-3 & 2.6 \\ \hline 1.6E-4 & 2.8 \\ \hline 7.0E-6 & 2.8 \\ \hline 8.8E-7 & 2.8 \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 5.3E-2 & 1.3 \\ \hline 2.0E-2 & 1.4 \\ \hline 7.3E-3 & 1.4 \\ \hline 3.6E-3 & 1.4 \\ \hline \hline \varepsilon_N & \varrho(2.1) \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N & \varrho(4.9) \\ \hline 1.1E-3 & 4.5 \\ 9.9E-6 & 4.9 \\ 8.4E-8 & 4.9 \\ 3.4E-9 & 4.9 \\ \hline \varepsilon'_N & \varrho'(1.7) \\ \hline 3.9E-2 & 1.5 \\ 8.2E-3 & 1.7 \\ \hline 1.7E-3 & 1.7 \\ 5.7E-4 & 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N & \varrho(4.3) \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7E-4 & 6.0 \\ 2.0E-6 & 7.5 \\ 4.6E-9 & 7.6 \\ \hline 7.9E-11 & 7.7 \\ \hline \varepsilon'_N & \varrho'(2.0) \\ \hline 2.7E-2 & 1.9 \\ 3.2E-3 & 2.0 \\ \hline 3.8E-4 & 2.0 \\ 9.0E-5 & 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N & \varrho(8.0) \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 3.3E-3 & 2.6 \\ \hline 1.6E-4 & 2.8 \\ \hline 7.0E-6 & 2.8 \\ \hline 8.8E-7 & 2.8 \\ \hline \varepsilon_N' & \varrho'(1.4) \\ \hline 5.3E-2 & 1.3 \\ \hline 2.0E-2 & 1.4 \\ \hline 7.3E-3 & 1.4 \\ \hline 3.6E-3 & 1.4 \\ \hline \hline \varepsilon_N & \varrho(2.1) \\ \hline 2.2E-3 & 1.7 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ 7.3E-4 3.1 \end{array}$	$\begin{array}{c c} r = 2.067 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 7.7\text{E-4} & 6.0 \\ 2.0\text{E-6} & 7.5 \\ 4.6\text{E-9} & 7.6 \\ \hline 7.9\text{E-11} & 7.7 \\ \hline \varepsilon_N' & \varrho'(2.0) \\ \hline 2.7\text{E-2} & 1.9 \\ 3.2\text{E-3} & 2.0 \\ \hline 3.8\text{E-4} & 2.0 \\ 9.0\text{E-5} & 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 4.5\text{E-4} & 4.7 \\ \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 3.3E-3 & 2.6 \\ \hline 1.6E-4 & 2.8 \\ \hline 7.0E-6 & 2.8 \\ \hline 8.8E-7 & 2.8 \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 5.3E-2 & 1.3 \\ \hline 2.0E-2 & 1.4 \\ \hline 7.3E-3 & 1.4 \\ \hline 3.6E-3 & 1.4 \\ \hline \hline \varepsilon_N & \varrho(2.1) \\ \hline 2.2E-3 & 1.7 \\ \hline 3.2E-4 & 2.0 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ 7.3E-4 3.1 \\ 1.2E-5 4.1 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7\text{E-4} 6.0 \\ 2.0\text{E-6} 7.5 \\ 4.6\text{E-9} 7.6 \\ \hline 7.9\text{E-11} 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7\text{E-2} 1.9 \\ \hline 3.2\text{E-3} 2.0 \\ \hline 3.8\text{E-4} 2.0 \\ \hline 9.0\text{E-5} 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5\text{E-4} 4.7 \\ \hline 1.7\text{E-6} 7.1 \\ \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \hline 1.6E-4 & 5.7 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 3.3E-3 & 2.6 \\ \hline 1.6E-4 & 2.8 \\ \hline 7.0E-6 & 2.8 \\ \hline 8.8E-7 & 2.8 \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 5.3E-2 & 1.3 \\ \hline 2.0E-2 & 1.4 \\ \hline 7.3E-3 & 1.4 \\ \hline 3.6E-3 & 1.4 \\ \hline \hline \varepsilon_N & \varrho(2.1) \\ \hline 2.2E-3 & 1.7 \\ \hline 3.2E-4 & 2.0 \\ \hline 3.3E-5 & 2.1 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ 7.3E-4 3.1 \\ 1.2E-5 4.1 \\ 1.5E-7 4.3 \\ \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ \hline 3.8E-4 2.0 \\ 9.0E-5 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5E-4 4.7 \\ 1.7E-6 7.1 \\ \hline 4.1E-9 7.5 \\ \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \hline 1.6E-4 & 5.7 \\ \hline 3.9E-7 & 7.9 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \\ 3.6E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ 2.2E-3 1.7 \\ 3.2E-4 2.0 \\ 3.3E-5 2.1 \\ 7.3E-6 2.1 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ 7.3E-4 3.1 \\ 1.2E-5 4.1 \\ 1.5E-7 4.3 \\ 8.1E-9 4.3 \\ \hline \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ \hline 3.8E-4 2.0 \\ 9.0E-5 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5E-4 4.7 \\ 1.7E-6 7.1 \\ \hline 4.1E-9 7.5 \\ \hline 7.2E-11 7.6 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \hline 1.6E-4 & 5.7 \\ \hline 3.9E-7 & 7.9 \\ \hline 6.0E-9 & 8.0 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \frac{\nu = \frac{9}{10}}{N} \\ \hline 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 7 \\ 7 \\ 7 \\ $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \\ 3.6E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ 2.2E-3 1.7 \\ 3.2E-4 2.0 \\ 3.3E-5 2.1 \\ 7.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ 7.3E-4 3.1 \\ 1.2E-5 4.1 \\ 1.5E-7 4.3 \\ 8.1E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline \varepsilon_N' \varphi'(1.1) \\ \hline \varepsilon_N' \varphi'(1$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ \hline 3.8E-4 2.0 \\ 9.0E-5 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5E-4 4.7 \\ 1.7E-6 7.1 \\ \hline 4.1E-9 7.5 \\ \hline 7.2E-11 7.6 \\ \hline \varepsilon_N' \varrho'(1.2) \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \hline 1.6E-4 & 5.7 \\ \hline 3.9E-7 & 7.9 \\ \hline 6.0E-9 & 8.0 \\ \hline \varepsilon_N & \varrho'(4.0) \\ \hline \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{\nu = \frac{9}{10}}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ \frac{1}{32} \\ 256 \\ 1024 \\ \frac{1}{N} \\ \frac{1}{32} \\ \frac$	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \\ 3.6E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ 2.2E-3 1.7 \\ 3.2E-4 2.0 \\ 3.3E-5 2.1 \\ 7.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 2.5E-2 1.2 \\ \hline z = 1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ 7.3E-4 3.1 \\ 1.2E-5 4.1 \\ 1.5E-7 4.3 \\ 8.1E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 3.4E-2 0.7 \\ \hline \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ \hline 3.8E-4 2.0 \\ 9.0E-5 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5E-4 4.7 \\ 1.7E-6 7.1 \\ \hline 4.1E-9 7.5 \\ \hline 7.2E-11 7.6 \\ \hline \varepsilon_N' \varrho'(1.2) \\ \hline 3.4E-2 0.9 \\ \hline \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \hline 1.6E-4 & 5.7 \\ \hline 3.9E-7 & 7.9 \\ \hline 6.0E-9 & 8.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 4.6E-2 & 1.0 \\ \hline \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{\nu = \frac{9}{10}}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256 \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256 \\ 1024 \\ \frac{1}{N} \\ \frac{4}{32} \\ 256 \\ 1024 \\ \frac{1}{N} \\ $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \\ 3.6E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ 2.2E-3 1.7 \\ 3.2E-4 2.0 \\ 3.3E-5 2.1 \\ 7.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 2.5E-2 1.2 \\ 3.5E-2 1.0 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ \hline 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ \hline 7.3E-4 3.1 \\ 1.2E-5 4.1 \\ 1.5E-7 4.3 \\ 8.1E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline 3.4E-2 0.7 \\ 2.7E-2 1.1 \\ \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7E-2 1.9 \\ 3.2E-3 2.0 \\ \hline 3.8E-4 2.0 \\ 9.0E-5 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5E-4 4.7 \\ 1.7E-6 7.1 \\ \hline 4.1E-9 7.5 \\ \hline 7.2E-11 7.6 \\ \hline \varepsilon_N' \varrho'(1.2) \\ \hline 3.4E-2 0.9 \\ \hline 2.0E-2 1.2 \\ \end{array}$	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 2.6E-3 & 4.5 \\ \hline 6.7E-6 & 7.8 \\ \hline 1.3E-8 & 8.0 \\ \hline 2.1E-10 & 8.1 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 2.0E-2 & 2.5 \\ \hline 3.6E-4 & 4.0 \\ \hline 5.6E-6 & 4.0 \\ \hline 3.5E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 5.9E-3 & 1.1 \\ \hline 1.6E-4 & 5.7 \\ \hline 3.9E-7 & 7.9 \\ \hline 6.0E-9 & 8.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 4.6E-2 & 1.0 \\ \hline 2.4E-3 & 3.9 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ 3.3E-3 2.6 \\ 1.6E-4 2.8 \\ 7.0E-6 2.8 \\ 8.8E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ 5.3E-2 1.3 \\ 2.0E-2 1.4 \\ 7.3E-3 1.4 \\ 3.6E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ 2.2E-3 1.7 \\ 3.2E-4 2.0 \\ 3.3E-5 2.1 \\ 7.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 2.5E-2 1.2 \\ 3.5E-2 1.0 \\ 3.0E-2 1.1 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 1.1E-3 4.5 \\ 9.9E-6 4.9 \\ 8.4E-8 4.9 \\ 3.4E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ \hline 3.9E-2 1.5 \\ 8.2E-3 1.7 \\ 1.7E-3 1.7 \\ 5.7E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ \hline 7.3E-4 3.1 \\ 1.2E-5 4.1 \\ 1.5E-7 4.3 \\ 8.1E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline 3.4E-2 0.7 \\ 2.7E-2 1.1 \\ 1.8E-2 1.1 \\ \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 7.7E-4 6.0 \\ 2.0E-6 7.5 \\ \hline 4.6E-9 7.6 \\ \hline 7.9E-11 7.7 \\ \hline \varepsilon_N' \varrho'(2.0) \\ \hline 2.7E-2 1.9 \\ \hline 3.2E-3 2.0 \\ \hline 3.8E-4 2.0 \\ \hline 9.0E-5 2.0 \\ \hline r = 2.818 \\ \hline \varepsilon_N \varrho(8.0) \\ \hline 4.5E-4 4.7 \\ \hline 1.7E-6 7.1 \\ \hline 4.1E-9 7.5 \\ \hline 7.2E-11 7.6 \\ \hline \varepsilon_N' \varrho'(1.2) \\ \hline 3.4E-2 0.9 \\ \hline 2.0E-2 1.2 \\ \hline 1.1E-2 1.2 \\ \hline \end{array}$	$\begin{array}{c} r=4\\ \hline \varepsilon_N & \varrho(8.0)\\ \hline 2.6E-3 & 4.5\\ \hline 6.7E-6 & 7.8\\ \hline 1.3E-8 & 8.0\\ \hline 2.1E-10 & 8.1\\ \hline \varepsilon_N' & \varrho'(4.0)\\ \hline 2.0E-2 & 2.5\\ \hline 3.6E-4 & 4.0\\ \hline 5.6E-6 & 4.0\\ \hline 3.5E-7 & 4.0\\ \hline \hline r=20\\ \hline \varepsilon_N & \varrho(8.0)\\ \hline 5.9E-3 & 1.1\\ \hline 1.6E-4 & 5.7\\ \hline 3.9E-7 & 7.9\\ \hline 6.0E-9 & 8.0\\ \hline \varepsilon_N' & \varrho'(4.0)\\ \hline 4.6E-2 & 1.0\\ \hline 2.4E-3 & 3.9\\ \hline 3.7E-5 & 4.0\\ \hline \end{array}$

