Proc. Estonian Acad. Sci. Phys. Math., 2004, **53**, 3, 165–176
https://doi.org/10.3176/phys.math.2004.3.04

A convergence theorem for approximate methods of tangent hyperbolas

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Received 18 March 2003, in revised form 22 July 2003

Abstract. For solving an operator equation $F(x) = 0$, where F is a nonlinear operator from a Banach space X into another Banach space Y, approximate variants of the method of tangent hyperbolas are developed, provided F is twice Frechet-differentiable and its first derivative has the uniformly bounded inverse. A local convergence theorem is provided for the methods under consideration and their computational aspects are briefly discussed.

Key words: nonlinear equation, Banach space, Newton's method, cubically convergent method, approximate variants of methods, midpoint method.

1. INTRODUCTION AND BASIC THEOREM

One of the central problems in numerical analysis is the efficient solution of a nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a mapping from a Banach space X into another Banach space Y, and it is as many times differentiable as necessary.

Methods with a high order of convergence enable sometimes finding solutions to (1.1) more rapidly and accurately, since they require for computing a solution with the prescribed accuracy, as a rule, less iterations than the ones with a lower convergence rate provided a proper initial guess is available and therefore likely less total arithmetic. Another reason for using methods with the convergence order higher than that of the Newton method or its variants is the fact that methods of order $p \geq 3$ do not break down if F' is singular or strongly ill-conditioned, because they are based on a quadratic model and even a rough approximation to the operator of second derivatives may provide their numerical stability [1–4].

One of the most popular methods of order three is the method of tangent hyperbolas (or the Chebyshev–Halley method)

$$x_{k+1} = x_k - T_k^{-1}\Gamma_k F(x_k), \tag{1.2}$$

where $\Gamma_k = [F'(x_k)]^{-1}$ and $T_k = I - \frac{1}{2}\Gamma_k F''(x_k)\Gamma_k F(x_k)$.

It can be rewritten as

$$x_{k+1} = x_k - \left[F'(x_k) - \frac{1}{2}F''(x_k)\Gamma_k F(x_k)\right]^{-1} F(x_k). \tag{1.3}$$

The approach adopted by this paper is the use of iterative methods to obtain approximations for T_k^{-1} and/or Γ_k or approximate solutions to the corresponding linear equations. But the approximations may also be regarded as a result of the inevitable inaccuracy of computation. If A_k and $L(x, x-y), (x,y) \in X$ approximate the operators Γ_k and $F''(x)(x-y)$, respectively, then in the special case $x_k - y_k = A_k F(x_k)$ and

$$L(x_k, A_k) := L(x_k, A_k F(x_k)) = L(x_k, x_k - y_k) \approx F''(x_k)A_k F(x_k)$$

it follows from (1.3) that

$$x_{k+1} = x_k - U_k^{-1} A_k F(x_k), \tag{1.4}$$

where

$$U_k = A_k F'(x_k) - \frac{1}{2}A_k L(x_k, A_k).$$

Further on we shall suppose the existence and boundedness of the operators $[F'(x_k)]^{-1}$ and U_k^{-1}. If in turn instead of U_k^{-1} its approximation V_k is used, we get the method

$$x_{k+1} = x_k - V_k A_k F(x_k). \tag{1.5}$$

Likewise we assume the existence of the constants $\alpha, \beta, \lambda, \mu, \Lambda,\ M, K, G, G_1$, and sequences $\{\gamma_{1k}\}$ and $\{\gamma_{2k}\}$ such that the following inequalities are valid:

$$\|F'(x)\| \leq M,\ \|F''(x)\| \leq K,\ \|A_k\| \leq \mu_k \leq \mu,\ \|A_k^{-1}\| \leq \beta_k \leq \beta,$$

$$\|V_k A_k\| \leq \lambda_k \|F(x_k)\| \leq \lambda \|F(x_k)\|,\ \|V_k\| \leq \Lambda_k \leq \Lambda,\ (\beta, \lambda, \Lambda, \mu < \infty), \tag{1.6}$$

$$\|I - U_k V_k\| \leq \gamma_{1k},\ \max\{\|I - A_k F'(x_k)\|,\ \|I - F'(x_k)A_k\|\} \leq \gamma_{2k}.$$

