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Optimization problems with points of discontinuity and discrete arguments

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Abstract. Minimization of such functions is considered, where some arguments are related to the final function by intermediate functions with discontinuity points, but other arguments have only 0 and 1 for the allowed values, although the theoretical generalization allows also intermediate values. Both of the circumstances create difficulties in the use of the gradient method. We solve the first problem by approximation, primarily by a square polynomial obtained using the integral form of the least squares method, and later by the partial sums of orthogonal series of the wave function treated with the logarithmic averages method. The second problem can be solved with the help of the planes, which have been taken in the *n*-dimensional space in such a way that any allowed point on the side of the space relative to this plane is better than all the points on the other side.

Key words: optimization, discontinuity points, discrete values, least squares method, wave functions, logarithmic average.

1. SET-UP OF THE PROBLEM

When optimizing electric regimes, discrete values become significant for changing the topology of the network, and to some extent, for switching generators on and off. Earlier [1], this optimization problem was solved mainly by the discretion of a network section and testing possible options. Recent works [2] show that the problem can be solved also by using continuous variables. In the present paper a procedure is given for solving the problem by means of continuous variables, keeping in view its applicability to solving the problems arising in [3].

Thus the present problem was initiated by a concrete need for optimizing the operating conditions of electrical networks. It is treated purely mathematically, so the methods given here can be used for solving also other problems with similar mathematical formulation.

Let us minimize the function

$$\Phi(x_\iota : \iota \in I_1; \varphi_\iota(y_\iota) : \iota \in I_2; z_\iota : \iota \in I_3).$$

Here Φ is a continuous function and the cost function for our electric regimes. The symbols in the brackets denote the following: $x_\iota,\ \iota \in I_1,$ are continuously varying arguments, which, for instance, in the present case could be node voltages and phases; $\varphi_\iota,\ \iota \in I_2,$ are the intermediate functions with possible discontinuity points; y_ι are continuously varying arguments, for instance, power injections of supply nodes. The node injections can be varied continuously, but the impact of injection on the general cost function is stepped in some points, namely when a new unit has to be started. The variables $z_\iota,\ \iota \in I_3,$ have only discrete value. We shall consider a case where the permitted value of any z_ι is 0 or 1. In theory, z_ι could have any other intermediate value, i.e., Φ is continuous relative to these values also, but we are looking for the solution in the form where any z_ι is either 1 or 0, corresponding, respectively, to switching the network lines on or out. The conductivity of each line will be multiplied by a factor $t_\iota,\ 0 \le t_\iota \le 1,$ where 0 stands for switching out and 1 for switching on, and the intermediate values correspond to the variation of conductivity, i.e., the increase in resistance. Since the situation changes too rapidly in the neighbourhood of 0, let us change the variables $t_\iota = e^{1-1/z_\iota},\ 0 \le z_\iota \le 1.$

Thus in our case the values of the function Φ are theoretically defined for all values $z_\iota \in [0,1].$ In practice only the values that correspond to integer values of variables z_ι will be taken into account.

Below, we shall assume that such a theoretical continuous idealization exists. We shall use the gradient method for optimization. However, to do this, we must overcome the discontinuity of the functions φ_ι and solve the discretion problem for z_ι.

2. ELIMINATION OF DISCONTINUITY POINTS

Let us start from the first option. In the course of optimization we shall replace each φ_ι with the continuous approximation $\overline{\varphi}_\iota$. Herewith it is reasonable not to take a fixed $\overline{\varphi}_\iota$ for the given ι throughout the solution process, but to start with a rougher and simpler approximation while specifying it later. Namely, there is no good reason to think that the instantaneous value of y_ι would be close to the final value from the beginning.

Obviously, there are several intermediate discontinuity points of the function φ_ι. A too precise approximation of $\overline{\varphi}_\iota$ would cause a considerable jerk at any discontinuity point and slow down the implementation of the gradient method. Therefore it is reasonable to make first a rough approximation that would mark general changing of $\overline{\varphi}_\iota,$ without paying attention to local variations. A square polynomial obtained by the least squares method in the integral form [4] could be suitable in this case.

When the differences of y_ι are so small that they do not jump over several discontinuity points at a time, more precise $\overline{\varphi}_\iota$ must be taken. Since we know definitely the discontinuity points of each φ_ι, this moment can be recognized when comparing the last approximations of φ_ι made up to now. As it is known, the values of φ_ι can be well approximated by the partial sums of the orthogonal series of wave functions [5,6]. However, the disadvantage is that although the value is well approximated, the highest frequency wave considered last determines the local behaviour significantly. As a result, the gradient starts to toss about. To suppress the highest frequency waves, the use of partial sums proper has not proved to be reasonable, but summing them up with the weighted averages method seems to be appropriate. The logarithmic averages method [7], where the number of terms considered by summing up can be increased gradually to improve the accuracy, suits for this purpose.