Table 4. Method 1; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{5}{6}$

Table 5. Method 1; η_1	$=\frac{3-\sqrt{3}}{6}, \eta_2 =$	$\frac{3+\sqrt{3}}{6}$
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$\nu = -\frac{1}{4}$	r = 1	r = 1.189	r = 1.378	r = 1.6
Ň	$\varepsilon_N \varrho(4.8)$	$\varepsilon_N \varrho(6.4)$	$\varepsilon_N \varrho(8.0)$	$\varepsilon_N \varrho(8.0)$
4	5.9E-4 4.5	3.3E-4 6.0	2.4E-4 7.5	3.4E-4 7.1
32	5.8E-6 4.7	1.3E-6 6.4	4.7E-7 8.0	7.0E-7 7.9
256	5.5E-8 4.8	5.2E-9 6.4	9.1E-10 8.0	1.4E-9 8.0
1024	2.4E-9 4.8	1.3E-10 6.4	1.4E-11 8.0	2.1E-11 8.0
N	$\varepsilon'_N \varrho'(2.4)$	$\varepsilon'_N \varrho'(2.8)$	$\varepsilon'_N \qquad \varrho'(3.3)$	$\varepsilon'_N \qquad \varrho'(4.0)$
4	1.1E-2 2.2	7.8E-3 2.6	5.7E-3 3.1	4.7E-3 3.9
32	8.4E-4 2.4	3.7E-4 2.8	1.6E-4 3.3	7.3E-5 4.0
256	6.3E-5 2.4	1.7E-5 2.8	4.6E-6 3.3	1.1E-6 4.0
1024	1.1E-5 2.4	2.2E-6 2.8	4.2E-7 3.3	7.2E-8 4.0
$\nu = 0$	r = 1	r = 1.275	r = 1.550	r = 2
N	$\varepsilon_N \varrho(4.0)$	$\varepsilon_N \varrho(5.9)$	$\varepsilon_N \varrho(8.0)$	$\varepsilon_N \varrho(8.0)$
4	3.4E-3 3.8	1.6E-3 5.6	1.1E-3 7.7	1.9E-3 6.9
32	5.6E-5 4.0	8.4E-6 5.8	2.2E-6 8.0	3.9E-6 8.0
256	8.8E-7 4.0	4.2E-8 5.9	4.2E-9 8.1	7.6E-9 8.0
1024	5.5E-8 4.0	1.2E-9 5.9	6.4E-11 8.1	1.2E-10 8.0
N	$\varepsilon'_N \varrho'(2.0)$	$\varepsilon'_N \varrho'(2.4)$	$\varepsilon'_N \varrho'(2.9)$	$\varepsilon'_N \varrho'(4.0)$
4	5.9E-2 1.9	4.1E-2 2.3	2.8E-2 2.8	2.0E-2 3.7
32	7.8E-3 2.0	3.0E-3 2.4	1.2E-3 2.9	3.4E-4 4.0
256	9.8E-4 2.0	2.1E-4 2.4	4.6E-5 2.9	5.2E-6 4.0
1024	2.4E-4 2.0	3.6E-5 2.4	5.4E-6 2.9	3.3E-7 4.0
$\nu = \frac{1}{2}$	r = 1	r = 1.533	r = 2.067	r = 4
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\varepsilon_N \varrho(2.8)$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$	$r = 4$ $\varepsilon_N \varrho(8.0)$
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = 1$ $\varepsilon_N \varrho(2.8)$ 2.7E-3 2.6	r = 1.533 $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6	r = 2.067 $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2	$r = 4$ $\varepsilon_N \varrho(8.0)$ 1.9E-3 4.8
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = 1$ $\varepsilon_N \varrho(2.8)$ 2.7E-3 2.6 1.3E-4 2.8	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1	$r = 4$ $\varepsilon_N \varrho(8.0)$ 1.9E-3 4.8 5.6E-6 7.7
	$r = 1$ $\overline{\varepsilon_N} \varrho(2.8)$ 2.7E-3 2.6 1.3E-4 2.8 5.8E-6 2.8	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E\text{-}3 & 4.8 \\ \hline 5.6E\text{-}6 & 7.7 \\ \hline 1.1E\text{-}8 & 8.0 \end{array}$
	$r = 1$ $\overline{\varepsilon_N} \varrho(2.8)$ 2.7E-3 2.6 1.3E-4 2.8 5.8E-6 2.8 7.2E-7 2.8	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2	$\begin{array}{c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ 1.9E-3 & 4.8 \\ 5.6E-6 & 7.7 \\ 1.1E-8 & 8.0 \\ 1.8E-10 & 8.0 \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{32}{256} $ $ \frac{1024}{N} $	$r = 1$ $\varepsilon_N \varrho(2.8)$ 2.7E-3 2.6 1.3E-4 2.8 5.8E-6 2.8 7.2E-7 2.8 $\varepsilon'_N \varrho'(1.4)$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ 5.6E-6 & 7.7 \\ 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{N}{4} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7\text{E-}3 2.6 \\ \hline 1.3\text{E-}4 2.8 \\ \hline 5.8\text{E-}6 2.8 \\ \hline 7.2\text{E-}7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1\text{E-}2 1.2 \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 9.4E-4 4.6 \\ 8.2E-6 4.9 \\ 6.8E-8 4.9 \\ 2.8E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ 3.0E-2 1.5 \end{array}$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ 5.6E-6 & 7.7 \\ 1.1E-8 & 8.0 \\ 1.8E-10 & 8.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1024}{N} $ $ \frac{4}{32} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ 5.6E-6 & 7.7 \\ 1.1E-8 & 8.0 \\ 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ 2.6E-4 & 4.0 \\ \end{array}$
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ 5.6E-6 & 7.7 \\ 1.1E-8 & 8.0 \\ 1.8E-10 & 8.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ 2.6E-4 & 4.0 \\ 4.1E-6 & 4.0 \\ \hline \end{array}$
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7	$r = 2.067$ $\overline{\varepsilon_N} \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\overline{\varepsilon'_N} \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0	$r = 4$ $\overline{\varepsilon_N} \varrho(8.0)$ 1.9E-3 4.8 5.6E-6 7.7 1.1E-8 8.0 1.8E-10 8.0 $\overline{\varepsilon'_N} \varrho'(4.0)$ 1.5E-2 3.0 2.6E-4 4.0 4.1E-6 4.0 2.6E-7 4.0
	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon'_N \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline r = 20 \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon'_N \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$ $\varepsilon_N \varrho(4.3)$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$	$r = 4$ $\frac{\varepsilon_N}{\varepsilon_N} \frac{\varrho(8.0)}{\varrho(8.0)}$ 1.9E-3 4.8 5.6E-6 7.7 1.1E-8 8.0 1.8E-10 8.0 $\frac{\varepsilon'_N}{\rho'(4.0)}$ 1.5E-2 3.0 2.6E-4 4.0 4.1E-6 4.0 2.6E-7 4.0 $r = 20$ $\frac{\varepsilon_N}{\varepsilon_N} \frac{\varrho(8.0)}{\varrho(8.0)}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ \hline N \\ 4 \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon'_N \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$ $\varepsilon_N \varrho(4.3)$ 6.3E-4 3.2	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ 256 \\ 1024 \\ \hline \\ 256 \\ 1024 \\ \hline \\ \nu = \frac{9}{10} \\ \hline \\ N \\ 4 \\ 32 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$ $\varepsilon_N \varrho(4.3)$ 6.3E-4 3.2 8.9E-6 4.3	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ \hline \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ \nu = \frac{9}{10} \\ \hline \\ N \\ 4 \\ 32 \\ 256 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \\ \hline 2.9E-5 2.1 \\ \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$ $\varepsilon_N \varrho(4.3)$ 6.3E-4 3.2 8.9E-6 4.3 1.1E-7 4.3	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8 1.8E-9 8.2	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \hline 3.7E-7 & 7.7 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \\ \hline 2.9E-5 2.1 \\ \hline 6.3E-6 2.1 \\ \end{array}$	$r = 1.533$ $\varepsilon_N \varrho(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\varepsilon'_N \varrho'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$ $\varepsilon_N \varrho(4.3)$ 6.3E-4 3.2 8.9E-6 4.3 1.1E-7 4.3 6.2E-9 4.3	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8 1.8E-9 8.2 2.6E-11 8.3	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline c_N & \varrho(8.0) \\ \hline \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \hline 3.7E-7 & 7.7 \\ \hline 5.9E-9 & 7.9 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 56 \\ 1024 \\ \hline N \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \\ \hline 2.9E-5 2.1 \\ \hline 6.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \end{array}$	$r = 1.533$ $\overline{\varepsilon}_{N} \underline{\rho}(4.9)$ 9.4E-4 4.6 8.2E-6 4.9 6.8E-8 4.9 2.8E-9 4.9 $\overline{\varepsilon}'_{N} \underline{\rho}'(1.7)$ 3.0E-2 1.5 6.4E-3 1.7 1.3E-3 1.7 4.5E-4 1.7 $r = 1.909$ $\overline{\varepsilon}_{N} \underline{\rho}(4.3)$ 6.3E-4 3.2 8.9E-6 4.3 1.1E-7 4.3 6.2E-9 4.3 $\overline{\varepsilon}'_{N} \underline{\rho}'(1.1)$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8 1.8E-9 8.2 2.6E-11 8.3 $\varepsilon'_N \varrho'(1.2)$	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline c_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \hline 3.7E-7 & 7.7 \\ \hline 5.9E-9 & 7.9 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 4 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \\ \hline 2.9E-5 2.1 \\ \hline 6.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline 2.5E-2 1.4 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 9.4E-4 4.6 \\ 8.2E-6 4.9 \\ 6.8E-8 4.9 \\ 2.8E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ \hline 3.0E-2 1.5 \\ 6.4E-3 1.7 \\ 1.3E-3 1.7 \\ 4.5E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ \hline 6.3E-4 3.2 \\ 8.9E-6 4.3 \\ 1.1E-7 4.3 \\ 6.2E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 2.7E-2 0.9 \end{array}$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8 1.8E-9 8.2 2.6E-11 8.3 $\varepsilon'_N \varrho'(1.2)$ 2.7E-2 0.9	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline c = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \hline 3.7E-7 & 7.7 \\ \hline 5.9E-9 & 7.9 \\ \hline \varepsilon_N' & \varrho'(4.0) \\ \hline 5.9E-2 & 1.0 \\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 56 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \\ \hline 2.9E-5 2.1 \\ \hline 6.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline 2.5E-2 1.4 \\ \hline 2.8E-2 1.0 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 9.4E-4 4.6 \\ 8.2E-6 4.9 \\ 6.8E-8 4.9 \\ 2.8E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ \hline 3.0E-2 1.5 \\ 6.4E-3 1.7 \\ 1.3E-3 1.7 \\ 4.5E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ \hline 6.3E-4 3.2 \\ 8.9E-6 4.3 \\ 1.1E-7 4.3 \\ 6.2E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 2.7E-2 0.9 \\ 2.2E-2 1.1 \\ \end{array}$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8 1.8E-9 8.2 2.6E-11 8.3 $\varepsilon'_N \varrho'(1.2)$ 2.7E-2 0.9 1.6E-2 1.2	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \hline 3.7E-7 & 7.7 \\ \hline 5.9E-9 & 7.9 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 5.9E-2 & 1.0 \\ \hline 2.8E-3 & 4.2 \\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N \varrho(2.8) \\ \hline 2.7E-3 2.6 \\ \hline 1.3E-4 2.8 \\ \hline 5.8E-6 2.8 \\ \hline 7.2E-7 2.8 \\ \hline \varepsilon_N' \varrho'(1.4) \\ \hline 4.1E-2 1.2 \\ \hline 1.6E-2 1.4 \\ \hline 5.7E-3 1.4 \\ \hline 2.9E-3 1.4 \\ \hline r = 1 \\ \hline \varepsilon_N \varrho(2.1) \\ \hline 1.9E-3 1.8 \\ \hline 2.7E-4 2.0 \\ \hline 2.9E-5 2.1 \\ \hline 6.3E-6 2.1 \\ \hline \varepsilon_N' \varrho'(1.1) \\ \hline 2.5E-2 1.4 \\ \hline 2.8E-2 1.0 \\ \hline 2.4E-2 1.1 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \varepsilon_N \varrho(4.9) \\ 9.4E-4 4.6 \\ 8.2E-6 4.9 \\ 6.8E-8 4.9 \\ 2.8E-9 4.9 \\ \hline \varepsilon_N' \varrho'(1.7) \\ \hline 3.0E-2 1.5 \\ 6.4E-3 1.7 \\ 1.3E-3 1.7 \\ 4.5E-4 1.7 \\ \hline r = 1.909 \\ \hline \varepsilon_N \varrho(4.3) \\ \hline 6.3E-4 3.2 \\ 8.9E-6 4.3 \\ 1.1E-7 4.3 \\ 6.2E-9 4.3 \\ \hline \varepsilon_N' \varrho'(1.1) \\ 2.7E-2 0.9 \\ 2.2E-2 1.1 \\ 1.5E-2 1.1 \\ \end{array}$	$r = 2.067$ $\varepsilon_N \varrho(8.0)$ 5.7E-4 6.2 1.2E-6 8.1 2.1E-9 8.2 3.1E-11 8.2 $\varepsilon'_N \varrho'(2.0)$ 2.1E-2 1.9 2.5E-3 2.0 3.0E-4 2.0 7.1E-5 2.0 $r = 2.818$ $\varepsilon_N \varrho(8.0)$ 3.6E-4 6.0 9.8E-7 7.8 1.8E-9 8.2 2.6E-11 8.3 $\varepsilon'_N \varrho'(1.2)$ 2.7E-2 0.9 1.6E-2 1.2 8.9E-3 1.2	$\begin{array}{c c} r = 4 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 1.9E-3 & 4.8 \\ \hline 5.6E-6 & 7.7 \\ \hline 1.1E-8 & 8.0 \\ \hline 1.8E-10 & 8.0 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 1.5E-2 & 3.0 \\ \hline 2.6E-4 & 4.0 \\ \hline 4.1E-6 & 4.0 \\ \hline 2.6E-7 & 4.0 \\ \hline \hline r = 20 \\ \hline \varepsilon_N & \varrho(8.0) \\ \hline 6.2E-3 & 1.1 \\ \hline 1.6E-4 & 5.6 \\ \hline 3.7E-7 & 7.7 \\ \hline 5.9E-9 & 7.9 \\ \hline \varepsilon'_N & \varrho'(4.0) \\ \hline 5.9E-2 & 1.0 \\ \hline 2.8E-3 & 4.2 \\ \hline 2.8E-5 & 4.3 \\ \end{array}$

Theorem 2. [⁴] Assume that $p, q \in C^{m+1,\nu}(0,T]$, $K \in W^{m+1,\nu}(\Delta_T)$, $m \in \mathbb{N} = \{1, 2, \ldots\}, \nu \in \mathbb{R} \setminus \mathbb{Z}, \nu < 1$ and that the parameters η_j are chosen so that the interpolatory quadrature approximation $\int_0^1 \varphi(s) ds \approx \sum_{j=1}^m A_j \varphi(\eta_j)$, with

appropriate weights $\{A_j\}$, is exact for all polynomials of degree m. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$ the error estimate

$$||u - y||_{\infty} \le c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \le r < \frac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1 + \log N) & \text{for } r = \frac{m+1}{2-\nu}, \\ N^{-m-1} & \text{for } r > \frac{m+1}{2-\nu} \end{cases}$$

holds, where c is a positive constant which is independent of N.

Numerical experiments in case $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{6}$, when the corresponding quadrature formula is exact for all polynomials up to order 2, are presented in Table 4. Numerical results show that the convergence rate is much better in this case (compared with Table 1) and agrees well with the estimate of Theorem 2.

Remark 1. If we use Gaussian parameters $\left(\eta_1 = \frac{3-\sqrt{3}}{6}, \eta_2 = \frac{3+\sqrt{3}}{6}\right)$ that are exact for polynomials of order 2m-1=3, we do not get any further improvement in the convergence rate, although the actual errors are slightly smaller due to a smaller error coefficient. The corresponding numerical experiments are presented in Table 5.

5. METHOD 2

The second reformulation of problem $\{(1),(2)\}\$ is based on the integration of both sides of (1) over (0, t). Using this and (2), Eq. (1) may be rewritten as a linear Volterra integral equation with respect to y:

$$y(t) = f_2(t) + \int_0^t K_2(t,s)y(s)ds, \qquad t \in [0,T],$$
(13)

where

$$f_2(t) = y_0 + \int_0^t q(s) \mathrm{d}s, \quad K_2(t,s) = p(s) + \int_s^t K(\tau,s) \mathrm{d}\tau.$$
 (14)

We look for an approximate solution u of Eq. (13) in $S_m^{(-1)}(\Pi_N^r)$, $m, N \in \mathbb{N}$: this approximation $u = u^{(N)} \in S_m^{-1}(\Pi_N^r)$ will be determined by the collocation method from the following conditions:

$$u_j(t_{jk}) = f_2(t_{jk}) + \int_0^{t_{jk}} K_2(t_{jk}, s)u(s)ds, \quad k = 1, \dots, m+1; \ j = 1, \dots, N.$$
(15)

Here, f_2 and K_2 are defined in (14), $u_j = u|_{[t_{j-1},t_j]}$ (j = 1,...,N) is the restriction of $u \in S_m^{(-1)}(\Pi_N^r)$ to $[t_{j-1},t_j]$, and the collocation points $\{t_{jk}\}$ are given by $t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1})$, where $\{t_j\}$ are the nodes of Π_N^r and $0 \le \eta_1 < \ldots < \eta_{m+1} \le 1$ is a fixed system of parameters which does not depend on j and N.

Theorem 3. [¹] Assume (3) and that the collocation points (6), where k = 1, ..., m + 1, are used. Then for all sufficiently large $N \in \mathbb{N}$ and for every choice of parameters $\{\eta_j\}$ $(0 \le \eta_1 < ... < \eta_{m+1} \le 1)$ the collocation conditions (15) define a unique approximation $u \in S_m^{(-1)}(\Pi_N^r)$ to y, the solution of the Cauchy problem $\{(1), (2)\}$. Then the following error estimates hold: 1) if $m < 1 - \nu$, then

$$\begin{split} ||u-y||_{\infty} &\leq c N^{-m-1} \quad for \quad r \geq 1, \\ ||u'-y'||_{\infty} &\leq c N^{-m} \quad for \quad r = 1, \\ ||u'-y'||_{\varepsilon,\infty} &\leq c_{\varepsilon} N^{-m} \quad for \quad r > 1; \end{split}$$

2) if $m = 1 - \nu$, then

$$\begin{aligned} ||u-y||_{\infty} &\leq c \begin{cases} N^{-m-1}(1+|\log N|) & for \quad r=1, \\ N^{-m-1} & for \quad r>1; \\ ||u'-y'||_{\infty} &\leq c N^{-m}(1+|\log N|) & for \quad r=1, \\ ||u'-y'||_{\varepsilon,\infty} &\leq c_{\varepsilon} N^{-m} & for \quad r>1; \end{aligned}$$

3) if $m > 1 - \nu$, then

$$\begin{aligned} ||u-y||_{\infty} &\leq c \begin{cases} N^{-r(2-\nu)} & for \quad 1 \leq r \leq \frac{m+1}{2-\nu}, \\ N^{-m-1} & for \quad r > \frac{m+1}{2-\nu}; \end{cases} \\ ||u'-y'||_{\infty} &\leq c N^{-(1-\nu)} & for \quad r = 1, \\ ||u'-y'||_{\varepsilon,\infty} &\leq c_{\varepsilon} \begin{cases} N^{-r(1-\nu)} & for \quad 1 \leq r \leq \frac{m}{1-\nu}, \\ N^{-m} & for \quad r > \frac{m}{1-\nu}. \end{cases} \end{aligned}$$

Here the constants c *and* c_{ε} *are independent of* N *and*

$$\|u' - y'\|_{\varepsilon,\infty} = \max_{j=1,\dots,N} \left(\max_{t \in [t_{j-1},t_j] \cap [\varepsilon,T]} |u'_j(t) - y'(t)| \right), \ 0 < \varepsilon < T.$$

The corresponding numerical results are presented in Table 6. This table is comparable to Table 4. As we can see, the agreement with theoretical estimates is very good for all values of r and ν and we may conclude that in case of the test equations, the error estimates of Theorem 3 correspond to the leading term of the errors $||u - y||_{\infty}$ and $||u' - y'||_{\infty}$, which are dominating even for small values of N. Moreover, it seems that the estimate for the derivative of the error,

$$||u' - y'||_{\infty} \le c \begin{cases} N^{-r(1-\nu)} & \text{for } 1 \le r \le \frac{m}{1-\nu}, \\ N^{-m} & \text{for } r > \frac{m}{1-\nu}, \end{cases}$$

may be valid.