Theorem 1. *Let* $x_0 \in X,\ S = \{x \in X : \|x - x_0\| \le \rho\}$ *and let the following conditions be valid on S*:
1° *operator F is twice Frechet-differentiable*;
2° *the second derivative satisfies a Lipschitz condition*

$$\|F''(x) - F''(y)\| \le L_2\|x - y\|;$$

3° $\|F''(x)(x-y) - L(x, x-y)\| \le G\|x-y\|^2, G < \infty$;
4° *there exist* $\Gamma(x)$ *and* $U^{-1}(x)$ *with* $\|\Gamma(x)\| \le C$ *and* $\|U^{-1}(x)\| \le C$, $C, C_1 < \infty$;
5° $\delta = \delta_0^{(i)} < 1,\ i = 1, 2, 3$ (*the quantity* δ_0 *is defined differently in cases* 1–3 *below*).

Then the following results are valid:
1. *If*

$$\gamma_{ik} \le \gamma_{i0} < 1,\ i = 1, 2$$

and $r_1 = \lambda\|F(x_0)\|/(1-\delta) \le \rho$, *then the equation* $F(x) = 0$ *has a solution* $x^\star$ *in S*, $\|x^\star - x_0\| \le r_1$, *to which the sequence* (1.5) *converges with*

$$\|x_k - x^\star\| \le r_1\delta^k,\ \delta = \delta_0^{(1)};$$

if $\gamma_{i0} \ge \gamma_{i1} \ge \ldots \ge \gamma_{in} \ge \ldots \ge 0$, *and* $\gamma_{ik} \to 0$ *as* $k \to \infty$, *then* $\delta_k^{(1)} \to 0$ *and the sequence* (1.5) *converges superlinearly with*

$$\|x_k - x^\star\| \le r_1 \prod_{m=0}^{k-1} \delta_m^{(1)},$$

where

$$\delta_k^{(1)} = \gamma_{1k}\beta_k\mu_k + \frac{1}{2}\lambda_k K(\gamma_{1k}\mu_k + \gamma_{2k}\lambda_k)\|F(x_k)\| + \left[\frac{1}{4}\mu_k^2\lambda_k^2 K(K + 2G\rho) + \frac{1}{2}\lambda_k\mu_k^2 G + \frac{1}{6}\lambda_k^3 L_2\right]\|F(x_k)\|^2.$$

2. *If* $\gamma_{1k} \le C_2\|F(x_k)\|,\ \ \gamma_{2k} \le \gamma_{20},\ \gamma_{20}, C_2 < \infty,\ \delta = \delta_0^{(2)} = d_0^{(2)}\|F(x_0)\| < 1,$

$$d = \lim_{k\to\infty} d_k^{(2)} > 0,$$

$$d_k^{(2)} = \beta_k\mu_k C_2 + \frac{1}{2}\gamma_{20}\lambda_k^2 K + \left[\frac{1}{2}\mu_k\lambda_k K C_2 + \frac{1}{4}\mu_k^2\lambda_k^2 K(K + 2G\rho) + \frac{1}{2}\lambda_K\mu_k^2 G + \frac{1}{6}\lambda_k^3 L_2\right]\|F(x_k)\|,$$

then Eq. (1.1) *has a solution* $x^\star$ *in S,* $\|x^\star - x_0\| \le r_2$, *to which the sequence* (1.5) *converges quadratically*:

$$\|x_k - x^\star\| \le \lambda H_k^{(2)}(\delta)/d,\ H_k^{(2)}(\delta) = \sum_{i=k}^{\infty} \delta^{2^i},\ r_2 = \lambda H_0^{(2)}(\delta)/d \le \rho.$$