After these changes we get the continuous objective function, and the gradient method can be used. However, the solution obtained may not be feasible since the values of z_ι are not 0 or 1.

3. FINDING ALLOWED SOLUTIONS

If $|I_3| = n$, then z_ι's form an n-dimensional cube. Our objective is to select the best of the 2^n-vertices of this cube. If n is not too small, checking these vertices one by one would be rather labour-consuming. Therefore we shall use the continuous idealization method mentioned above, which would allow us to eliminate the unmatching vertices by batches. Thereby we presume that the finite sequence of x_ι and y_ι for each concrete point in the cube has been chosen so that it is the best for each point. This finite sequence can be found using the method given in the previous section. Thus, optimizing in any region of the cube brings along optimization also in the space of x_ι and y_ι. As the values of all x_ι and y_ι have been determined uniquely by the values of z_ι in the mentioned way, from now on we can consider the target function Φ as depending only on the arguments z_ι.

Let us assume that the level surfaces of the target function are convex in our cube. With the use of the gradient method, we shall reach the point given at the end of the previous section, which is the minimum point of this target function on the whole cube. Let us denote it P_0. Let $< a_\iota : \iota \in I_3 >$ be the coordinates of this point. If P_0 is a vertex, the problem will be solved. Let us consider the opposite situation. Let as assume that $\frac{1}{2} \geq a_1 \geq a_2 \geq ... \geq a_n \geq 0$. It is not a constraint, since we can choose the coordinate system so that the origin of coordinates is the vertex of this cube, which is the closest to the point P_0. This provides $0 \leq a_\iota \leq \frac{1}{2}$ for each ι. We can also choose the order of coordinates that will guarantee the above inequalities. Next let us put a check plane through the cube. This is an $(n-1)$-dimensional plane, with its position defined by n cube vertices on it. We shall look for the initial position of the check plane in the form $\sum_{i=k+1}^{n} x_i = m,$ where $0 \le k < n$ and m is the least integer such that $\sum_{i=k+1}^{n} a_i < m.$ The last inequality provides that $\sum_{i=k+1}^{n} x_i < m$ is valid at the point P_0, i.e., P_0 remains on the same side of the check plane with the origin of coordinates. Let us note that $m \le n-k,$ since $a_i \le \frac{1}{2}$ for each $i.$ Further we shall name the side of the space of z_ι's relative to the check plane where P_0 is located **the space on this side**, and the opposite side **the space on the other side**. For the initial position the number of vertices in the space on this side together with those on the check plane is $2^k \sum_{j=0}^{m} (n-k)!/j!(n-k-j)!.$ We shall choose k so that this number of vertices would be as small as possible. A too great k would turn the factor 2^k great, while the minimum k could give a too great $m.$ Proceeding from the given numbers $< a_\iota : \iota \in I_3 >,$ k must be chosen in the optimal way.

Let us consider a 4-dimensional space with the coordinates $x, y, z,$ and $w.$ Let the coordinates of P_0 be $(1/3, 1/4, 1/10, 0).$ In this case it is suitable to take $k = 0,$ and then m obtains the value 1 since $1/3 + 1/4 + 1/10 + 0 < 1.$ The equation of the check plane will then be $x + y + z + w = 1.$ In our case the number of vertices in the space on this side is 5 (with the points on the check plane). If the coordinates of P_0 were $(1/2, 1/3, 1/3, 1/4)$ for example, then in case of $k = 0,$ m would be 2, since $1 < 1/2 + 1/3 + 1/3 + 1/4 < 2.$ The equation $x + y + z + w = 2$ of the check plane means that the number of vertices in the space on this side is 11 (with the points on the check plane). Now it is suitable to take $k = 1$ with $m = 1,$ since $1/3 + 1/3 + 1/4 < 1.$ The equation of the check plane will be $y + z + w = 1,$ and the number of the vertices in the space on this side is 8 (with the points on the check plane).

If the number of vertices on the check plane and in the space on this side is moderate, the value of the target function can be calculated for all of them. Otherwise we shall do it for a moderate number of vertices, preferring vertices closest to the point $P_0.$ We shall name the preliminary preferred vertex the vertex among them where the value of the target function is the best.