Table 6. Method 2;	η_1	=	$\frac{1}{6}, \eta_2$	=	$\frac{1}{2}, \eta_3$	=	$\frac{5}{6}$
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$\nu = -\frac{1}{4}$	r = 1		r = 1.1	189	r = 1.3	878	r = 1	.6
4 N	$\varepsilon_N \rho(4.8)$)	$\varepsilon_N c$	(6.4)	$\varepsilon_N 0$	(8.0)	ε_N ((8.0)
4	2.0E-4 4.4	1	1.1E-4	5.9	9.8E-5	8.1	1.4E-4	7.4
32	2.0E-6 4.	7	4.6E-7	6.3	1.8E-7	8.1	2.9E-7	8.0
256	1.9E-8 4.	3	1.8E-9	6.4	3.6E-10	8.0	5.6E-10	8.0
1024	8.3E-10 4.	3	4.4E-11	6.4	5.6E-12	8.0	8.7E-12	8.0
N	$\varepsilon'_N \varrho'(2.4)$)	$\varepsilon'_N \varrho'$	(2.8)	$\varepsilon'_N \varrho'$	(3.3)	$\varepsilon'_N \varrho$	(4.0)
4	1.7E-2 2.	3	1.2E-2	2.8	8.8E-3	3.3	7.2E-3	3.9
32	1.3E-3 2.4	1	5.6E-4	2.8	2.5E-4	3.3	1.1E-4	4.0
256	9.4E-5 2.4	1	2.6E-5	2.8	6.9E-6	3.3	1.8E-6	4.0
1024	1.7E-5 2.4	1	3.3E-6	2.8	6.3E-7	3.3	1.1E-7	4.0
$\nu = 0$	r = 1		r = 1.2	275	r = 1.5	550	r =	2
N	$\varepsilon_N \qquad \varrho(4.0)$)	ε_N ℓ	(5.9)	$\varepsilon_N = \varrho$	(8.0)	ε_N (2(8.0)
4	1.1E-3 3.	3	5.1E-4	5.5	3.8E-4	7.5	7.1E-4	6.3
32	1.8E-5 4.)	2.7E-6	5.8	7.4E-7	8.0	1.6E-6	7.9
256	2.8E-7 4.)	1.3E-8	5.9	1.4E-9	8.0	3.1E-9	8.0
1024	1.8E-8 4.)	3.9E-10	5.9	2.3E-11	8.0	4.9E-11	8.0
N	$\varepsilon'_N \varrho'(2.0)$)	$\varepsilon'_N \varrho'$	(2.4)	$\varepsilon'_N \varrho'$	(2.9)	$\varepsilon'_N \varrho$	'(4.0)
4	9.1E-2 2.)	6.2E-2	2.4	4.3E-2	2.9	3.3E-2	3.9
32	1.1E-2 2.) .	4.4E-3	2.4	1.7E-3	2.9	5.2E-4	4.0
256	1.4E-3 2.)	3.1E-4	2.4	6.8E-5	2.9	8.1E-6	4.0
1024	3.6E-4 2.)	5.3E-5	2.4	7.9E-6	2.9	5.1E-7	4.0
$\nu = \frac{1}{2}$	r = 1		r = 1.5	533	r = 2.0)67	r =	4
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\varepsilon_N \varrho(2.8)$)	$r = 1.5$ $\varepsilon_N \varrho$	533 9(4.9)	$r = 2.0$ $\varepsilon_N \varrho$)67 (8.0)	$r = \varepsilon_N $	4 p(8.0)
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = 1$ $\varepsilon_N \varrho(2.8)$ $6.8E-4 2.2$)	$r = 1.5$ $\varepsilon_N \varrho$ 2.4E-4	533 p(4.9) 4.4	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4)67 (8.0) 7.3	$r = \frac{\varepsilon_N}{8.4\text{E-4}}$	4 2(8.0) 3.8
$\frac{\nu = \frac{1}{2}}{N}$ $\frac{1}{4}$ $\frac{32}{32}$	r = 1 $\varepsilon_N \varrho(2.8)$ 6.8E-4 2. 3.3E-5 2.)	$r = 1.5$ $\varepsilon_N \underline{\rho}$ 2.4E-4 2.1E-6	533 9(4.9) 4.4 4.9	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4 3.3E-7)67 (8.0) 7.3 8.1	$r = \frac{\varepsilon_N}{8.4\text{E-4}}$ 2.3E-6	4 (8.0) 3.8 7.8
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{1}{4} $ $ \frac{32}{256} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8 \\ \hline 6.8E-4 & 2. \\ 3.3E-5 & 2. \\ 1.5E-6 & 2. \end{array}$) (5 (3 (3 (r = 1.5 ε_N (2) 2.4E-4 2.1E-6 1.8E-8	533 5(4.9) 4.4 4.9 4.9 4.9	r = 2.0 $\varepsilon_N \varrho$ 1.8E-4 3.3E-7 6.4E-10	067 (8.0) 7.3 8.1 8.0	$r = \frac{\varepsilon_N}{8.4\text{E-4}}$ 8.4E-4 2.3E-6 4.7E-9	4 p(8.0) 3.8 7.8 8.0
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1}{2} + \frac$	r = 1 $\varepsilon_N \varrho(2.8)$ 6.8E-4 2. 3.3E-5 2. 1.5E-6 2. 1.9E-7 2.) (5 (3 (3 (3 (3 ()	r = 1.5 ε_N Q 2.4E-4 2.1E-6 1.8E-8 7.3E-10	533 (4.9) 4.4 4.9 4.9 4.9 4.9	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4 3.3E-7 6.4E-10 1.0E-11	067 (8.0) 7.3 8.1 8.0 8.0	$r = \frac{\varepsilon_N}{8.4\text{E-4}}$ 8.4E-4 2.3E-6 4.7E-9 7.3E-11	4 (8.0) 3.8 7.8 8.0 8.0 8.0
	$\begin{array}{c} r=1\\ \hline \varepsilon_{N} \varrho(2.8\\ 6.8E\text{-}4 2.\\ 3.3E\text{-}5 2.\\ 1.5E\text{-}6 2.\\ 1.9E\text{-}7 2.\\ \hline \varepsilon_{N}' \varrho'(1.4) \end{array}$)	$r = 1.5$ $\varepsilon_N \varrho$ 2.4E-4 2.1E-6 1.8E-8 7.3E-10 $\varepsilon'_N \varrho'$	533 (4.9) 4.4 4.9 4.9 4.9 (1.7)	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4 3.3E-7 6.4E-10 1.0E-11 $\varepsilon'_N \varrho'$)67 (8.0) 7.3 8.1 8.0 8.0 (2.0)	$r = \frac{\varepsilon_N}{\varepsilon_N} \frac{\varrho}{\varepsilon_N}$ 8.4E-4 2.3E-6 4.7E-9 7.3E-11 $\varepsilon'_N - \varrho$	4 p(8.0) 3.8 7.8 8.0 8.0 7(4.0)
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1}{N} $ $ \frac{4}{4} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2 \\ 3.3E-5 & 2 \\ 1.5E-6 & 2 \\ 1.9E-7 & 2 \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 6.3E-2 & 1 \\ \hline \end{array}$) (5 (3 (3 (3 (3 () (1 ()	$\frac{r = 1.5}{\varepsilon_N} \frac{\rho}{2}$ 2.4E-4 2.1E-6 1.8E-8 7.3E-10 $\frac{\varepsilon'_N}{4} \frac{\rho'}{4}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ $	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 3.0E-2 \end{array}$	067 (8.0) 7.3 8.1 8.0 8.0 (2.0) 2.0	$r = \frac{\varepsilon_N}{\varepsilon_N} \frac{\varrho}{\varepsilon_N}$ 8.4E-4 2.3E-6 4.7E-9 7.3E-11 $\frac{\varepsilon'_N}{\varepsilon'_N} \frac{\varrho}{\varepsilon'_N}$ 2.5E-2	4 2(8.0) 3.8 7.8 8.0 8.0 7(4.0) 3.3
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1024}{N} $ $ \frac{4}{32} $ $ \frac{1024}{32} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2 \\ 3.3E-5 & 2 \\ 1.5E-6 & 2. \\ 1.9E-7 & 2. \\ \hline \varepsilon_N' & \varrho'(1.4) \\ \hline 6.3E-2 & 1 \\ 2.2E-2 & 1 \\ \end{array}$) 5 5 2 3 3 3 7 1 4 4 7	$\frac{r = 1.5}{\varepsilon_N} \frac{\rho}{2.4E-4}$ 2.4E-4 2.1E-6 1.8E-8 7.3E-10 $\frac{\varepsilon'_N}{\varepsilon'_N} \frac{\rho'}{4.4E-2}$ 8.9E-3	533 (4.9) 4.4 4.9 4.9 (1.7) 1.7 1.7	$\frac{r = 2.0}{\varepsilon_N \ \varrho}$ 1.8E-4 3.3E-7 6.4E-10 1.0E-11 $\frac{\varepsilon'_N \ \varrho'}{3.0E-2}$ 3.5E-3	067 (8.0) 7.3 8.1 8.0 8.0 (2.0) 2.0 2.0	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \end{array}$	$ \frac{4}{2(8.0)} \\ 3.8 \\ 7.8 \\ 8.0 \\ 8.0 \\ 7(4.0) \\ 3.3 \\ 4.0 \\ 4.0 \\ 5.3 \\ 4.0 \\ 5.3 \\ 4.0 \\ 5.3$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1024}{N} $ $ \frac{4}{32} $ $ \frac{256}{256} $ $ \frac{1025}{256} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2. \\ \hline 3.3E-5 & 2. \\ \hline 1.5E-6 & 2. \\ \hline 1.9E-7 & 2. \\ \hline \varepsilon_N' & \varrho'(1.4) \\ \hline 6.3E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 7.9E-3 & 1. \\ \hline 4.6E-2 & 1. \\ \hline \end{array}$) 5 5 3 3 3) 4 4 4	$\frac{r = 1.5}{\varepsilon_N} \frac{\rho}{2.4E-4}$ 2.4E-4 2.1E-6 1.8E-8 7.3E-10 $\frac{\varepsilon'_N}{\varepsilon'_N} \frac{\rho'}{2}$ 4.4E-2 8.9E-3 1.8E-3	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ $	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4 3.3E-7 6.4E-10 1.0E-11 $\varepsilon'_N \varrho'$ 3.0E-2 3.5E-3 4.1E-4	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline \end{array}$	$r = \frac{\varepsilon_N}{\varepsilon_N} \frac{\varepsilon}{\varepsilon_N} \frac{\varepsilon}{\varepsilon_N$	$ \frac{4}{2(8.0)} 3.8 7.8 8.0 8.0 7(4.0) 3.3 4.0 4.0 4.0 $
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2. \\ \hline 3.3E-5 & 2. \\ \hline 1.5E-6 & 2. \\ \hline 1.9E-7 & 2. \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 6.3E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 7.9E-3 & 1. \\ \hline 4.0E-3 & 1. \\ \end{array}$) 5 3 3 3 3 4 4 4 4	$r = 1.5$ $\varepsilon_N \varrho$ 2.4E-4 2.1E-6 1.8E-8 7.3E-10 $\varepsilon'_N \varrho'$ 4.4E-2 8.9E-3 1.8E-3 6.2E-4	533 (4.9) 4.4 4.9 4.9 (1.7) 1.7 1.7 1.7 1.7	$\begin{array}{c c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 3.0E-2 \\ 3.5E-3 \\ 4.1E-4 \\ 9.8E-5 \end{array}$	067 (8.0) 7.3 8.1 8.0 (2.0) 2.0 2.0 2.0 2.0	$r = \frac{\varepsilon_N}{\varepsilon_N} \frac{\varepsilon}{\varepsilon_N}$ 8.4E-4 2.3E-6 4.7E-9 7.3E-11 $\frac{\varepsilon'_N}{\varepsilon'_N} \frac{\varrho}{\varepsilon'_N}$ 4.0E-4 6.2E-6 3.9E-7	$ \frac{4}{2(8.0)} 3.8 7.8 8.0 8.0 7(4.0) 3.3 4.0 4.0 4.0 $
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1024}{4} $ $ \frac{1024}{1024} $ $ \frac{1024}{1024} $ $ \frac{1024}{1024} $	$\begin{array}{c} r=1\\ \hline \varepsilon_{N} \varrho(2.8\\ \hline 6.8E-4 2.\\ \hline 3.3E-5 2.\\ \hline 1.5E-6 2.\\ \hline 1.9E-7 2.\\ \hline \varepsilon_{N}' \varrho'(1.4\\ \hline 6.3E-2 1.\\ \hline 2.2E-2 1.\\ \hline 7.9E-3 1.\\ \hline 4.0E-3 1.\\ \hline r=1 \end{array}$) (55) (33) (33) (33) (4) (4) (4) (4) (4) (4) (4) (4	$\frac{r = 1.5}{\varepsilon_N} \frac{\rho}{2.4E-4}$ 2.4E-4 2.1E-6 1.8E-8 7.3E-10 $\frac{\varepsilon'_N}{4.4E-2}$ 8.9E-3 1.8E-3 6.2E-4 $r = 1.5$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 009$	$\begin{array}{c} r = 2.0\\ \hline \varepsilon_N & \varrho\\ 1.8E-4\\ 3.3E-7\\ 6.4E-10\\ 1.0E-11\\ \hline \varepsilon'_N & \varrho'\\ 3.0E-2\\ 3.5E-3\\ 4.1E-4\\ 9.8E-5\\ \hline r = 2.8\end{array}$	067 (8.0) 7.3 8.1 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 318	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \end{array}$	$ \frac{4}{2(8.0)} \\ \frac{3.8}{7.8} \\ 8.0 \\ 8.0 \\ 7(4.0) \\ 3.3 \\ 4.0 \\ 4.0 \\ 4.0 \\ 20 $
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2. \\ \hline 3.3E-5 & 2. \\ \hline 1.5E-6 & 2. \\ \hline 1.9E-7 & 2. \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 6.3E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 7.9E-3 & 1. \\ \hline 4.0E-3 & 1. \\ \hline r = 1 \\ \hline \varepsilon_N & \varrho(2.1) \\ \end{array}$) 5 3 3 3 3 3 4 4 4 4 4 4 4 1 1 1 1 1 1 1 1 1 1 1 1 1	$\begin{array}{c} r = 1.5 \\ \varepsilon_N & \varrho \\ 2.4E-4 \\ 2.1E-6 \\ 1.8E-8 \\ 7.3E-10 \\ \varepsilon'_N & \varrho' \\ 4.4E-2 \\ 8.9E-3 \\ 1.8E-3 \\ 6.2E-4 \\ \hline r = 1.5 \\ \varepsilon_N & \varrho \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 0.9 \\ 0(4.3)$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ 3.0E-2 \\ 3.5E-3 \\ 4.1E-4 \\ 9.8E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \end{array}$	$\begin{array}{c} 0.067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 318 \\ \hline (8.0) \end{array}$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \end{array}$	$ \frac{4}{2(8.0)} \\ \frac{3.8}{7.8} \\ 8.0 \\ 8.0 \\ 7(4.0) \\ 3.3 \\ 4.0 \\ 4.0 \\ 4.0 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 8 \\ 8 \\ 1 \\ 1 \\ 1 \\ 1 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20 \\ 20$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ \hline \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_{N} \varrho(2.8\\ \hline 6.8E-4 2.\\ \hline 3.3E-5 2.\\ \hline 1.5E-6 2.\\ \hline 1.9E-7 2.\\ \hline \varepsilon_{N}' \varrho'(1.4\\ \hline 6.3E-2 1.\\ \hline 2.2E-2 1.\\ \hline 7.9E-3 1.\\ \hline 4.0E-3 1.\\ \hline \hline \varepsilon_{N} \varrho(2.1\\ \hline 4.2E-4 2.