3. *If*

$$\gamma_{1k} \le C_3\|F(x_k)\|^2,\ \gamma_{2k} \le C_4\|F(x_k)\|,\ C_3, C_4 < \infty,\ r_3 = H_0^{(3)}(\delta)/d \le \rho,$$

where

$$H_k^{(3)}(\delta) = \sum_{i=k}^{\infty} \delta^{3^i},\ \delta = \delta_0^{(3)} = \sqrt{d}\,\|F(x_0)\| < 1$$

and

$$\begin{aligned} d =& d_0^{(3)} = \beta_0\mu_0 C_3 \\ &+ \frac{1}{2}\lambda_0 K(\mu_3 C_3 + \lambda_0 C_4) + \frac{1}{4}\mu_0^2\lambda_0 K(K + 2G\rho) + \frac{1}{2}\lambda_0\mu_0^2 G + \frac{1}{6}\lambda_0^3 L_2, \end{aligned}$$

then the sequence (1.5) *converges cubically*

$$\|x_k - x^\star\| \le (\lambda/\sqrt{d}) H_k^{(3)}(\delta).$$

Proof. Under our assumptions on the basis of a Banach theorem there exist bounded A_k^{-1} and V_k^{-1} provided $\gamma_{1k}, \gamma_{2k} < 1$. According to the Taylor formula and the equality

$$F'(x_k)(x_{k+1} - x_k) - \frac{1}{2}L(x_k, A_k)(x_{k+1} - x_k) = A_k^{-1}U_k(x_{k+1} - x_k),$$

we get

$$\begin{aligned} F(x_{k+1}) =& F(x_k) + F'(x_k)(x_{k+1} - x_k) \\ &+ \int_0^1 F''(x_k + t(x_{k+1} - x_k))(x_{k+1} - x_k)^2(1-t)dt \\ =& F(x_k) + A_k^{-1}U_k(x_{k+1} - x_k) + \frac{1}{2}[F''(x_k)(y_k - x_k) \\ &+ L(x_k, A_k)](x_{k+1} - x_k) \\ &+ \frac{1}{2}[F''(x_k)(x_{k+1} - x_k) - F''(x_k)(y_k - x_k)](x_{k+1} - x_k) \\ &+ \int_0^1 [F''(x_k + t(x_{k+1} - x_k)) - F''(x_k)](x_{k+1} - x_k)^2(1-t)dt \\ &+ F(x_k) + A_k^{-1}V_k^{-1}(x_{k+1} - x_k) - A_k^{-1}V_k^{-1}(x_{k+1} - x_k) \\ &+ A_k^{-1}U_k(x_{k+1} - x_k) + R_k, \end{aligned} \tag{1.7}$$

where

$$
\begin{aligned}
R_k =& \frac{1}{2}[F''(x_k)(y_k - x_k) + L(x_k, A_k)](x_{k+1} - x_k) \\
&+ \frac{1}{2}F''(x_k)(x_{k+1} - y_k)(x_{k+1} - x_k) \\
&+ \int_0^1 [F''(x_k + t(x_{k+1} - x_k)) - F''(x_k)](x_{k+1} - x_k)^2(1-t)dt. \quad (1.8)
\end{aligned}
$$

Due to the relations (1.4)–(1.6) we have

$$
F(x_k) + A_k^{-1}V_k^{-1}(x_{k+1} - x_k) = F(x_k) - A_k^{-1}V_k^{-1}V_kA_kF(x_k) = 0,
$$

$$
\begin{aligned}
-A_k^{-1}V^{-1}(x_{k+1} - x_k) + A_k^{-1}U_k(x_{k+1} - x_k)& \\
= -A_k^{-1}(I - U_kV_k)V_k^{-1}(x_{l+1} - x_k)& \\
= A_k^{-1}(I - U_kV_k)A_kF(x_k),& \quad (1.9)
\end{aligned}
$$

$$
\begin{aligned}
x_{k+1} - y_k &= A_kF(x_k) - V_kA_kF(x_k) \\
&= A_kF(x_k) - U_kV_kA_kF(x_k) + U_kV_kA_kF(x_k) - V_kA_kF(x_k) \\
&= (I - U_kV_k)A_kF(x_k) + (U_k - I)V_kA_kF(x_k).
\end{aligned}
$$