Let us find the minimum point of the target function in the part of the check plane, which remains inside the cube and denote it as $P,$ which again may not be a vertex. Let us find the value of the target function at the point $P.$ Since the level surface of the target function osculates the check plane (or the part of the plane that remains inside the cube), we can consider the following statement valid: for all vertices, which remain on the other side of the space, the value of the target function is worse than at the point $P.$

Conclusions. If the value of the target function at the preferred vertex is at least as good as at the point $P,$ also in case P itself is a vertex, the best vertex of the cube cannot be placed on the other side of the space, and cannot be at some vertex different from the point P on the check plane.

In case none of the mentioned conclusions is satisfied, the check plane should be shifted in such a way that some of the assumptions would be satisfied and so that a minimum number of new vertices would be added to the space on this side. Since the level surface of the target function osculates the check plane at the point P, the level surfaces of the target function are obviously extended in the P_0P direction. Thus it seems promising to shift this edge of the check plane, which is located in the direction of the elongation of P_0P, since namely in this direction worsening of the target function is the slowest with respect to its value at the point P_0. The check plane is shifted in such a way that a new vertex is found in the space on the other side that the check plane must pass, and the $(n-2)$-dimensional opposite edge will be preserved. The check plane must move to the new vertex by turning around the edge. Let us explain the procedure of finding a new vertex first. To do this, we shall elongate the interval P_0P over the point P till it intersects some $(n-1)$-dimensional face of the cube at the point Q. If P lies on the face, then $P = Q$. Let us fix the face with Q on it and which lies at least partly in the space on the other side. In case Q lies at the $(n-2)$-dimensional edge of two faces, we shall select among these two faces the one, which meets the above mentioned condition. If both faces meet this condition, also their $(n-2)$-dimensional common edge will meet it, and in this case we shall fix the edge instead of the face. In case there are more edges than one, which pass Q and meet this condition, we can fix already the $(n-3)$-dimensional boundary that passes Q and is at least partly located in the space on the other side, etc. So, going on with the process as long as possible, we come to a k-dimensional boundary, $k \le n-1$, where Q lies and which lies at least partially in the space on the other side.

For the example above, where the equation of the check plane in the 4-dimensional $xyzw$-space is $y + z + w = 2$ in case the coordinates of Q are $(2/3, 1, 1/2, 3/4)$, we shall fix the face $y = 1$ on which the vertex of the new check plane, which was with respect to the previous one in the space on the other side, must lie. If $P = Q = (2/3, 1, 1, 0)$, then Q will lie simultaneously on the faces $y = 1$, $z = 1$, and $w = 0$. The first two faces lie partially in the space on the other side with respect to the check plane, while the third one lies in the space on this side, reaching the check plane only by its edge and nonexisting in the space on the other side. Thus, the third face can be neglected and we shall fix the common edge of the faces $y = 1$ and $z = 1$ where the searched vertex of the new check plane must lie.

Having fixed the k-dimensional boundary, we shall continue moving on it, but staying in the part of the cube that does not belong to the space on this side. It means that by starting to move on the k-dimensional boundary as well as at any moment when continuation in the chosen directon will take us out of the allowed part of the cube, we shall find a new direction so that the angle with respect to the present direction will remain minimal.

In case of our last example we must continue moving in the xw-space, having fixed $y = z = 1$. The square $0 \le x,\ w \le 1$ will remain in the 4-dimensional cube. Only one edge of this square, namely $w = 0$, will lie on the check plane. The remaining part will be in the space on the other side. We have to continue from the point $x = 2/3$, $w = 0$ so that the angle between the previous direction and the new one were minimal. As x was increased and w decreased, we should continue in the same way, but it would take us out of the allowed region. Therefore, the allowed direction that follows the previous one maximally is such that $w = 0$ and x increases.

This way we shall reach the point R, which lies on the $(k-1)$-dimensional edge of the k-dimensional boundary, which in turn is at least partially located in the space on the other side. For our example we shall come to the point $x = 1$. The face with the equation $x = 1$ is partially in the space on the other side, and so intersection of the faces $x = 1$, $y = 1$, and $z = 1$ where the searched vertex will lie is fixed.

Further, let us apply the same procedure that we used for Q to the point R, only instead of the $(n-1)$-dimensional face, we shall have now the $(k-1)$-dimensional boundary. Continuing like that, we shall reach a one-dimensional boundary where we shall choose this end point, which lies in the space on the other side. In our example this point will be $x = 1$, $y = 1$, $z = 1$, $w = 1$.