\\ \end{array}$) 5 3 3 3 3 3 4 4 4 4 4 4 4 4 4 4 4 4 4	$\begin{array}{c c} r = 1.5 \\ \varepsilon_N & \varrho \\ \hline 2.4E-4 \\ 2.1E-6 \\ 1.8E-8 \\ 7.3E-10 \\ \varepsilon'_N & \varrho' \\ \hline 4.4E-2 \\ 8.9E-3 \\ 1.8E-3 \\ 6.2E-4 \\ \hline r = 1.5 \\ \varepsilon_N & \varrho \\ \hline 1.2E-4 \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ $	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ 3.0E-2 \\ 3.5E-3 \\ 4.1E-4 \\ 9.8E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ 1.7E-4 \end{array}$	067 (8.0) 7.3 8.1 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 2.0 2.0 2.0 2.0 7.9	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 9(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ \hline 8.0\\ \hline 7(4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 20\\ \hline 0(8.0)\\ \hline 1.0\\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ \hline 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline 32 \\ \hline \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2. \\ \hline 3.3E-5 & 2. \\ \hline 1.5E-6 & 2. \\ \hline 1.9E-7 & 2. \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 6.3E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 7.9E-3 & 1. \\ \hline 4.0E-3 & 1. \\ \hline \hline \varepsilon_N & \varrho(2.1) \\ \hline 4.2E-4 & 2. \\ \hline 5.3E-5 & 2. \\ \end{array}$) (5 3 3 3 3 3 3 3 1 4 4 4 4 4 4 1 1 1 1 1 1 1 1 1 1 1 1 1	$\begin{array}{c c} r = 1.5 \\ \hline \varepsilon_N & \underline{\ell} \\ 2.4E-4 \\ 2.1E-6 \\ 1.8E-8 \\ 7.3E-10 \\ \hline \varepsilon'_N & \underline{\ell}' \\ 4.4E-2 \\ 8.9E-3 \\ 1.8E-3 \\ 6.2E-4 \\ \hline r = 1.5 \\ \hline \varepsilon_N & \underline{\ell} \\ 1.2E-4 \\ 1.8E-6 \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.4 \\ 0.9 \\ (4.3) \\ 5.2 \\ 4.3 \\ 1.4 $	$\begin{array}{c c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 3.0E-2 \\ 3.5E-3 \\ 4.1E-4 \\ 9.8E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.7E-4 \\ 2.4E-7 \\ \end{array}$	067 (8.0) 7.3 8.1 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 2.0 2.0 318 (8.0) 7.9 8.6	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \\ 8.8E-5 \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ 8.0\\ \hline 7(4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ 4.0\\ \hline 20\\ \hline 0(8.0)\\ \hline 1.0\\ 6.6 \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ \end{array} $	$\begin{array}{c} r = 1 \\ \hline \varepsilon_N & \varrho(2.8) \\ \hline 6.8E-4 & 2. \\ \hline 3.3E-5 & 2. \\ \hline 1.5E-6 & 2. \\ \hline 1.9E-7 & 2. \\ \hline \varepsilon'_N & \varrho'(1.4) \\ \hline 6.3E-2 & 1. \\ \hline 2.2E-2 & 1. \\ \hline 7.9E-3 & 1. \\ \hline 4.0E-3 & 1. \\ \hline \hline \varepsilon_N & \varrho(2.1) \\ \hline 4.2E-4 & 2. \\ \hline 5.3E-5 & 2. \\ \hline 5.8E-6 & 2. \\ \end{array}$) (5 3 3 3 3 3 3 3 4 4 4 4 4 4 4 4 4 4 4 4 4	$\begin{array}{c c} r = 1.5\\ \hline \varepsilon_N & \underline{\varrho}\\ 2.4E-4\\ 2.1E-6\\ 1.8E-8\\ 7.3E-10\\ \hline \varepsilon'_N & \underline{\varrho'}\\ 4.4E-2\\ 8.9E-3\\ 1.8E-3\\ 6.2E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \underline{\varrho}\\ 1.2E-4\\ 1.8E-6\\ 2.3E-8\\ \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.4 \\ 4.3 \\ 4.3 \\ 4.3 \\ 4.3 \\ 1000$	$\begin{array}{c} r = 2.0\\ \hline \varepsilon_N & \varrho\\ 1.8E-4\\ 3.3E-7\\ 6.4E-10\\ 1.0E-11\\ \hline \varepsilon'_N & \varrho'\\ 3.0E-2\\ 3.5E-3\\ 4.1E-4\\ 9.8E-5\\ \hline r = 2.8\\ \hline \varepsilon_N & \varrho\\ 1.7E-4\\ 2.4E-7\\ 4.3E-10\\ \end{array}$	$\begin{array}{c} 0.067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 8.1 \\ \hline (8.0) \\ \hline 7.9 \\ 8.6 \\ 8.1 \\ \end{array}$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \\ 8.8E-5 \\ 1.6E-7 \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ 8.0\\ \hline 7(4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ 4.0\\ \hline 20\\ \hline 0(8.0)\\ \hline 1.0\\ 6.6\\ 8.2 \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_{N} \varrho(2.8\\ \hline 6.8E-4 2\\ \hline 3.3E-5 2.\\ \hline 1.5E-6 2.\\ \hline 1.9E-7 2.\\ \hline \varepsilon_{N}' \varrho'(1.4\\ \hline 6.3E-2 1.\\ \hline 2.2E-2 1.\\ \hline 7.9E-3 1.\\ \hline 4.0E-3 1.\\ \hline \hline r=1\\ \hline \varepsilon_{N} \varrho(2.1\\ \hline 4.2E-4 2.\\ \hline 5.3E-5 2.\\ \hline 5.8E-6 2.\\ \hline 1.3E-6 2.\\ \end{array}$) 5 3 3 3 3 3 3 3 3 3 3 3 3 3	$\begin{array}{c c} r = 1.5\\ \hline \varepsilon_N & \underline{\ell}\\ 2.4E-4\\ 2.1E-6\\ 1.8E-8\\ 7.3E-10\\ \hline \varepsilon_N' & \underline{\ell'}\\ 4.4E-2\\ 8.9E-3\\ 1.8E-3\\ 6.2E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \underline{\ell}\\ 1.2E-4\\ 1.8E-6\\ 2.3E-8\\ 1.2E-9\\ \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.4 \\ 4.3 \\ $	$\begin{array}{c c} r = 2.0\\ \hline \varepsilon_N & \varrho\\ \hline 1.8E-4\\ 3.3E-7\\ 6.4E-10\\ 1.0E-11\\ \hline \varepsilon'_N & \varrho'\\ \hline 3.0E-2\\ 3.5E-3\\ 4.1E-4\\ 9.8E-5\\ \hline r = 2.8\\ \hline \varepsilon_N & \varrho\\ \hline 1.7E-4\\ 2.4E-7\\ 4.3E-10\\ 6.8E-12\\ \end{array}$	067 (8.0) 7.3 8.1 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 2.0 2.0 318 (8.0) 7.9 8.6 8.1 7.9	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \\ 8.8E-5 \\ 1.6E-7 \\ 2.4E-9 \\ \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ \hline 8.0\\ \hline 7(4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 20\\ \hline 0(8.0)\\ \hline 1.0\\ \hline 6.6\\ 8.2\\ 8.1\\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_{N} \varrho(2.8\\ \hline 6.8E-4 2.\\ \hline 3.3E-5 2.\\ \hline 1.5E-6 2.\\ \hline 1.9E-7 2.\\ \hline \varepsilon_{N}' \varrho'(1.4\\ \hline 6.3E-2 1.\\ \hline 2.2E-2 1.\\ \hline 7.9E-3 1.\\ \hline 4.0E-3 1.\\ \hline \hline r=1\\ \hline \varepsilon_{N} \varrho(2.1\\ \hline 4.2E-4 2.\\ \hline 5.3E-5 2.\\ \hline 5.8E-6 2.\\ \hline 1.3E-6 2.\\ \hline \varepsilon_{N}' \varrho'(1.1\\ \hline \end{array}$) 5 3 3 3 3 3 4 4 4 4 4 4 4 4 1 1 1 1 1 1 1 1 1 1 1 1 1	$\begin{array}{c c} r = 1.5 \\ \hline \varepsilon_N & \underline{\ell} \\ 2.4E-4 \\ 2.1E-6 \\ 1.8E-8 \\ 7.3E-10 \\ \hline \varepsilon'_N & \underline{\rho'} \\ 4.4E-2 \\ 8.9E-3 \\ 1.8E-3 \\ 6.2E-4 \\ \hline r = 1.5 \\ \hline \varepsilon_N & \underline{\ell} \\ \hline r = 1.5 \\ \hline \varepsilon_N & \underline{\ell} \\ 1.2E-4 \\ 1.8E-6 \\ 2.3E-8 \\ 1.2E-9 \\ \hline \varepsilon'_N & \underline{\rho'} \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.4 \\ 4.3 \\ 4.3 \\ 4.3 \\ 4.3 \\ (1.1)$	$\begin{array}{c c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 3.0E-2 \\ 3.5E-3 \\ 4.1E-4 \\ 9.8E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.7E-4 \\ 2.4E-7 \\ 4.3E-10 \\ 6.8E-12 \\ \hline \varepsilon'_N & \varrho' \\ \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 2.0 \\ \hline 2.0 \\ \hline 318 \\ \hline (8.0) \\ \hline 7.9 \\ \hline 8.6 \\ 8.1 \\ \hline 7.9 \\ \hline (1.2) \\ \hline \end{array}$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \\ 8.8E-5 \\ 1.6E-7 \\ 2.4E-9 \\ \hline \varepsilon'_N & \varrho \\ \hline \varepsilon'_N & \varrho \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ \hline 8.0\\ \hline (4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 20\\ \hline 0(8.0)\\ \hline 1.0\\ \hline 6.6\\ 8.2\\ 8.1\\ \hline 7(4.0)\\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ 256 \\ 1024 \\ \hline \\ \nu = \frac{9}{10} \\ \hline \\ \nu = \frac{9}{10} \\ \hline \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ N \\ 4 \\ \hline \\ N \\ 4 \\ \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_N & \varrho(2.8\\ \hline 6.8E-4 & 2.\\ \hline 3.3E-5 & 2.\\ \hline 1.5E-6 & 2.\\ \hline 1.9E-7 & 2.\\ \hline \varepsilon_N' & \varrho'(1.4\\ \hline 6.3E-2 & 1.\\ \hline 2.2E-2 & 1.\\ \hline 7.9E-3 & 1.\\ \hline 4.0E-3 & 1.\\ \hline r=1\\ \hline \varepsilon_N & \varrho(2.1\\ \hline 4.2E-4 & 2.\\ \hline 5.3E-5 & 2.\\ \hline 5.8E-6 & 2.\\ \hline 1.3E-6 & 2.\\ \hline \varepsilon_N' & \varrho'(1.1\\ \hline 4.5E-2 & 0.\\ \end{array}$) 5 3 3 3 3 3 3 4 4 4 4 4 4 4 1 1 1 1 1 1 1 1 1 1 1 1 1	$\begin{array}{c c} r = 1.5\\ \hline \varepsilon_N & \underline{\ell}\\ 2.4E-4\\ 2.1E-6\\ 1.8E-8\\ 7.3E-10\\ \hline \varepsilon'_N & \underline{\rho'}\\ 4.4E-2\\ 8.9E-3\\ 1.8E-3\\ 6.2E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \underline{\ell}\\ 1.2E-4\\ 1.8E-6\\ 2.3E-8\\ 1.2E-9\\ \hline \varepsilon'_N & \underline{\rho'}\\ 4.2E-2\\ \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.1 \\ $	$\begin{array}{c c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.3E-7 \\ 6.4E-10 \\ 1.0E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 3.0E-2 \\ 3.5E-3 \\ 4.1E-4 \\ 9.8E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.7E-4 \\ 2.4E-7 \\ 4.3E-10 \\ 6.8E-12 \\ \hline \varepsilon'_N & \varrho' \\ \hline 3.8E-2 \\ \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 2.0 \\ \hline 2.0 \\ \hline 2.0 \\ \hline 318 \\ \hline (8.0) \\ \hline 7.9 \\ \hline 8.6 \\ 8.1 \\ \hline 7.9 \\ \hline (1.2) \\ \hline 1.2 \\ \end{array}$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \\ 8.8E-5 \\ 1.6E-7 \\ 2.4E-9 \\ \hline \varepsilon'_N & \varrho \\ 3.6E-2 \\ \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ \hline 8.0\\ \hline (4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 0(8.0)\\ \hline 1.0\\ \hline 6.6\\ 8.2\\ 8.1\\ \hline (4.0)\\ \hline 1.0\\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ 256 \\ 1024 \\ \hline \\ \nu = \frac{9}{10} \\ \hline \\ \nu = \frac{9}{10} \\ \hline \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \\ N \\ 4 \\ 32 \\ \hline \\ N \\ 4 \\ 32 \\ \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_N & \varrho(2.8\\ \hline 6.8E-4 & 2.\\ \hline 3.3E-5 & 2.\\ \hline 1.5E-6 & 2.\\ \hline 1.9E-7 & 2.\\ \hline \varepsilon_N' & \varrho'(1.4\\ \hline 6.3E-2 & 1.\\ \hline 2.2E-2 & 1.\\ \hline 7.9E-3 & 1.\\ \hline 4.0E-3 & 1.\\ \hline r=1\\ \hline \varepsilon_N & \varrho(2.1\\ \hline 4.2E-4 & 2.\\ \hline 5.3E-5 & 2.\\ \hline 5.8E-6 & 2.\\ \hline 1.3E-6 & 2.\\ \hline \varepsilon_N' & \varrho'(1.1\\ \hline 4.5E-2 & 0.\\ \hline 4.0E-2 & 1.\\ \end{array}$) 5 3 3 3 3 3 3 3 3 1 4 4 4 4 4 4 4 4 4 4 4 4 4	$\begin{array}{c c} r = 1.5\\ \hline \varepsilon_N & \underline{\ell}\\ 2.4E-4\\ 2.1E-6\\ 1.8E-8\\ 7.3E-10\\ \hline \varepsilon'_N & \underline{\varrho'}\\ 4.4E-2\\ 8.9E-3\\ 1.8E-3\\ 6.2E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \underline{\varrho'}\\ 1.2E-4\\ 1.8E-6\\ 2.3E-8\\ 1.2E-9\\ \hline \varepsilon'_N & \underline{\varrho'}\\ 4.2E-2\\ 2.9E-2\\ \end{array}$	$533 \\ (4.9) \\ 4.4 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.1 \\ $	$\begin{array}{c c} r = 2.0\\ \hline \varepsilon_N & \varrho\\ \hline 1.8E-4\\ 3.3E-7\\ 6.4E-10\\ 1.0E-11\\ \hline \varepsilon'_N & \varrho'\\ \hline 3.0E-2\\ 3.5E-3\\ 4.1E-4\\ 9.8E-5\\ \hline r = 2.8\\ \hline \varepsilon_N & \varrho\\ \hline 1.7E-4\\ 2.4E-7\\ 4.3E-10\\ 6.8E-12\\ \hline \varepsilon'_N & \varrho'\\ \hline 3.8E-2\\ 2.1E-2\\ \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 1.2 \\$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ 2.5E-2 \\ 4.0E-4 \\ 6.2E-6 \\ 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ 3.5E-3 \\ 8.8E-5 \\ 1.6E-7 \\ 2.4E-9 \\ \hline \varepsilon'_N & \varrho \\ 3.6E-2 \\ 2.7E-3 \\ \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ \hline 8.0\\ \hline (4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 0(8.0)\\ \hline 1.0\\ \hline 6.6\\ 8.2\\ \hline 8.1\\ \hline (4.0)\\ \hline 1.0\\ 3.9\\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r=1\\ \hline \varepsilon_N & \varrho(2.8\\ \hline 6.8E-4 & 2.\\ \hline 3.3E-5 & 2.\\ \hline 1.5E-6 & 2.\\ \hline 1.9E-7 & 2.\\ \hline \varepsilon_N' & \varrho'(1.4\\ \hline 6.3E-2 & 1.\\ \hline 2.2E-2 & 1.\\ \hline 7.9E-3 & 1.\\ \hline 4.0E-3 & 1.\\ \hline r=1\\ \hline \varepsilon_N & \varrho(2.1\\ \hline 4.2E-4 & 2.\\ \hline 5.3E-5 & 2.\\ \hline 5.8E-6 & 2.\\ \hline 1.3E-6 & 2.\\ \hline \varepsilon_N' & \varrho'(1.1\\ \hline 4.5E-2 & 0.\\ \hline 4.0E-2 & 1.\\ \hline 3.2E-2 & 1.\\ \end{array}$) 5 3 3 3 3 3 3 1 4 4 4 4 4 4 4 4 4 4 4 4 4	$\begin{array}{c c} r = 1.5\\ \hline \varepsilon_N & \underline{\ell}\\ 2.4E-4\\ 2.1E-6\\ 1.8E-8\\ 7.3E-10\\ \hline \varepsilon'_N & \underline{\ell'}\\ 4.4E-2\\ 8.9E-3\\ 1.8E-3\\ 6.2E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \underline{\ell'}\\ 1.2E-4\\ 1.2E-4\\ 1.2E-4\\ 1.2E-9\\ \hline \varepsilon'_N & \underline{\ell'}\\ 4.2E-2\\ 2.9E-2\\ 2.0E-2\\ \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ \hline 4.4 \\ 4.9 \\ 4.9 \\ \hline 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.1$	$\begin{array}{c} r = 2.0\\ \hline \varepsilon_N & \varrho\\ 1.8E-4\\ 3.3E-7\\ 6.4E-10\\ 1.0E-11\\ \hline \varepsilon_N' & \varrho'\\ 3.0E-2\\ 3.5E-3\\ 4.1E-4\\ 9.8E-5\\ \hline r = 2.8\\ \hline \varepsilon_N & \varrho\\ 1.7E-4\\ 2.4E-7\\ 4.3E-10\\ 6.8E-12\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 2.1E-2\\ 1.2E-2\\ \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.3 \\ 8.1 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 1.2 \\$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 8.4E-4 \\ 2.3E-6 \\ 4.7E-9 \\ \hline 7.3E-11 \\ \hline \varepsilon'_N & \varrho \\ \hline 2.5E-2 \\ 4.0E-4 \\ \hline 6.2E-6 \\ \hline 3.9E-7 \\ \hline r = 2 \\ \hline \varepsilon_N & \varrho \\ \hline 3.5E-3 \\ \hline 8.8E-5 \\ \hline 1.6E-7 \\ \hline 2.4E-9 \\ \hline \varepsilon'_N & \varrho \\ \hline 3.6E-2 \\ \hline 2.7E-3 \\ \hline 4.2E-5 \\ \end{array}$	$\begin{array}{r} 4\\ \hline 4\\ \hline 0(8.0)\\ \hline 3.8\\ 7.8\\ 8.0\\ \hline 8.0\\ \hline (4.0)\\ \hline 3.3\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 4.0\\ \hline 0(8.0)\\ \hline 1.0\\ 6.6\\ 8.2\\ \hline 8.1\\ \hline (4.0)\\ \hline 1.0\\ 3.9\\ 4.0\\ \end{array}$