On the basis of assumption 3° and the inequalities $\|F''(x)\| \leq K$ and $\|x - x_0\| \leq \rho$, we obtain

$$
\begin{aligned}
\|L(x,y)\| &= \|F''(x)(x-y) + L(x, x-y) - F''(x)(x-y)\| \\
&\leq K\|x-y\| + G\|x-y\|^2 \leq (K + 2G\rho)\|x-y\| = G_1\|x-y\|. \quad (1.10)
\end{aligned}
$$

Bearing in mind the inequalities (1.6)–(1.10), we get the following estimates:

$$
\|I - U_k\| = \left\| I - A_kF'(x_k) + \frac{1}{2}A_kL(x_k, A_k) \right\| \leq \gamma_{2k} + \frac{1}{2}\mu_k^2G_1\|F(x_k)\|,
$$

$$
\left\{
\begin{aligned}
\|x_{k+1} - y_k\| &= \|(I - U_kV_k)A_kF(x_k) + (U_k - I)V_kA_kF(x_k)\| \\
&\leq \gamma_{1k}\mu_k\|F(x_k)\| + \lambda_k\left(\gamma_{2k} + \frac{1}{2}\mu_k^2G_1\|F(x_k)\|\right)\|F(x_k)\| \\
&= (\gamma_{1k}\mu_k + \gamma_{2k}\lambda_k)\|F(x_k)\| + \frac{1}{2}\mu_k^2\lambda_kG_1\|F(x_k)\|^2, \\
\|F(x_{k+1})\| &= \|A_k^{-1}(I - U_kV_k)A_kF(x_k) + R_k\| \\
&\leq \gamma_{1k}\beta_k\mu_k\|F(x_k)\| + \frac{1}{2}\lambda_kK(\gamma_{1k}\mu_k + \gamma_{2k}\lambda_k)\|F(x_k)\|^2 \\
&+ \frac{1}{4}\mu_k^2\lambda_k^2KG_1\|F(x_k)\|^3 + \frac{1}{2}\lambda_k\mu_k^2G\|F(x_k)\|^3 + \frac{1}{6}\lambda_k^3L_2\|F(x_k)\|^3.
\end{aligned}
\right. \quad (1.11)
$$

1. If $\gamma_{ik} \le \gamma_{i0} < 1, \;\; i = 1, 2$, then

$$\|F(x_{k+1})\| \le \delta_k^{(1)} \|F(x_k)\|,$$

where

$$\begin{aligned}\delta_k^{(1)} =& \gamma_{1k}\beta_k\mu_k + \frac{1}{2}\lambda_k K(\gamma_{1k}\mu_k + \gamma_{2k}\lambda_k)\|F(x_k)\| \\ &+ \left[\frac{1}{4}\mu_k^2\lambda_k^2 K(K + 2G\rho) + \frac{1}{2}\lambda_k\mu_k^2 G + \frac{1}{3}\lambda_k^3 L_2\right] \|F(x_k)\|^2,\end{aligned}$$

and

$$\|F(x_{k+1})\| \le \delta_k^{(1)}\|F(x_k)\| \le \|F(x_0)\| \prod_{i=1}^{k} \delta_i^{(1)} = \|F(x_0)\|\delta^{k+1},$$

$$\|x_{k+1} - x_k\| \le \lambda\|F(x_0)\|\delta^k,$$

$$\|x_m - x_k\| \le \sum_{l=k}^{m-1} \|x_{l+1} - x_l\| \le r_1(\delta^k - \delta^m), \; m \ge k.$$

This means that the sequence (1.5) is fundamental and consequently

$$x^* = \lim_{k\to\infty} x_k, \; \|x^* - x_k\| \le r_1\delta_k,$$

$$\lim_{k\to\infty} \|F(x_k)\| = \|F(\lim_{k\to\infty} x_k)\| = \|F(x^*)\| = 0.$$

In case $\gamma_{ik} \to 0 \; (i = 1, \; 2)$, as $k \to \infty$, obviously $\delta_k^{(1)} \to 0$ as well, and therefore the sequence (1.5) converges at least superlinearly to a solution of (1.1).