We remind of the necessity to change sometimes the direction not only in case we are on some face of the cube, but having reached the check plane. For example, if $P_0 = (1/2, 1/2, 2/5, 1/4)$, the check plane $x + y + z + w = 2$ and $P = (1/2, 5/12, 1/3, 3/4)$, then $Q = (1/2, 3/8, 3/10, 1)$. Further, we shall continue moving along the projection of the previous straight line onto the face $w = 1$. But having contacted the check plane before reaching a new face, we have to change the direction, continuing to move on the face $w = 1$ in such a way that we do not pass the check plane and turn off the least from the previous direction.

Having chosen a vertex in the space on the other side, we have to choose a vertex to be rejected on the present check plane. We shall choose it in such a way that it would be dimensionally possibly close to the new vertex, i.e., a boundary of the cube with a possibly small dimension would exist, which passes through both, the new and rejected vertices. Since the coordinates of the vertices are 0 or 1, it is required to change the minimum number of coordinates of the new vertex, so that the obtained point would satisfy the equation of the present check plane.

Next, let us find the $n-1$ vertices on the present check plane, which together with the rejected vertex will fix the present check plane. It means that in the form of vertices we shall find n linearly independent solutions for the equation of the check plane, including the rejected vertex. The remaining $n-1$ vertices form the $(n-2)$-dimensional axis, and we shall turn the check plane around this axis. These $n-1$ vertices together with the new vertex fix the new position of the check plane.

Let us now present the whole plane shifting procedure using our example where $P_0 = (1/3, 1/4, 1/10, 0)$ and the check plane $x + y + z + w = 1$. If we choose $P = (1/4, 1/6, 1/3, 1/4)$, then $Q = (1/12, 0, 4/5, 3/4)$. Let us fix the face $y = 0$. Moving on this face and using the simple projection of the present direction, we shall reach the point $R = (1/84, 0, 1, 27/28)$. Now we shall fix additionally the face $z = 1$ and moving along the common edge of the faces $y = 0$ and $z = 1$, once again projecting the present direction onto it, we shall reach the vertex (0, 0, 1, 1). It will be our new vertex. The present check plane $x + y + z + w = 1$ is determined by the vertices (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), and (0, 0, 0, 1). We have to reject one of them. We shall choose the vertex that is dimensionally the closest to the new vertex, i.e., has the largest number of coinciding coordinates. Such are the two latter vertices, differing from the new vertex by one coordinate. The other two have three such coordinates. For example, let us reject the vertex (0, 0, 0, 1). The equation of the new check plane that passes through three previous vertices and the new vertex is $x + y + z = 1$.

After finding the new position of the check plane we shall find the values of the target function at the vertices, which were added in the space on this side, at least at these vertices, which are in the immediate neighbourhood of the new vertex. The immediate neighbourhood of the named vertex is of interest relative to also these vertices, which were already in the part of the space on this side, but were not considered. When doing so, we can find a new preferred vertex where the value of the target function is better than at the previous one. On the other side, since the check plane moves away from the point P_0 when turning, the best point of the new check plane will obviously be worse than P. Thus it will be more evident that the best point of the new check plane is already worse than the present preferred vertex, i.e., the assumption given in the conclusion could be fulfilled now.

When doing so, we shall reach such a check plane, for which the value of the target function in the whole space on the other side is worse than at the best point of the check plane. Then the best vertex of the cube is in the space on this side, perhaps on the check plane. The latter case can be considered only if the best point on the check plane turns out to be a vertex. When in the space on this side the number of vertices is moderate, so that they can be checked one by one, the problem can be solved. If not, we shall consider each $(n-1)$-dimensional face or part of the face in the space on this side separately. On each such face or part of the face we shall find separately its best point and compare the values of the target function at this point and at the already found preferred vertex. The faces, where the target function at its best point is worse than at the preferred vertex, will be totally rejected. It should also be considered that if any $(n-1)$-dimensional face falls out, its neighbouring face will degenerate into an $(n-2)$-dimensional opposite edge, which must now be checked instead of the neighbouring face. So, a certain number of cube boundaries will remain to check, each of them having a smaller dimension than n. For each such k-dimensional boundary we shall use the check plane method again, taking this time the $(k-1)$-dimensional secant line in the role of the check plane, but in the role of the preferred vertex using the up to now best checked vertex, even when it does not lie on this k-dimensional boundary.