In Table 7 the Gaussian parameters are used. We can see that the agreement with the theoretical estimates is very good, and there is no further improvement in the convergence rate.

Table 7. Method 2; $\eta_1 = \frac{5}{2}$	$\frac{-\sqrt{15}}{10}, \eta_2 =$	$\frac{1}{2}$, $\eta_3 =$	$\frac{5+\sqrt{15}}{10}$
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$\nu = -\frac{1}{4}$	r = 1	1	r = 1.1	89	r = 1.3	878	r = 1	6
Ń	ε_N ϱ	(4.8)	$\varepsilon_N \varrho$	(6.4)	$\varepsilon_N \varrho$	(8.0)	ε_N (2(8.0)
4	2.2E-4	4.7	1.2E-4	6.3	1.0E-4	7.7	1.5E-4	7.1
32	2.0E-6	4.7	4.7E-7	6.4	2.1E-7	8.0	3.2E-7	7.9
256	1.9E-8	4.8	1.8E-9	6.4	4.0E-10	8.0	6.3E-10	8.0
1024	8.4E-10	4.8	4.4E-11	6.4	6.3E-12	8.0	9.8E-12	8.0
N	$\varepsilon'_N \varrho'$	(2.4)	$\varepsilon'_N \varrho'$	(2.8)	$\varepsilon'_N \varrho'$	(3.3)	$\varepsilon'_N \varrho$	'(4.0)
4	1.5E-2	2.3	1.1E-2	2.7	7.8E-3	3.2	6.4E-3	3.9
32	1.1E-3	2.4	5.0E-4	2.8	2.2E-4	3.3	1.0E-4	4.0
256	8.4E-5	2.4	2.3E-5	2.8	6.1E-6	3.3	1.6E-6	4.0
1024	1.5E-5	2.4	2.9E-6	2.8	5.6E-7	3.3	9.9E-8	4.0
$\nu = 0$	r = 1	1	r = 1.2	275	r = 1.5	550	r =	2
N	$\varepsilon_N = \varrho$	(4.0)	$\varepsilon_N \varrho$	(5.9)	$\varepsilon_N \varrho$	(8.0)	ε_N b	2(8.0)
4	1.1E-3	4.0	5.2E-4	5.8	4.2E-4	7.7	7.9E-4	6.6
32	1.7E-5	4.0	2.6E-6	5.9	8.3E-7	8.0	1.8E-6	7.9
256	2.7E-7	4.0	1.3E-8	5.9	1.6E-9	8.0	3.5E-9	8.0
1024	1.7E-8	4.0	3.8E-10	5.9	2.5E-11	8.0	5.5E-11	8.0
N	$\varepsilon'_N \varrho'$	(2.0)	$\varepsilon'_N \varrho'$	(2.4)	$\varepsilon'_N \varrho'$	(2.9)	$\varepsilon'_N \varrho$	'(4.0)
4	8.0E-2	2.0	5.5E-2	2.4	3.8E-2	2.9	2.9E-2	3.9
32	1.0E-2	2.0	3.9E-3	2.4	1.5E-3	2.9	4.6E-4	4.0
256	1.3E-3	2.0	2.8E-4	2.4	6.0E-5	2.9	7.2E-6	4.0
1024	3.2E-4	2.0	4.7E-5	2.4	7.0E-6	2.9	4.5E-7	4.0
$\nu = \frac{1}{2}$	r = 1	1	r = 1.5	533	r = 2.0)67	r =	4
$\nu = \frac{1}{2}$ N	$r = \frac{1}{\varepsilon_N}$	1 (2.8)	$r = 1.5$ $\varepsilon_N \varrho$	533 (4.9)	$r = 2.0$ $\varepsilon_N \varrho$)67 (8.0)	$r = \varepsilon_N$	4 2(8.0)
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = \frac{\varepsilon_N}{6.8\text{E-4}}$	1 (2.8) 2.8	$r = 1.5$ $\varepsilon_N \varrho$ 2.2E-4	533 (4.9) 4.9	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4)67 (8.0) 7.4	$r = \frac{\varepsilon_N}{8.7\text{E-4}}$	4 2(8.0) 4.2
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $	$r = \frac{\varepsilon_N}{\epsilon_N} \frac{\rho}{\rho}$ 6.8E-4 3.0E-5	$ \frac{1}{2.8} \\ 2.8 \\ 2.8 \\ 2.8 $	r = 1.5 $\varepsilon_N \varrho$ 2.2E-4 1.9E-6	533 (4.9) 4.9 4.9	$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4 3.7E-7	067 (8.0) 7.4 8.0	$r = \frac{\varepsilon_N}{8.7\text{E-4}}$ 2.6E-6	$ \frac{4}{2(8.0)} \frac{4.2}{7.6} $
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ 256 $	$r = \frac{\varepsilon_N}{\varepsilon_N} \frac{\varrho}{\varrho}$ 6.8E-4 3.0E-5 1.3E-6	1 (2.8) 2.8 2.8 2.8 2.8	r = 1.5 $\varepsilon_N \varrho$ 2.2E-4 1.9E-6 1.6E-8	533 (4.9) 4.9 4.9 4.9	r = 2.0 $\varepsilon_N \varrho$ 1.8E-4 3.7E-7 7.2E-10	067 (8.0) 7.4 8.0 8.0	$r = \frac{\varepsilon_N}{8.7\text{E-4}}$ 8.7E-4 2.6E-6 5.2E-9	$ \frac{4}{2(8.0)} 4.2 7.6 8.0 $
$\nu = \frac{1}{2}$ N 4 32 256 1024	r = 1 $\varepsilon_N \varrho$ 6.8E-4 3.0E-5 1.3E-6 1.6E-7	1 (2.8) 2.8 2.8 2.8 2.8 2.8	$r = 1.5$ $\varepsilon_N \varrho$ 2.2E-4 1.9E-6 1.6E-8 6.4E-10	533 (4.9) 4.9 4.9 4.9 4.9 4.9	r = 2.0 $\varepsilon_N \varrho$ 1.8E-4 3.7E-7 7.2E-10 1.1E-11	067 (8.0) 7.4 8.0 8.0 8.0	$r = \frac{\varepsilon_N}{8.7\text{E-4}}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11	$ \frac{4}{p(8.0)} 4.2 7.6 8.0 8.0 8.0 $
	$r = \frac{\varepsilon_N}{\varepsilon_N} \frac{\varrho}{\varepsilon_N}$ 6.8E-4 3.0E-5 1.3E-6 1.6E-7 $\varepsilon'_N \varrho'$	$ \begin{array}{r} 1 \\ \hline (2.8) \\ 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ \hline (1.4) $	$r = 1.5$ $\varepsilon_N \varrho$ 2.2E-4 1.9E-6 1.6E-8 6.4E-10 $\varepsilon'_N \varrho'$		$r = 2.0$ $\varepsilon_N \varrho$ 1.8E-4 3.7E-7 7.2E-10 1.1E-11 $\varepsilon'_N \varrho'$	$ \begin{array}{r} 067 \\ \hline (8.0) \\ 7.4 \\ 8.0 \\ 8.0 \\ 8.0 \\ (2.0) \\ \hline $	$r = \frac{\varepsilon_N}{8.7E-4}$ $2.6E-6$ $5.2E-9$ $8.2E-11$ $\varepsilon'_N \varrho$	$ \frac{4}{2(8.0)} \\ \frac{4.2}{7.6} \\ \frac{8.0}{8.0} \\ \frac{6}{7(4.0)} $
	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \varepsilon'_N & \varrho' \\ 5.5E-2 \end{array}$	$ \begin{array}{r} 1 \\ \hline $	$r = 1.5$ $\varepsilon_N \varrho$ 2.2E-4 1.9E-6 1.6E-8 6.4E-10 $\varepsilon'_N \varrho'$ 3.8E-2	$533 \\ (4.9) \\ 4.9 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ $	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon'_N & \varrho' \\ 2.7E-2 \\ \hline \end{array}$	$\begin{array}{r} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ \hline \end{array}$	$r = \frac{\varepsilon_N}{8.7E-4}$ $\frac{2.6E-6}{5.2E-9}$ $\frac{8.2E-11}{\varepsilon'_N}$ $\frac{\varphi'_N}{2.2E-2}$	$ \frac{4}{2(8.0)} \frac{4.2}{7.6} 8.0 \frac{8.0}{7(4.0)} 3.2$
	$r = \frac{\varepsilon_N}{6.8E-4}$ 6.8E-4 3.0E-5 1.3E-6 1.6E-7 $\frac{\varepsilon'_N}{5.5E-2}$ 2.0E-2 2.0E-2	$ \frac{1}{(2.8)} \\ 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ 1.4 \\$	$\begin{array}{c} r = 1.5 \\ \hline \varepsilon_N & \varrho \\ \hline 2.2E-4 \\ 1.9E-6 \\ 1.6E-8 \\ 6.4E-10 \\ \hline \varepsilon_N' & \varrho' \\ \hline 3.8E-2 \\ 7.8E-3 \\ \hline 7.8E-3 \end{array}$	$ \begin{array}{r} 533 \\ \hline 4.9 \\ 4.9 \\ 4.9 \\ 4.9 \\ 4.9 \\ \hline 1.7 \\ $	$r = 2.0$ $\varepsilon_{N} \varrho$ 1.8E-4 3.7E-7 7.2E-10 1.1E-11 $\varepsilon'_{N} \varrho'$ 2.7E-2 3.1E-3 3.1E-3	$ \begin{array}{r} 067 \\ \overline{(8.0)} \\ 7.4 \\ 8.0 \\ 8.0 \\ 8.0 \\ \overline{(2.0)} \\ 2.0 \\ $	$r = \frac{\varepsilon_N}{8.7E-4}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4	$ \frac{4}{2(8.0)} \frac{4.2}{7.6} 8.0 8.0 7(4.0)3.24.04.27.68.0$
	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \hline \varepsilon'_N & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 2.0E-3 \\ \hline 0.5E-2 \\ \hline 0.5E-2$	$ \begin{array}{r} 1 \\ \hline (2.8) \\ 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ 1.4 \\ 1.$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 1.6E-3 \end{array}$	533 (4.9) 4.9 4.9 4.9 (1.7) 1.7 1.7 1.7	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N' & \varrho' \\ 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 5.6E-4 \\ 5$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline (2.0) \\ \hline 2.0 \\$	$r = \frac{\varepsilon_N}{2.6E-6}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 5.4	$ \frac{4}{2(8.0)} \frac{4.2}{7.6} 8.0 7(4.0)3.24.0$
	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \hline \varepsilon'_N & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \end{array}$	$ \begin{array}{r} 1 \\ \hline (2.8) \\ 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ 1.4 \\ 1.$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4 \end{array}$	533 (4.9) 4.9 4.9 4.9 (1.7) 1.7 1.7 1.7 1.7	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N' & \varrho' \\ 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \end{array}$	067 (8.0) 7.4 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0	$r = \frac{\varepsilon_N}{8.7E-4}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7	$ \begin{array}{r} 4 \\ \hline 2(8.0) \\ \hline 4.2 \\ 7.6 \\ 8.0 \\ 8.0 \\ 7(4.0) \\ \hline 3.2 \\ 4.0 \\ 4.0 \\ 4.0 \\ \end{array} $
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \end{array} $	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \varepsilon'_N & \varrho' \\ 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = 1 \end{array}$	$ \begin{array}{r} 1 \\ \hline (2.8) \\ 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ \hline (1.4) \\ 1.4 \\ 1.4 \\ 1$	$\begin{array}{c c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ \hline 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon'_N & \varrho'\\ \hline 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \end{array}$	$533 \\ (4.9) \\ 4.9 \\ 4.9 \\ 4.9 \\ 4.9 \\ (1.7) \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 009 $	$\begin{array}{c c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon'_N & \varrho' \\ 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \end{array}$	067 (8.0) 7.4 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 318	$r = \frac{\varepsilon_N}{8.7E-4}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = 2$	$\begin{array}{c} 4\\ \hline 2(8.0)\\ \hline 4.2\\ 7.6\\ 8.0\\ 8.0\\ \hline 7(4.0)\\ \hline 3.2\\ 4.0\\ 4.0\\ \hline 4.0\\ \hline 20\\ \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \end{array} $	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \varepsilon'_N & \varrho' \\ 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = \\ \varepsilon_N & \varrho \end{array}$	$ \frac{1}{2.8} \\ 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ 1.4 \\ 1$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\end{array}$	533 (4.9) 4.9 4.9 4.9 (1.7) 1.7 1.7 1.7 1.7 1.7 (1.7) (4.3)	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \end{array}$	067 (8.0) 7.4 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 318 (8.0)	$r = \frac{\varepsilon_N}{8.7E-4}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{c}{2}$	$ \begin{array}{r} 4 \\ 2(8.0) \\ 4.2 \\ 7.6 \\ 8.0 \\ 8.0 \\ 7(4.0) \\ 3.2 \\ 4.0 \\ 4.0 \\ 4.0 \\ 20 \\ 2(8.0) \\ \end{array} $
$ \begin{array}{r} \nu = \frac{1}{2} \\ N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ N \\ 4 \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline 256 \\ 1024 \\ 256 \\ 1024 \\ 1024 \\ 256 \\ 1024 $	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \varepsilon'_N & \varrho' \\ 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = \\ \varepsilon_N & \varrho \\ 5.2E-4 \\ \hline \end{array}$	$ \frac{1}{2.8} \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ 1.4 \\ 1$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.2E-4\\ \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ 4.9 \\ 4.9 \\ 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ \hline 1.7 \\ 1.7 \\ \hline (4.3) \\ \hline (4.3) \\ \hline 4.7 \\ \hline \end{array}$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon'_N & \varrho' \\ \hline 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ \hline \end{array}$	067 (8.0) 7.4 8.0 8.0 (2.0) 2.0 2.0 2.0 2.0 2.0 2.0 318 (8.0) 8.8	$r = \frac{\varepsilon_N}{8.7E-4}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{2}{2}$ $\frac{\varepsilon_N}{2.9E-3}$	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ (4.0)\\ \hline \\ 3.2\\ 4.0\\ 4.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ \hline \\ 20\\ \hline \\ 2(8.0)\\ \hline \\ 1.0\\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ \hline N \\ 4 \\ 32 \\ 56 \\ \hline N \\ 4 \\ 32 \\ \hline N \\ 4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1$	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \varepsilon'_N & \varrho' \\ 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = 1 \\ \varepsilon_N & \varrho \\ 5.2E-4 \\ 4.8E-5 \\ \end{array}$	$ \frac{1}{2.8} \\ 2.8 \\ 2.8 \\ 2.8 \\ (1.4) \\ 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ 1.2 \\ 2.2 \\ 2.2 \\ 2.2 \\ 1.4 \\ 1$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ \hline \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ 4.9 \\ 4.9 \\ 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ \hline 1.7 \\ 1.7 \\ \hline (4.3) \\ \hline (4.3) \\ \hline 4.7 \\ 4.3 \\ \hline \end{array}$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N' & \varrho' \\ \hline 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ 2.6E-7 \\ \hline \end{array}$	067 (8.0) 7.4 8.0 8.0 (2.0) 2.0 2.0 2.0 8.18 (8.0) 8.8 8.2	$r = \frac{\varepsilon_N}{2.6E-6}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{c}{2}$ $\frac{\varepsilon_N}{2.9E-3}$ 6.8E-5	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ (4.0)\\ \hline \\ 3.2\\ 4.0\\ 4.