2. If $\gamma_{1k} \le C_2\|F(x_k)\|, \; \gamma_{2k} \le \gamma_{20} < \infty$, then

$$\|F(x_{k+1})\| \le d_k^{(2)}\|F(x_k)\|^2,$$

where

$$\begin{aligned}d_k^{(2)} = \beta_k\mu_k C_2 + \frac{1}{2}\gamma_{20}\lambda_k^2 K + \Bigg[&\frac{1}{2}\lambda_k K\mu_k C_2 + \frac{1}{4}\mu_k^2\lambda_k^2 K(K + 2G\rho) \\ &+ \frac{1}{2}\lambda_k\mu_k^2 G + \frac{1}{6}\lambda_k^3 L_2\Bigg] \|F(x_k)\|.\end{aligned}$$

Obviously, $d_0^{(2)} \ge d_1^{(2)} \ge ... \ge k_k^{(2)} ... > 0$ and $\lim_{k\to\infty} d_k = d > 0$.

Therefore

$$\|F(x_{k+1})\| \le d_k^{(2)}\|F(x_k)\| \le d^{-1}(d_k^{(2)}\|F(x_k)\|) \le ... \le d^{-1}\delta^{2^{k+1}},$$

since for $p \geq 2$

$$(d_k^{1/(p-1)}\|F(x_k)\|)^p \leq (d_k^{1/(p-1)} d_{k-1}\|F(x_{k-1})\|^p)^p$$
$$\leq (d_{k-1}^{p/(p-1)}\|F(x_{k-1})\|^p)^p \leq (d_{k-1}^{1/(p-1)}\|F(x_{k-1})\|)^{p^2}$$
$$\leq ... \leq (d_0^{1/(p-1)}\|F(x_0)\|)^{p^{k+1}} = \delta^{p^{k+1}}$$

provided $d_k \leq d_{k-1}$.

Thus

$$\|x_{k+1} - x_k\| \leq \lambda d^{-1}\delta^{2^k},\ \|x_n - x_k\| \leq \lambda d^{-1}[H_k^{(2)}(\delta) - H_n^{(2)}(\delta)]$$

with $n \geq k$, i.e. the sequence is fundamental and therefore it has a limit point $x^* = \lim_{k\to\infty} x_k$. It is easy to see that

$$\|F(x^*)\| = 0,\ \|x_k - x^*\| \leq \lambda d^{-1}H_k^{(2)}(\delta) \leq \rho,\ \|x_0 - x^*\| \leq \lambda d^{-1}H_0^2(\delta).$$

3. If $\gamma_{1k} \leq C_2\|F(x_k)\|^2$ and $\gamma_{2k} \leq C_4\|F(x_k)\|$, then

$$\|F(x_{k+1})\| \leq d_k^{(3)}\|F(x_k)\| \leq d\|F(x_k)\|^3,$$

where

$$d_k^{(3)} = \beta_k\mu_k C_3 + \frac{1}{2}\lambda_k K(\mu_k C_3 + \lambda_k C_4) + \frac{1}{4}\mu_k^2\lambda_k^2 K(K + 2G\rho)$$
$$+ \frac{1}{2}\lambda_k\mu_k^2 K + \frac{1}{6}\lambda_k^3 L_2$$

and we can take $d = d_0^{(3)}$. In a similar way as before we can prove that

$$\|x_n - x_k\| \leq \lambda d^{-1/(p-1)}[H_k^{(3)}(\delta) - H_n^{(3)}(\delta)],\ n \geq k,$$

$$F(x^*) = 0,\ \|x_k - x^*\| \leq \lambda d^{-1/(p-1)}H_k^{(3)}(\delta) \leq \rho,$$
$$\|x_0 - x^*\| \leq \lambda^{-1/(p-1)}H_0^{(3)}(\delta).$$