4. POLYOPTIMIZATION

Let us have several such functions Φ_k expressing various interests of different stakeholders, which should be minimized. Therewith we shall assume that the values of different Φ_k are given in the same units, so that their changes can be compared with each other. As changes can be made based on the consensus only, the value of any single objective function cannot deteriorate. Instead of the optimum, we are now looking for the Pareto optimum, where the value of the objective function of all parties against the initial value has improved. In addition, we shall consider the min-max principle of the improvement rates: even the least improved objective function must improve as much as possible.

For the objective functions the following is valid [3]. If we build a linear function $\sum_k \alpha_k \Phi_k$ so that for each k, $0 \le \alpha_k \le 1$, while $\sum_k \alpha_k = 1$, and choose α_k under these conditions so that $\left\|\sum_k \alpha_k \Phi_k'\right\|$ is minimal, the optimization direction of $\sum_k \alpha_k \Phi_k$ will be exactly of the kind where the max–min principle is valid for recovering velocities: any deviation from this direction will decrease the recovering speed of some objective function among the still most slowly recovering functions.

In the discrete case we shall replace φ_ι with $\overline{\varphi}_\iota$ in each objective function separately. For the cube of z_ι's, we shall choose α_k's for the initial point, which is also the first preferred vertex based on the above principle. Further, the check plane is placed and shifted proceeding from the objective function $\sum_k \alpha_k \Phi_k$ as long as the preferred vertex remains the same. We shall consider as a new and more preferred vertex the one where each objective function is better (or at least as good as they were). It is evident that the new vertex is better for $\sum_k \alpha_k \Phi_k$ also in case of the existing combination and any other combination consisting of non-negative α_k. If we can select freely, we shall proceed from the max-min principle again when selecting the preferred vertex. After we have chosen the new preferred vertex, we shall find the α_k in the way that $\left\|\sum_k \alpha_k \Phi_k'\right\|$ is minimal in the new vertex, and from that moment on we shall consider $\sum_k \alpha_k \Phi_k$ an objective function for these α_k.

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REFERENCES

1. Koglin, H. J. and Müller, H. Overload reduction through corrective switching actions. In *Proc. Intern. Conf. on Power System Monitoring and Control, 24–26 June, 1980*. London, 1980, 159–164.
2. Torres, G. L. and Quintana, V. H. An interior-point method for non-linear optimal power flow using voltage rectangular coordinates. *IEEE Trans. Power Syst*., 1998, **13**, 1211–1218.
3. Krumm, L., Kurrel, Ü., Tauts, A., Terno, O., Zeidmanis, I., Krišans, Z., Dale, V., Putnynsh, V. and Oleinikova, I. Theory and methods of complex optimization of the interconnected power system. In *World Energy Council Regional Forum "Central and East European Energy Policies, Markets and Technologies for the 21st Century" (September 16–18, 1999, Lithuania, Vilnius), Proceedings*. Vilnius, 1999, 198–210.
4. Levin, M. and Ulm, S. *Arvutusmeetodite käsiraamat*. Valgus, Tallinn, 1966.
5. Kivinukk, A. and Verhovitch, J. Summation methods of trigonometric Fourier series defined by the Zak transform. *Proc. Estonian Acad. Sci. Phys. Math*., 2001, **50**, 5–15.
6. Kukk, V. Modelling non-linear systems by extended Fourier series. *Proc. Estonian Acad. Sci. Eng.*, 2001, **7**, 289–308.
7. Baron, A. *Introduction to the Theory of Summability of Series*. Valgus, Tallinn, 1977 (in Russian).

Optimiseerimisprobleemid katkevuspunktide ja diskreetsete argumentide korral

Ants Tauts

Vaadeldakse selliste funktsioonide minimeerimist, milles osa argumente on lõppfunktsiooniga seotud katkevuspunkte omavate ühekohaliste vahefunktsioonidega, osa omab aga lubatavate väärtustena vaid väärtusi 0 ja 1, kuigi teoreetiline üldistus võimaldab ka vahepealseid väärtusi. Kumbki asjaolu tekitab probleemi gradientmeetodi kasutamisel. Esimene probleem on lahenduv pideva aproksimatsiooniga esialgu ruutpolünoomiga, mis on saadud vähimruutude meetodi integraalkuju abil, ja hiljem lainefunktsioonide ortogonaalridade osasummadega, mida on töödeldud logaritmiliste keskmiste menetluse abil. Teine probleem on lahenduv tasandite abil, mis on võetud *n*-mõõtmelises ruumis selliselt, et mingi lubatud punkt ruumi ühel poolel selle tasandi suhtes on parem kui kõik punktid teisel poolel.