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ \hline \\ 20\\ \hline \\ \hline \\ (8.0)\\ \hline \\ 1.0\\ \hline \\ 6.5\\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 4 \\ 32 \\ 56 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ \hline 1.3E-6 \\ \hline 1.6E-7 \\ \hline \varepsilon_N' & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ \hline 3.5E-3 \\ \hline r = \\ \hline \varepsilon_N & \varrho \\ \hline 5.2E-4 \\ \hline 4.8E-5 \\ \hline 4.8E-6 \\ \hline 1.6E-7 \\ \hline 0.6E-7 \\ \hline $	$\begin{array}{c} 1 \\ \hline (2.8) \\ \hline 2.8 \\ 2.8 \\ 2.8 \\ \hline 2.8 \\ \hline (1.4) \\ \hline 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ \hline 1.4 \\ \hline 1.4 \\ \hline 1.4 \\ \hline 2.2 \\ 2.2 \\ 2.1 \\ \hline 1.4 \\ \hline 1.4$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ 1.9E-8\\ 0 \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ 4.9 \\ 4.9 \\ 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ \hline 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 4.3 \\ 4.3 \\ 4.3 \\ 4.3 \\ 1.3 \\ 1.5 \\$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N' & \varrho' \\ \hline 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ 2.6E-7 \\ 4.9E-10 \\ \hline \end{array}$	067 (8.0) 7.4 8.0 8.0 (2.0) 2.0 2.0 2.0 318 (8.0) 8.2 8.0 8.0	$r = \frac{\varepsilon_N}{2.6E-6}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{c}{2.9E-3}$ 6.8E-5 1.7E-7	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ 4.0\\ \hline \\ 20\\ \hline \\ 2(8.0)\\ \hline \\ 1.0\\ \hline \\ 6.5\\ 7.9\\ \hline \\ 7.9\\ \hline \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline 1024 \\ \hline \end{array} $	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ \hline 1.3E-6 \\ \hline 1.6E-7 \\ \hline \varepsilon_N' & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ \hline 7.0E-3 \\ \hline 3.5E-3 \\ \hline r = \\ \hline \varepsilon_N & \varrho \\ \hline 5.2E-4 \\ \hline 4.8E-5 \\ \hline 4.8E-6 \\ \hline 1.0E-6 \\ \hline \end{array}$	$\begin{array}{c} 1 \\ \hline (2.8) \\ \hline 2.8 \\ 2.8 \\ 2.8 \\ \hline 2.8 \\ \hline (1.4) \\ \hline 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ \hline 1.4 \\ 1.4 \\ \hline 1.4 \\ \hline 2.2 \\ 2.1 \\ 2.1 \\ \hline 2.1 \\ 1.1 \\ \hline 1.1 \\ 1.1 \\ \hline 1$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ 1.9E-8\\ 1.0E-9\\ \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ 4.9 \\ 4.9 \\ 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.7 \\ 1.4 \\ 4.3 \\ 4$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N & \varrho' \\ 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ 2.6E-7 \\ 4.9E-10 \\ 7.6E-12 \\ \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 2.0 \\ \hline 8.8 \\ 8.2 \\ 8.0 \\ 8.8 \\ 8.2 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline 8.1 \\ \hline 8.1 \\ \hline 8.2 \\ 8.0 \\ \hline 8.1 \\ \hline 8.1 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.1 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.1 \\ \hline 8.1 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.2 \\ \hline 8.2 \\ \hline 8.0 \\ \hline 8.2 \\ \hline 8.2$	$r = \frac{\varepsilon_N}{2.6E-6}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{2}{2.9E-3}$ 6.8E-5 1.7E-7 2.7E-9	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ 4.0\\ \hline \\ 6.5\\ \hline \\ 7.9\\ \hline \\ 8.0\\ \hline \\ \hline \\ 0(8.0)\\ \hline \\ 1.0\\ \hline \\ 6.5\\ \hline \\ 7.9\\ \hline \\ 8.0\\ \hline \\ \hline \\ 0(8.0)\\ \hline \\ 1.0\\ \hline \\ 6.5\\ \hline \\ 7.9\\ \hline \\ 8.0\\ \hline \\ \hline \\ 0(8.0)\\ \hline 0(8.0)\\ \hline \\ 0(8.0)\\ \hline 0(8$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ \end{array} $	$\begin{array}{c} r = \\ \varepsilon_N & \varrho \\ 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \varepsilon'_N & \varrho' \\ 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = \\ \varepsilon_N & \varrho \\ 5.2E-4 \\ 4.8E-5 \\ 4.8E-6 \\ 1.0E-6 \\ \varepsilon'_N & \varrho' \\ \hline \end{array}$	$\begin{array}{c} 1 \\ \hline (2.8) \\ \hline 2.8 \\ 2.8 \\ 2.8 \\ \hline 2.8 \\ \hline 2.8 \\ \hline (1.4) \\ \hline 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ \hline 1.4 \\ \hline 1.4 \\ \hline 2.2 \\ 2.1 \\ \hline 2.2 \\ 2.1 \\ \hline 2.1 \\ \hline (1.1) \\ \hline \end{array}$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ 1.9E-8\\ 1.0E-9\\ \hline \varepsilon_N' & \varrho'\\ \hline \varepsilon_N' & \varrho'\\ \hline \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ \hline 4.9 \\ 4.9 \\ \hline 4.9 \\ \hline 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ \hline 4.3 \\ \hline (1.1) \\ \hline 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N' & \varrho' \\ \hline 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ 2.6E-7 \\ 4.9E-10 \\ 7.6E-12 \\ \hline \varepsilon_N' & \varrho' \\ \hline \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline (2.0) \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 2.0 \\ \hline 8.0 \\ \hline 8.8 \\ 8.2 \\ 8.0 \\ \hline 8.0 \\ \hline 8.0 \\ \hline (1.2) \\ \hline \end{array}$	$r = \frac{\varepsilon_N}{2}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2}$ 2.2E-2 3.5E-4 5.5E-6 3.4E-7 $r = \frac{2}{2}$ $\frac{\varepsilon_N}{2}$ 2.9E-3 6.8E-5 1.7E-7 2.7E-9 $\frac{\varepsilon'_N}{2}$	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ 4.0\\ \hline \\ 20\\ \hline \\ 8.0\\ \hline \\ 7.9\\ \hline \\ 8.0\\ \hline \\ \hline \\ (4.0)\\ \hline \\ 1.0\\ \hline $
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5$	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ \hline 1.3E-6 \\ \hline 1.6E-7 \\ \hline \varepsilon_N' & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ \hline 7.0E-3 \\ \hline 3.5E-3 \\ \hline r = \\ \hline \varepsilon_N & \varrho \\ \hline 5.2E-4 \\ \hline 4.8E-5 \\ \hline 4.8E-5 \\ \hline 4.8E-6 \\ \hline 1.0E-6 \\ \hline \varepsilon_N' & \varrho' \\ \hline 3.8E-2 \\ \hline 2.5E-2 \\ \hline \end{array}$	$\begin{array}{c} 1 \\ \hline (2.8) \\ \hline 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ \hline 2.8 \\ \hline 2.8 \\ \hline 2.1 \\ \hline (1.4) \\ \hline 1.4 \\ 1.4 \\ 1.4 \\ 1.4 \\ \hline 1.4 \\ \hline 2.2 \\ 2.1 \\ \hline 2.2 \\ 2.1 \\ \hline 2.1 \\ \hline (1.1) \\ 0.9 \\ 0.1 \\ 0$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ 1.9E-8\\ 1.0E-9\\ \hline \varepsilon_N' & \varrho'\\ 3.7E-2\\ 3.7E-2\\ \hline 0.5E-2\\ \hline 0.5E-2$	$\begin{array}{c} 533 \\ \hline (4.9) \\ \hline 4.9 \\ \hline 4.9 \\ \hline 4.9 \\ \hline 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ \hline 4.3 \\ \hline (1.1) \\ \hline 1.0 \\ \hline 1.0 \\ \hline \end{array}$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N' & \varrho' \\ \hline 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ 2.6E-7 \\ 4.9E-10 \\ 7.6E-12 \\ \hline \varepsilon_N' & \varrho' \\ \hline 3.3E-2 \\ 1.9E-7 \\ \hline r = 2.8 \\ \hline r$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 8.0 \\ \hline 8.8 \\ 8.2 \\ 8.0 \\ \hline 8.0 \\ \hline 8.0 \\ \hline 8.0 \\ \hline 1.2 \\ 1.2 \\ \hline 1.2 \\$	$r = \frac{\varepsilon_N}{2.8 \times 7E-4}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{c}{2.9E-3}$ 6.8E-5 1.7E-7 2.7E-9 $\frac{\varepsilon'_N}{2.9E-3}$ 3.8E-2 3.8E-2	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ 4.0\\ \hline \\ 20\\ \hline \\ 2(8.0)\\ \hline \\ 1.0\\ \hline \\ 6.5\\ 7.9\\ \hline \\ 8.0\\ \hline \\ 7.9\\ \hline \\ 8.0\\ \hline \\ 7(4.0)\\ \hline \\ 1.0\\ \hline \\ 0 \end{array}$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 32 \\ 56 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \hline \varepsilon_N' & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = \\ \hline \varepsilon_N & \varrho \\ \hline 5.2E-4 \\ 4.8E-5 \\ 4.8E-5 \\ 4.8E-5 \\ 4.8E-6 \\ 1.0E-6 \\ \hline \varepsilon_N' & \varrho' \\ 3.8E-2 \\ 3.5E-2 \\ 2.0E-2 \\ \hline r = \\ r = $	$\begin{array}{c} 1 \\ \hline (2.8) \\ \hline 2.8 \\ 2.8 \\ 2.8 \\ 2.8 \\ \hline 2.8 \\ \hline 2.8 \\ \hline 2.1 \\ \hline (1.4) \\ \hline 1.4 \\ 1.4 \\ 1.4 \\ \hline 1.4 \\ \hline (2.1) \\ \hline 2.2 \\ 2.1 \\ \hline 2.1 \\ \hline 2.1 \\ \hline (1.1) \\ 0.9 \\ 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ 1.9E-8\\ 1.0E-9\\ \hline \varepsilon_N' & \varrho'\\ 3.7E-2\\ 2.6E-2\\ 1.6E-2\\ \hline \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ \hline 4.9 \\ 4.9 \\ \hline 4.9 \\ \hline 4.9 \\ \hline (1.7) \\ \hline 1.7 \\ \hline 4.3 \\ \hline 1.1 \\ \hline 1.0 \\ \hline 1.1 \\ \hline \end{array}$	$\begin{array}{c} r = 2.0 \\ \hline \varepsilon_N & \varrho \\ \hline 1.8E-4 \\ 3.7E-7 \\ 7.2E-10 \\ 1.1E-11 \\ \hline \varepsilon_N & \varrho' \\ 2.7E-2 \\ 3.1E-3 \\ 3.6E-4 \\ 8.7E-5 \\ \hline r = 2.8 \\ \hline \varepsilon_N & \varrho \\ \hline 1.3E-4 \\ 2.6E-7 \\ 4.9E-10 \\ 7.6E-12 \\ \hline \varepsilon_N & \varrho' \\ 3.3E-2 \\ 1.9E-2 \\ 1.9E-2 \\ \end{array}$	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 8.0 \\ \hline 8.8 \\ 8.2 \\ 8.0 \\ \hline 8.0 \\ \hline 8.0 \\ \hline 1.2 \\ 1.2$	$r = \frac{\varepsilon_N}{2.6E-6}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{2}{2.5E-3}$ 6.8E-5 1.7E-7 2.7E-9 $\frac{\varepsilon'_N}{2.3.8E-2}$ 2.5E-3 3.8E-2 2.5E-3	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 8.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ 4.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ \hline \\ 6.5\\ 7.9\\ \hline \\ 8.0\\ \hline \\ 7.9\\ \hline 7.9\\$
$ \begin{array}{r} \nu = \frac{1}{2} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \nu = \frac{9}{10} \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline N \\ 4 \\ 32 \\ 256 \\ 1024 \\ \hline \end{array} $	$\begin{array}{c} r = \\ \hline \varepsilon_N & \varrho \\ \hline 6.8E-4 \\ 3.0E-5 \\ 1.3E-6 \\ 1.6E-7 \\ \hline \varepsilon_N' & \varrho' \\ \hline 5.5E-2 \\ 2.0E-2 \\ 7.0E-3 \\ 3.5E-3 \\ \hline r = \\ \hline \varepsilon_N & \varrho \\ \hline 5.2E-4 \\ 4.8E-5 \\ 4.8E-5 \\ 4.8E-5 \\ 4.8E-6 \\ 1.0E-6 \\ \hline \varepsilon_N' & \varrho' \\ \hline 3.8E-2 \\ 3.5E-2 \\ 2.8E-2 \\ 2.8E-2 \\ 2.8E-2 \\ 2.6E-2 \\ \hline \end{array}$	$\begin{array}{c} 1\\ \hline (2.8)\\ \hline 2.8\\ 2.8\\ 2.8\\ \hline 2.8\\ 2.8\\ \hline (1.4)\\ \hline 1.4\\ 1.4\\ 1.4\\ \hline 1.4\\ \hline (2.1)\\ \hline 2.2\\ 2.2\\ 2.1\\ \hline 2.1\\ \hline (1.1)\\ \hline 0.9\\ 1.1\\ 1.1\\ 1.1\\ \end{array}$	$\begin{array}{c} r = 1.5\\ \hline \varepsilon_N & \varrho\\ 2.2E-4\\ 1.9E-6\\ 1.6E-8\\ 6.4E-10\\ \hline \varepsilon_N' & \varrho'\\ 3.8E-2\\ 7.8E-3\\ 1.6E-3\\ 5.5E-4\\ \hline r = 1.5\\ \hline \varepsilon_N & \varrho\\ 1.2E-4\\ 1.5E-6\\ 1.9E-8\\ 1.0E-9\\ \hline \varepsilon_N' & \varrho'\\ 3.7E-2\\ 2.6E-2\\ 1.7E-2\\ 1.7E-2\\ 1.7E-2\\ \end{array}$	$\begin{array}{c} 533 \\ \hline (4.9) \\ \hline 4.9 \\ 4.9 \\ \hline 4.9 \\ \hline 4.9 \\ \hline 4.9 \\ \hline 1.7 \\ \hline 4.3 \\ \hline 1.1 \\ \hline 1.0 \\ \hline 1.1 \\ 1.1 \\ \hline 1$	$\begin{array}{c} r = 2.0\\ \hline \varepsilon_N & \varrho\\ 1.8E-4\\ 3.7E-7\\ 7.2E-10\\ 1.1E-11\\ \hline \varepsilon_N' & \varrho'\\ 2.7E-2\\ 3.1E-3\\ 3.6E-4\\ 8.7E-5\\ \hline r = 2.8\\ \hline \varepsilon_N & \varrho\\ 1.3E-4\\ 2.6E-7\\ 4.9E-10\\ 7.6E-12\\ \hline \varepsilon_N' & \varrho'\\ 3.3E-2\\ 1.9E-2\\ 1.0E-2\\ \hline r = 2.8\\ \hline \varepsilon_N' & \varrho'\\ \hline r = 2.8\\ $	$\begin{array}{c} 067 \\ \hline (8.0) \\ \hline 7.4 \\ 8.0 \\ 8.0 \\ \hline 8.0 \\ \hline 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ \hline 2.0 \\ 2.0 \\ \hline 1.2 \\ 1$	$r = \frac{\varepsilon_N}{2.6E-6}$ 8.7E-4 2.6E-6 5.2E-9 8.2E-11 $\frac{\varepsilon'_N}{2.2E-2}$ 3.5E-4 5.5E-6 3.4E-7 $r = \frac{2}{2.5E-3}$ 6.8E-5 1.7E-7 2.7E-9 $\frac{\varepsilon'_N}{2.5E-3}$ 3.8E-2 2.5E-3 3.7E-5	$\begin{array}{c} 4\\ \hline \\ 2(8.0)\\ \hline \\ 4.2\\ 7.6\\ 8.0\\ \hline \\ 7.6\\ 8.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ 4.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ \hline \\ 4.0\\ \hline \\ 6.5\\ 7.9\\ 8.0\\ \hline \\ \hline \\ 7.9\\ 8.0\\ \hline \\ \hline \\ 7.9\\ 8.0\\ \hline \\ \hline \\ (4.0)\\ \hline \\ 1.0\\ \hline \\ 3.8\\ 4.0\\ \hline \\ \end{array}$