2. PARTICULAR METHODS

In this section we shall consider some approximate variants of (1.2) avoiding the computation of F''. Approximating the term $F''(x_k)A_kF(x_k)$ by the expression

$$L(x_k, A_k) = 2\left[F'(x_k) - F'\left(x_k - \frac{1}{2}A_kF(x_k)\right)\right]$$

and using

$$F'(x_k) - \frac{1}{2}F''(x_k)\Gamma_k F(x_k) \approx F'(x_k) - \frac{1}{2}L(x_k, A_k) = F'\left(x_k - \frac{1}{2}A_k F(x_k)\right),$$

we get the method (1.2) in the form

$$x_{k+1} = x_k - \left[F'\left(x_k - \frac{1}{2}A_k F(x_k)\right)\right]^{-1} F(x_k), \tag{2.1}$$

which with $A_k = \Gamma_k$ coincides with the well-known midpoint method.

Since

$$\begin{aligned}\left[F'\left(x_k - \frac{1}{2}A_k F(x_k)\right)\right]^{-1} &= \left\{A_k^{-1}\left[A_k F'(x_k) - \frac{1}{2}A_k L(x_k, A_k)\right]\right\}^{-1} \\ &= U_k^{-1}A_k, \end{aligned} \tag{2.2}$$

Eq. (2.1) can be rewritten as

$$x_{k+1} = x_k - U_k^{-1}A_k F(x_k). \tag{2.3}$$

Suppose now that W_k approximates the inverse $[F'(x_k - \frac{1}{2}A_k F(x_k))]^{-1}$. Then on the basis of (2.2) it can be presented in the form $W_k = V_k A_k$, where $V_k \approx U_k^{-1}$, and (2.3) becomes

$$x_{k+1} = x_k - V_k A_k F(x_k). \tag{2.4}$$

We shall now derive a relation like (1.11).

Putting $P(x) = F'(x),\ \Delta x = -\frac{1}{2}A_k F(x_k)$ and $x_k = x_0$, on the basis of the formula

$$P(x_0 + \Delta x) = P(x) + \int_0^1 P'(x_0 + t\Delta x)\Delta x dt$$

we get

$$\begin{aligned}F'\left(x_k - \frac{1}{2}A_k F(x_k)\right) &= F'(x) - \frac{1}{2}\int_0^1 F''\left(x_k - \frac{t}{2}A_k F(x_k)\right) A_k F(x_k) dt \\ &= F'(x) - \frac{1}{2}F''(x_k)A_k F(x_k) \\ &\quad + \frac{1}{2}\int_0^1 \left[F''(x_k) - F''\left(x_k - \frac{t}{2}A_k F(x_k)\right)\right] \\ &\quad \times A_k F(x_k) dt,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}F''(x_k)A_kF(x_k) - 2\left[F'(x_k) - F'\left(x_k - \frac{1}{2}A_kF(x_k)\right)\right] \\ = F''(x_k)A_kF(x_k) - L(x_k, A_k) \\ = \int_0^1 \left[F''(x_k) - F''\left(x_k - \frac{t}{2}A_kF(x_k)\right)\right] A_kF(x_k)dt.\end{aligned}$$