However, if we use collocation parameters that are exact for polynomials of order m+1, it is possible to get superconvergence at the collocation points. Brunner et al. [⁸] have proved the following result (see Theorem 2.3 in [⁸]) for nonlinear Volterra integral equation.

Theorem 4. (adapted to the linear case) Let the following conditions be fulfilled:

- (V1) The kernel K(t, s) is $m' + \mu' + 1$ times $(m', \mu' \in \mathbb{Z}, m' \ge 1, 0 \le \mu' \le m' 1)$ continuously differentiable with respect to t, s for $t \in [0, T], s \in [0, t)$, and satisfies (4) with $i + j \le m' + \mu' + 1, \nu' \in (-\infty, 1)$. (V2) $f \in C^{m' + \mu' + 1, \nu'}(0, T]$.
- (V3) The collocation points (6) are generated by the knots η_j , j = 1, ..., m' of a quadrature formula $\int_0^1 \phi(\xi) d\xi \approx \sum_{i=1}^m w_i \phi(\eta_i)$, $0 \le \eta_1 < ... < \eta_{m'} \le 1$, which is exact for all polynomials of degree $m' + \mu'$, $0 \le \mu' \le m' 1$.

(V4) The scaling parameter $r = r(m', \nu', \mu') \ge 1$ is subject to the restrictions

$$r > \frac{m'}{1 - \nu'}, r \ge \frac{m' + 1 - \nu'}{2 - \nu'} \quad if \quad 1 - \nu' < \mu' + 1,$$

$$r > \frac{m'}{1 - \nu'}, r > \frac{m' + \mu' + 1}{2 - \nu'} \quad if \quad \mu' + 1 \le 1 - \nu' < m',$$

$$r \ge \frac{m' + \mu' + 1}{2 - \nu'}, r > 1 \qquad if \quad 1 - \nu' = m',$$

$$r \ge \frac{m' + \mu' + 1}{2 - \nu'} \qquad if \quad 1 - \nu' > m'.$$
(16)

Then the approximate solution $u \in S_{m'}^{(-1)}(\Pi_N^r)$ of the equation

$$y(t) = \int_{0}^{t} K(t,s)y(s)\mathrm{d}s + f(t), \quad 0 \le t \le T,$$

satisfies the error estimate

$$\max_{\substack{k=1,\dots,m;\\j=1,\dots,N}} |u(t_{jk}) - y(t_{jk})| \le cN^{-m'} \begin{cases} N^{-1} & \text{if } \nu' < 0, \\ N^{-1}(1 + \log N) & \text{if } \nu' = 0, \\ N^{-(1-\nu')} & \text{if } \nu' > 0, \end{cases}$$
(17)

where c is a positive constant which is independent of N.

For superconvergence at the collocation points for sufficiently large values of r, in case of Volterra integro-differential equations (1), we obtain the following result from Theorem 4.

Theorem 5. Let the following conditions be fulfilled: 1) $p \in C^{m+2}[0,T], q \in C^{m+1,\nu}[0,T], K \in \mathcal{W}^{m+2,\nu}(\Delta_T), m \in \mathbb{N}, -\infty < \nu < 1.$

- 2) The collocation points (6), where k = 1, ..., m + 1, are generated by the grid points $t_j = T(j/N)^r$, j = 0, ..., N, and by the knots η_j , j = 1, ..., m + 1, of a quadrature approximation $\int_0^1 \phi(s) ds \approx \sum_{q=1}^{m+1} A_q \phi(\eta_q)$, $0 \le \eta_1 < ... < \eta_{m+1} \le 1$, with appropriate weights $\{A_q\}$, which is exact for all polynomials of degree m + 1.
- 3) The scaling parameter $r = r(m, \nu) \ge 1$ satisfies the inequality $r > \frac{m+1}{2-\nu}$. Then, with the notation of Theorem (3), the error estimate

$$\max_{k=1,\dots,m+1; j=1,\dots,N} |u(t_{jk}) - y(t_{jk})| \le cN^{-m-2}$$
(18)

holds, where c is a positive constant which is independent of N.

Proof. We consider the linear Volterra integral equation (13)

$$y(t) = f_2(t) + \int_0^t K_2(t,s)y(s)ds, \quad t \in [0,T],$$

where

$$f_2(t) = y_0 + \int_0^t q(s) ds, \qquad K_2(t,s) = p(s) + \int_s^t K(\tau,s) d\tau.$$

It is easy to check that if $p \in C^{m+2}[0,T]$, $K \in \mathcal{W}^{m+2,\nu}(\Delta_T)$, then $K_2 \in \mathcal{W}^{m+2,\nu-1}(\Delta_T)$ and if $q \in C^{m+1,\nu}[0,T]$, then $f_2 \in C^{m+2,\nu-1}[0,T]$.

In our notation, the assumptions (V1)–(V3) of Theorem 4 are satisfied if m' = m + 1, $\nu' = \nu - 1 < 0$, and $\mu' = 0$. It remains to check the restrictions on the scaling parameter $r = r(m, \nu) \ge 1$. Assume that $r = r(m, \nu) \ge 1$ satisfies the inequality $r > \frac{m+1}{2-\nu}$. We shall show that then it also satisfies the conditions (16).

We see that the first case $1 - \nu' < \mu' + 1$ (i.e. $2 - \nu < 1$) never holds.

For the second case $\mu' + 1 \le 1 - \nu' < m'$, i.e. $1 \le 2 - \nu < m + 1$, i.e. $0 \le 1 - \nu < m$ we must show that

$$r > \frac{m'}{1-\nu'} = \frac{m+1}{2-\nu},$$

which is satisfied, and

$$r > \frac{m' + \mu' + 1}{2 - \nu'} = \frac{m + 2}{3 - \nu}.$$
(19)

For the last equality we find the difference

$$\frac{m+1}{2-\nu} - \frac{m+2}{3-\nu} = \frac{m+\nu-1}{(2-\nu)(2-\nu)},$$

which is greater than 0, because $m > 1-\nu$, i.e. $m+\nu-1 > 0$, and the denominator is positive as well. It means that the condition (19) is also satisfied.

If $1 - \nu' = m'$, i.e. $2 - \nu = m + 1$, then $r \ge \frac{m' + \mu' + 1}{2 - \nu'} = \frac{1 - \nu' + 1}{2 - \nu'} = 1$ and other restriction r > 1 gives us the condition r > 1, which is satisfied, since in our case $r > \frac{m+1}{2-\nu} = 1$.

If $1 - \nu' > m'$, then $2 - \nu' > m' + 1$ and we get that $r \ge \frac{m' + \mu' + 1}{2 - \nu'} = \frac{m' + 1}{2 - \nu'}$ follows from the condition $r \ge 1$.

We have shown that the restrictions (16) are satisfied. Since $\nu' = \nu - 1 < 0$, we get from (17) the error estimate (18).

Numerical results about superconvergence are presented in Tables 8–10. The errors at the collocation points are denoted by

$$\delta_N = \{ \max |u(t_{jk}) - y(t_{jk})| : k = 1, \dots, m+1; \ j = 1, \dots, N \}.$$

In Table 8 we can see that for sufficiently large values of r $(r > \frac{m+1}{2-\nu})$ the numerical results are in good accordance with the theoretical error estimate of Theorem 5. The notation $\rho(?)$ in this table indicates that the theoretical convergence rate is unknown for the corresponding values of r and ν .

$\nu = -\frac{1}{4}$	r = 1	r = 1.189	r = 1.378	r = 1.6
N	$\delta_N \qquad \varrho(?)$	$\delta_N \qquad \varrho(?)$	$\delta_N \qquad \varrho(16.0)$	$\delta_N \varrho(16.0)$
4	2.3E-5 9.1	1.1E-5 12.6	1.3E-5 14.9	2.1E-5 13.2
32	2.8E-8 9.4	4.1E-9 14.2	3.4E-9 15.8	5.7E-9 15.8
256	3.3E-11 9.5	1.4E-12 14.3	8.5E-13 15.9	1.4E-12 16.0
1024	3.7E-13 9.5	6.9E-15 14.4	3.1E-15 16.8	4.4E-15 19.4
$\nu = 0$	r = 1	r = 1.275	r = 1.550	r=2
N	$\delta_N \qquad \varrho(?)$	$\delta_N \qquad \varrho(?)$	$\delta_N \qquad \varrho(16.0)$	$\delta_N \varrho(16.0)$
4	9.9E-5 6.4	3.4E-5 11.3	2.9E-5 15.3	7.1E-5 12.2
32	2.4E-7 7.7	1.8E-8 13.0	7.0E-9 16.1	1.6E-8 16.5
256	4.9E-10 8.0	7.0E-12 13.7	1.7E-12 16.0	3.8E-12 16.1
1024	7.8E-12 8.0	3.7E-14 13.8	6.7E-15 15.9	1.5E-14 16.0
$\nu = \frac{1}{2}$	r = 1	r = 1.533	r = 2.067	r = 4
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\delta_N \varrho(?)$	$r = 1.533$ $\delta_N \qquad \varrho(?)$	$r = 2.067$ $\delta_N \varrho(16.0)$	$r = 4$ $\delta_N \varrho(16.0)$
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = 1$ $\delta_N \varrho(?)$ 1.2E-4 5.4	r = 1.533 $\delta_N \varrho(?)$ 3.1E-5 11.8	r = 2.067 $\delta_N \varrho(16.0)$ 5.1E-5 13.2	r = 4 $\delta_N \varrho(16.0)$ 3.7E-4 5.8
$\frac{\nu = \frac{1}{2}}{\frac{N}{4}}$	$r = 1$ $\delta_N \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0	$r = 1.533$ $\delta_N \varrho(?)$ 3.1E-5 11.8 1.5E-8 13.1	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0	$r = 4$ $\delta_N \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{1}{4} $ $ \frac{32}{256} $	$r = 1$ $\delta_{N} \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0 2.9E-9 5.9	$r = 1.533$ $\frac{\delta_N}{3.1\text{E-5}} \frac{\varrho(?)}{11.8}$ $1.5\text{E-8} 13.1$ $6.2\text{E-12} 13.6$	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0	$r = 4$ $\delta_{N} \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $	$r = 1$ $\delta_N \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0 2.9E-9 5.9 8.6E-11 5.8	$\begin{array}{c} r = 1.533 \\ \hline \delta_N & \varrho(?) \\ \hline 3.1E-5 & 11.8 \\ \hline 1.5E-8 & 13.1 \\ \hline 6.2E-12 & 13.6 \\ \hline 3.3E-14 & 13.7 \end{array}$	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0 1.1E-14 16.5	$r = 4$ $\overline{\delta_N} \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9 1.6E-13 16.3
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{\nu} $	$r = 1$ $\delta_{N} \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0 2.9E-9 5.9 8.6E-11 5.8 $r = 1$	$\begin{array}{c} r = 1.533 \\ \hline \delta_N & \varrho(?) \\ 3.1E-5 & 11.8 \\ 1.5E-8 & 13.1 \\ 6.2E-12 & 13.6 \\ 3.3E-14 & 13.7 \\ \hline r = 1.909 \end{array}$	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0 1.1E-14 16.5 $r = 2.818$	$r = 4$ $\delta_N \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9 1.6E-13 16.3 $r = 20$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{N} $	$r = 1$ $\delta_{N} \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0 2.9E-9 5.9 8.6E-11 5.8 $r = 1$ $\delta_{N} \varrho(?)$	$\begin{array}{c c} r = 1.533 \\ \hline \delta_N & \varrho(?) \\ \hline 3.1E-5 & 11.8 \\ \hline 1.5E-8 & 13.1 \\ \hline 6.2E-12 & 13.6 \\ \hline 3.3E-14 & 13.7 \\ \hline r = 1.909 \\ \hline \delta_N & \varrho(?) \end{array}$	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0 1.1E-14 16.5 $r = 2.818$ $\delta_N \varrho(16.0)$	$r = 4$ $\delta_N \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9 1.6E-13 16.3 $r = 20$ $\delta_N \varrho(16.0)$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{N} $ $ \frac{4}{4} $	$r = 1$ $\delta_{N} \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0 2.9E-9 5.9 8.6E-11 5.8 $r = 1$ $\delta_{N} \varrho(?)$ 3.3E-4 2.7	$\begin{array}{c c} r = 1.533 \\ \hline \delta_N & \varrho(?) \\ \hline 3.1E-5 & 11.8 \\ \hline 1.5E-8 & 13.1 \\ \hline 6.2E-12 & 13.6 \\ \hline 3.3E-14 & 13.7 \\ \hline r = 1.909 \\ \hline \delta_N & \varrho(?) \\ \hline 5.4E-5 & 6.9 \\ \end{array}$	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0 1.1E-14 16.5 $r = 2.818$ $\delta_N \varrho(16.0)$ 7.0E-5 9.4	$r = 4$ $\delta_N \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9 1.6E-13 16.3 $r = 20$ $\delta_N \varrho(16.0)$ 1.9E-3 1.0
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{1024}{10} $ $ \frac{1024}{N} $ $ \frac{1024}{10} $ $ \frac{1024}{10} $	$r = 1$ $\delta_{N} \varrho(?)$ 1.2E-4 5.4 6.0E-7 6.0 2.9E-9 5.9 8.6E-11 5.8 $r = 1$ $\delta_{N} \varrho(?)$ 3.3E-4 2.7 5.6E-6 4.3	$\begin{array}{c c} r = 1.533 \\ \hline \delta_N & \varrho(?) \\ \hline 3.1E-5 & 11.8 \\ \hline 1.5E-8 & 13.1 \\ \hline 6.2E-12 & 13.6 \\ \hline 3.3E-14 & 13.7 \\ \hline r = 1.909 \\ \hline \delta_N & \varrho(?) \\ \hline 5.4E-5 & 6.9 \\ \hline 3.0E-8 & 13.1 \\ \end{array}$	$r = 2.067$ $\delta_N \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0 1.1E-14 16.5 $r = 2.818$ $\delta_N \varrho(16.0)$ 7.0E-5 9.4 1.7E-8 16.4	$r = 4$ $\delta_N \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9 1.6E-13 16.3 $r = 20$ $\delta_N \varrho(16.0)$ 1.9E-3 1.0 3.3E-5 8.0
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{N} $ $ \frac{4}{32} $ $ \frac{32}{256} $	$\begin{array}{c} r = 1 \\ \hline \delta_N & \varrho(?) \\ 1.2E-4 & 5.4 \\ 6.0E-7 & 6.0 \\ 2.9E-9 & 5.9 \\ 8.6E-11 & 5.8 \\ \hline r = 1 \\ \hline \delta_N & \varrho(?) \\ 3.3E-4 & 2.7 \\ 5.6E-6 & 4.3 \\ 6.7E-8 & 4.5 \\ \end{array}$	$\begin{array}{c c} r = 1.533 \\ \hline \delta_N & \varrho(?) \\ \hline 3.1E-5 & 11.8 \\ \hline 1.5E-8 & 13.1 \\ \hline 6.2E-12 & 13.6 \\ \hline 3.3E-14 & 13.7 \\ \hline r = 1.909 \\ \hline \delta_N & \varrho(?) \\ \hline 5.4E-5 & 6.9 \\ \hline 3.0E-8 & 13.1 \\ \hline 1.2E-11 & 14.0 \\ \end{array}$	$r = 2.067$ $\overline{\delta_N} \varrho(16.0)$ 5.1E-5 13.2 1.2E-8 16.0 3.1E-12 16.0 1.1E-14 16.5 $r = 2.818$ $\overline{\delta_N} \varrho(16.0)$ 7.0E-5 9.4 1.7E-8 16.4 4.0E-12 16.0	$r = 4$ $\overline{\delta_N} \varrho(16.0)$ 3.7E-4 5.8 1.6E-7 15.0 4.1E-11 15.9 1.6E-13 16.3 $r = 20$ $\overline{\delta_N} \varrho(16.0)$ 1.9E-3 1.0 3.3E-5 8.0 9.9E-9 16.7

Table 8. Method 2; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{3}{4}$

By analysing the numerical results corresponding to smaller values of r ($r < \frac{m+1}{2-\nu}$) we can deduce the following conjecture:

Conjecture 1. *Let the conditions* 1 *and* 2 *of Theorem* 5 *be fulfilled. Then, with the notation of Theorem* 3, *the error estimate*

$$\max_{\substack{k=1,\dots,m+1;\\j=1,\dots,N}} |u(t_{jk}) - y(t_{jk})| \le c \begin{cases} N^{-r(3-\nu)} & for \quad 1 \le r < \frac{m+2}{3-\nu}, \\ N^{-r(3-\nu)}(1+\log N) & for \quad r = \frac{m+2}{3-\nu}, \\ N^{-m-2} & for \quad r > \frac{m+2}{3-\nu} \end{cases}$$
(20)

holds, where c is a positive constant, which is independent of N.

To confirm the error estimate (20), Table 9 is presented, where the theoretical convergence rate corresponding to Conjecture 1 is typed in **bold-face**.

As we can see in Table 9, the observed errors are in good agreement with the estimate (20). Similarly to Table 1, if r is close to the value $\frac{m+2}{3-\nu} = 1.6$, after which the maximal convergence rate is achieved, the observed convergence rate is smaller than the predicted one, but approaches slowly to the predicted theoretical value.

In Table 10 we have used the Gaussian parameters. As we can see, the numerical experiments in this case are in good agreement with the error estimate (20) and do not give any further improvement in the convergence rate. In this table the theoretical convergence rates that do not follow from Theorem 5 are again typed in bold-face.

	r = 1	r = 1.1	r = 1.2	r = 1.3
N	$\delta_N \qquad \varrho(5.7)$	$\delta_N \qquad \varrho(6.7)$	$\delta_N \qquad \varrho(8.0)$	$\delta_N \varrho(9.5)$
4	1.2E-4 5.4	8.7E-5 6.4	6.1E-5 7.7	4.3E-5 9.1
32	6.0E-7 6.0	2.4E-7 7.2	1.0E-7 8.5	4.6E-8 10.0
256	2.9E-9 5.9	7.1E-10 6.9	1.7E-10 8.2	5.1E-11 9.6
1024	8.6E-11 5.8	1.5E-11 6.8	2.6E-12 8.1	5.6E-13 9.5
	r = 1.4	r = 1.5	r = 1.6	r = 1.7
N	$\delta_N \varrho(11.3)$	$\delta_N \varrho(13.5)$	$\delta_N \varrho(16.0)$	$\delta_N \varrho(16.0)$
4	3.6E-5 9.1	3.2E-5 10.9	3.1E-5 12.8	3.3E-5 13.9
32	2.6E-8 11.4	1.7E-8 12.7	1.3E-8 13.9	1.1E-8 14.8
256	1.8E-11 11.3	7.7E-12 13.1	4.3E-12 14.5	3.1E-12 15.3
1024	1.4E-13 11.3	4.5E-14 13.2	2.0E-14 14.7	1.3E-14 15.6
	r = 1.8	r = 1.9	r = 2.0	r = 2.1
N	$\delta_N \varrho(16.0)$	$\delta_N \varrho(16.0)$	$\delta_N \varrho(16.0)$	$\delta_N \varrho(16.0)$
4	3.4E-5 14.8	3.9E-5 14.5	4.6E-5 13.7	5.4E-5 13.0
32	1.0E-8 15.3	1.1E-8 15.7	1.1E-8 15.9	1.3E-8 16.0
256	2.7E-12 15.8	2.6E-12 15.9	2.8E-12 16.0	3.2E-12 16.0
1024	1.0E-14 16.0	1.0E-14 16.2	1.1E-14 16.2	1.2E-14 16.6

Table 9. Method 2; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{3}{4}$, and $\nu = \frac{1}{2}$

$\nu = -\frac{1}{4}$	r = 1	r = 1.189	r = 1.378	r = 1.6
$\frac{4}{N}$	$\delta_N \varrho(9.5)$	$\delta_N \varrho(14.6)$	$\delta_N = \varrho(16.0)$	$\delta_N = \varrho(16.0)$
4	7.3E-6 9.2	3.1E-6 14.2	4.7E-6 14.3	7.8E-6 12.6
32	8.7E-9 9.5	1.0E-9 14.5	1.2E-9 15.8	2.2E-9 15.7
256	1.0E-11 9.5	3.4E-13 14.6	3.1E-13 15.8	5.6E-13 15.9
1024	1.1E-13 9.5	4.1E-15 5.7	4.8E-15 4.7	5.9E-15 6.5
$\nu = 0$	r = 1	r = 1.275	r = 1.550	r=2
N	$\delta_N \varrho(8.0)$	$\delta_N \varrho(14.2)$	$\delta_N \varrho(16.0)$	$\delta_N \varrho(16.0)$
4	3.6E-5 7.1	1.2E-5 12.6	1.9E-5 12.4	4.3E-5 9.8
32	7.9E-8 7.8	4.6E-9 14.0	5.6E-9 15.6	1.5E-8 15.2
256	1.6E-10 8.0	1.6E-12 14.2	1.4E-12 16.0	3.9E-12 15.9
1024	2.5E-12 8.0	8.1E-15 14.2	5.5E-15 15.8	1.5E-14 16.0
$\nu = \frac{1}{2}$	r = 1	r = 1.533	r = 2.067	r = 4
$\frac{\nu = \frac{1}{2}}{N}$	$r = 1$ $\delta_N \varrho(5.7)$	r = 1.533 $\delta_N \varrho(14.2)$	$r = 2.067$ $\delta_N \varrho(16.0)$	$r = 4$ $\delta_N \varrho(16.0)$
$\frac{\nu = \frac{1}{2}}{N}$	r = 1 $\delta_N \varrho(5.7)$ 3.8E-5 5.8	$r = 1.533$ $\delta_N \varrho(14.2)$ 6.9E-6 14.0	r = 2.067 $\delta_N \varrho(16.0)$ 1.6E-5 12.9	r = 4 $\delta_N \varrho(16.0)$ 1.2E-4 5.9
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $	$r = 1$ $\delta_N \varrho(5.7)$ 3.8E-5 5.8 1.9E-7 5.8	$r = 1.533$ $\delta_N \varrho(14.2)$ 6.9E-6 14.0 2.0E-9 14.8	$\frac{r = 2.067}{\delta_N} \frac{\rho(16.0)}{\rho(16.0)}$ 1.6E-5 12.9 3.7E-9 16.4	$r = 4$ $\delta_N \varrho(16.0)$ 1.2E-4 5.9 5.1E-8 15.4
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{1}{4} $ $ \frac{32}{256} $	$\begin{array}{c} r = 1 \\ \hline \delta_N \varrho(\textbf{5.7}) \\ \hline 3.8E-5 5.8 \\ 1.9E-7 5.8 \\ 9.7E-10 5.7 \end{array}$	$\begin{array}{c} r = 1.533 \\ \overline{\delta_N} \varrho(\textbf{14.2}) \\ 6.9\text{E-}6 14.0 \\ 2.0\text{E-}9 14.8 \\ 6.7\text{E-}13 14.4 \end{array}$	$\begin{array}{c} r = 2.067 \\ \overline{\delta_N} \varrho(16.0) \\ 1.6E-5 12.9 \\ 3.7E-9 16.4 \\ 8.7E-13 16.1 \end{array}$	$r = 4$ $\delta_N \varrho(16.0)$ 1.2E-4 5.9 5.1E-8 15.4 1.2E-11 16.2
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ 256 $ $ 1024 $	$\begin{array}{c} r = 1 \\ \hline \delta_N \varrho(\textbf{5.7}) \\ \hline 3.8E-5 5.8 \\ 1.9E-7 5.8 \\ 9.7E-10 5.7 \\ \hline 3.0E-11 5.7 \end{array}$	$\begin{array}{c} r = 1.533 \\ \overline{\delta_N} \varrho(\textbf{14.2}) \\ 6.9E-6 14.0 \\ 2.0E-9 14.8 \\ 6.7E-13 14.4 \\ 3.3E-15 14.3 \end{array}$	$\begin{array}{c c} r = 2.067 \\ \hline \delta_N & \varrho(16.0) \\ \hline 1.6E-5 & 12.9 \\ \hline 3.7E-9 & 16.4 \\ \hline 8.7E-13 & 16.1 \\ \hline 5.3E-15 & 10.8 \end{array}$	$\begin{array}{c c} r = 4 \\ \hline \delta_N & \varrho(16.0) \\ \hline 1.2E-4 & 5.9 \\ \hline 5.1E-8 & 15.4 \\ \hline 1.2E-11 & 16.2 \\ \hline 5.1E-14 & 15.0 \\ \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{\nu} $	$r = 1$ $\delta_{N} \varrho(5.7)$ 3.8E-5 5.8 1.9E-7 5.8 9.7E-10 5.7 3.0E-11 5.7 $r = 1$	$\begin{array}{c} r = 1.533 \\ \overline{\delta_N} \varrho(\textbf{14.2}) \\ 6.9E-6 14.0 \\ 2.0E-9 14.8 \\ 6.7E-13 14.4 \\ 3.3E-15 14.3 \\ \hline r = 1.909 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \delta_N & \varrho(16.0) \\ 1.6E-5 & 12.9 \\ 3.7E-9 & 16.4 \\ 8.7E-13 & 16.1 \\ 5.3E-15 & 10.8 \\ \hline r = 2.818 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \delta_N \varrho(16.0) \\ 1.2E-4 5.9 \\ 5.1E-8 15.4 \\ 1.2E-11 16.2 \\ 5.1E-14 15.0 \\ \hline r = 20 \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{N} $	$\begin{array}{c} r = 1 \\ \hline \delta_N \varrho(\textbf{5.7}) \\ 3.8E-5 5.8 \\ 1.9E-7 5.8 \\ 9.7E-10 5.7 \\ 3.0E-11 5.7 \\ \hline r = 1 \\ \hline \delta_N \varrho(\textbf{4.3}) \end{array}$	$\begin{array}{c} r = 1.533 \\ \overline{\delta_N} \varrho(\textbf{14.2}) \\ 6.9E-6 14.0 \\ 2.0E-9 14.8 \\ 6.7E-13 14.4 \\ 3.3E-15 14.3 \\ \hline r = 1.909 \\ \overline{\delta_N} \varrho(\textbf{16.0}) \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \delta_N \varrho(16.0) \\ 1.6E-5 12.9 \\ 3.7E-9 16.4 \\ 8.7E-13 16.1 \\ 5.3E-15 10.8 \\ \hline r = 2.818 \\ \hline \delta_N \varrho(16.0) \end{array}$	$\begin{array}{c} r = 4 \\ \hline \delta_N & \varrho(16.0) \\ 1.2E-4 & 5.9 \\ 5.1E-8 & 15.4 \\ 1.2E-11 & 16.2 \\ 5.1E-14 & 15.0 \\ \hline r = 20 \\ \hline \delta_N & \varrho(16.0) \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{N} $ $ \frac{4}{4} $	$\begin{array}{c} r = 1 \\ \overline{\delta_N} \varrho(\textbf{5.7}) \\ 3.8E-5 5.8 \\ 1.9E-7 5.8 \\ 9.7E-10 5.7 \\ 3.0E-11 5.7 \\ \hline r = 1 \\ \overline{\delta_N} \varrho(\textbf{4.3}) \\ 1.5E-4 3.5 \end{array}$	$\begin{array}{c} r = 1.533 \\ \overline{\delta_N} \varrho(\textbf{14.2}) \\ 6.9E-6 14.0 \\ 2.0E-9 14.8 \\ 6.7E-13 14.4 \\ 3.3E-15 14.3 \\ \hline r = 1.909 \\ \overline{\delta_N} \varrho(\textbf{16.0}) \\ 2.4E-5 15.7 \end{array}$	$\begin{array}{c} r = 2.067 \\ \overline{\delta_N} \varrho(16.0) \\ 1.6E-5 12.9 \\ 3.7E-9 16.4 \\ 8.7E-13 16.1 \\ 5.3E-15 10.8 \\ \hline r = 2.818 \\ \overline{\delta_N} \varrho(16.0) \\ 9.1E-5 10.0 \end{array}$	$\begin{array}{c} r = 4 \\ \hline \delta_N & \varrho(16.0) \\ \hline 1.2E-4 & 5.9 \\ 5.1E-8 & 15.4 \\ 1.2E-11 & 16.2 \\ 5.1E-14 & 15.0 \\ \hline r = 20 \\ \hline \delta_N & \varrho(16.0) \\ \hline 2.5E-3 & 1.0 \\ \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} $ $ \frac{4}{32} $ $ \frac{256}{1024} $ $ \frac{\nu = \frac{9}{10}}{N} $ $ \frac{4}{32} $	$\begin{array}{c} r = 1 \\ \hline \delta_N \varrho(\textbf{5.7}) \\ \hline 3.8E-5 5.8 \\ 1.9E-7 5.8 \\ 9.7E-10 5.7 \\ \hline 3.0E-11 5.7 \\ \hline r = 1 \\ \hline \delta_N \varrho(\textbf{4.3}) \\ \hline 1.5E-4 3.5 \\ 2.1E-6 4.4 \end{array}$	$\begin{array}{c} r = 1.533 \\ \overline{\delta_N} \varrho(\textbf{14.2}) \\ 6.9E-6 14.0 \\ 2.0E-9 14.8 \\ 6.7E-13 14.4 \\ 3.3E-15 14.3 \\ \hline r = 1.909 \\ \overline{\delta_N} \varrho(\textbf{16.0}) \\ 2.4E-5 15.7 \\ 4.5E-9 17.0 \end{array}$	$\begin{array}{c} r = 2.067 \\ \hline \delta_N \varrho(16.0) \\ \hline 1.6E-5 12.9 \\ \hline 3.7E-9 16.4 \\ \hline 8.7E-13 16.1 \\ \hline 5.3E-15 10.8 \\ \hline r = 2.818 \\ \hline \delta_N \varrho(16.0) \\ \hline 9.1E-5 10.0 \\ \hline 2.1E-8 16.3 \\ \end{array}$	$\begin{array}{c} r = 4 \\ \hline \delta_N & \varrho(16.0) \\ \hline 1.2E-4 & 5.9 \\ \hline 5.1E-8 & 15.4 \\ \hline 1.2E-11 & 16.2 \\ \hline 5.1E-14 & 15.0 \\ \hline r = 20 \\ \hline \delta_N & \varrho(16.0) \\ \hline 2.5E-3 & 1.0 \\ \hline 4.0E-5 & 8.8 \\ \end{array}$
$ \frac{\nu = \frac{1}{2}}{N} \\ \frac{4}{32} \\ 256} \\ 1024 \\ \frac{\nu = \frac{9}{10}}{N} \\ \frac{4}{32} \\ 256 $	$\begin{array}{c} r = 1 \\ \hline \delta_N \varrho(\textbf{5.7}) \\ \hline 3.8E-5 5.8 \\ 1.9E-7 5.8 \\ 9.7E-10 5.7 \\ \hline 3.0E-11 5.7 \\ \hline r = 1 \\ \hline \delta_N \varrho(\textbf{4.3}) \\ \hline 1.5E-4 3.5 \\ \hline 2.1E-6 4.4 \\ \hline 2.3E-8 4.5 \\ \end{array}$	$\begin{array}{c} r = 1.533 \\ \hline \delta_N \varrho(\textbf{14.2}) \\ \hline 6.9\text{E-}6 14.0 \\ 2.0\text{E-}9 14.8 \\ \hline 6.7\text{E-}13 14.4 \\ 3.3\text{E-}15 14.3 \\ \hline r = 1.909 \\ \hline \delta_N \varrho(\textbf{16.0}) \\ \hline 2.4\text{E-}5 15.7 \\ \hline 4.5\text{E-}9 17.0 \\ 9.9\text{E-}13 15.8 \\ \end{array}$	$\begin{array}{c c} r = 2.067 \\ \hline \delta_N & \varrho(16.0) \\ \hline 1.6E-5 & 12.9 \\ \hline 3.7E-9 & 16.4 \\ \hline 8.7E-13 & 16.1 \\ \hline 5.3E-15 & 10.8 \\ \hline r = 2.818 \\ \hline \delta_N & \varrho(16.0) \\ \hline 9.1E-5 & 10.0 \\ \hline 2.1E-8 & 16.3 \\ \hline 4.6E-12 & 16.6 \\ \end{array}$	$\begin{array}{c} r = 4 \\ \hline \delta_N & \varrho(16.0) \\ \hline 1.2E-4 & 5.9 \\ \hline 5.1E-8 & 15.4 \\ \hline 1.2E-11 & 16.2 \\ \hline 5.1E-14 & 15.0 \\ \hline r = 20 \\ \hline \delta_N & \varrho(16.0) \\ \hline 2.5E-3 & 1.0 \\ \hline 4.0E-5 & 8.8 \\ \hline 1.2E-8 & 15.6 \\ \end{array}$

Table 10. Method 2; $\eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$

6. COMPARISON OF METHODS 1 AND 2

Theoretical estimates and numerical experiments show that, in terms of uniform convergence, Method 2, with arbitrary collocation parameters, for computing approximate solution, as well as an approximation for the derivative of the solution, is equivalent to Method 1 if the conditions of Theorem 2 are satisfied.

An advantage of Method 2 is that if we use a special choice of collocation parameters, it is possible to obtain faster convergence (superconvergence) at the collocation points.

Considering that in the case of Method 2 with $u \in S_m^{(-1)}(\Pi_N)$, the complexity of implementation and computation time are comparable to those of Method 1 with $v \in S_m^{(-1)}(\Pi_N)$ ($u \in S_{m+1}^{(0)}(\Pi_N)$), Method 1 seems to be preferable to Method 2 if the assumptions of Theorem 2 hold.

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Teoreetiliste veahinnangute optimaalsus lineaarse nõrgalt singulaarse Volterra integro-diferentsiaalvõrrandi lahendamisel splain-kollokatsioonimeetoditega

Inga Parts

On vaadeldud kahte splain-kollokatsioonimeetodit Volterra integro-diferentsiaalvõrrandi lahendamiseks, toodud ära vastavad koonduvusteoreemid ja sooritatud hulgaliselt numbrilisi eksperimente teoreetiliste hinnangute optimaalsuse kontrollimiseks. Teoreetiliste tulemuste puudumise korral on numbriliste eksperimentide baasil püstitatud hüpotees meetodi koonduvuskiiruse kohta.