Consequently,

$$\begin{aligned}\|F''(x_k)A_kF(x_k) - L(x_k, A_k)\| &\le L_2 \int_0^1 \frac{t}{2}\|A_kF(x_k)\|^2 dt \\ &\le \frac{1}{4}\mu_k^2 L_2 \|F(x_k)\|^2 \end{aligned} \tag{2.5}$$

and here G can be taken equal to $L_2/4$. Further,

$$\|L(x_k, A_k)\| \le \mu_k K \|F(x_k)\| + \frac{1}{4}\mu_k^2 L_2 \|F(x_k)\|^2,$$

$$\begin{aligned}\|I - U_k\| &= \left\| I - A_kF'(x_k) + \frac{1}{2}A_kL(x_k, A_k)\right\| \\ &\le \gamma_{2k} + \frac{1}{2}\mu_k^2 K\|F(x_k)\| + \frac{1}{8}\mu_k^3 L_2 \|F(x_k)\|^2,\end{aligned}$$

$$\begin{aligned}\|x_{k+1} - y_k\| =& \|(I - U_kV_k)A_kF(x_k) + (U_k - I)V_kA_kF(x_k)\| \\ &\le \gamma_{1k}\mu_k\|F(x_k)\| + \gamma_{2k}\lambda_k\|F(x_k)\| + \frac{1}{2}\lambda_k\mu_k^2 K\|F(x_k)\|^2 \\ &+ \frac{1}{8}\lambda_k\mu_k^3 L_2\|F(x_k)\|^3,\end{aligned}$$

and

$$\begin{aligned}\|F(x_{k+1})\| \le& \|A_k^{-1}(I - U_kV_k)A_kF(x_k) + R_k\| \\ &\le \gamma_{1k}\mu_k\beta_k\|F(x_k)\| + \frac{1}{8}\lambda_k\mu_k^2 L_2\|F(x_k)\|^3 \\ &+ \frac{1}{2}\gamma_{1k}\lambda_k\mu_k K\|F(x_k)\|^2 + \frac{1}{2}\gamma_{2k}\lambda_k^2 K\|F(x_k)\|^2 \\ &+ \frac{1}{4}\lambda_k^2\mu_k^2 K^2\|F(x_k)\|^3 + \frac{1}{16}\lambda_k^2\mu_k^3 L_2 K\|F(x_k)\|^4 \\ &+ \frac{1}{6}L_2\lambda_k^3\|F(x_k)\|^3.\end{aligned}$$

Ignoring the terms of higher order than $O(\|F(x_k)\|^3)$, we get the inequality

$$\|F(x_{k+1})\| \le \omega_{11}\gamma_{1k}\|F(x_k)\| + \omega_{12}\gamma_{1k}\|F(x_k)\|^2 + \omega_{21}\gamma_{2k}\|F(x_k)\|^2 + C_3\|F(x_k)\|^3, \tag{2.6}$$

where $\omega_{11} = \mu_k\beta_k,\ \omega_{12} = \frac{1}{2}\lambda_k\mu_k K,\ \omega_{21} = \frac{1}{2}\lambda_k^2 K$, and

$$C_3 = \frac{1}{8}\lambda_k\mu_k^2 L_2 + \frac{1}{4}\lambda_k^2\mu_k^2 K + \frac{1}{6}\lambda_k^3 L_2.$$

Since (2.6) is similar to (1.11), analogous statements as established in Theorem 1 are valid for the methods (2.1) and (2.4) as well and, in particular, they are cubically convergent.

Another possibility of getting rid of the evaluation of F'' is replacing it by a bilinear operator:

$$x_{k+1} = x_k - \left[I - \frac{1}{2}A_k\Phi A_k F(x_k)\right]^{-1} A_k F(x_k), \tag{2.7}$$

where $\Phi : X \times X \to Y$ is a general bounded bilinear operator. The method (2.7) has similar computational cost as the Newton method, but it remains faster than the Newton method as shown in [5].

One more possibility of avoiding the evaluation of F'' is replacing in (2.1) A_k by $[F'(\Theta_{k-1})]^{-1}$ and $[F'(x_k - \frac{1}{2}A_kF(x_k))]^{-1}$ by $[F'(\Theta_k^{-1})]$, where

$$\Theta_k = \begin{cases} x_0 & \text{if } k = 0; \\ x_k - \frac{1}{2}[F'(\Theta_{k-1})]^{-1}F(x_k), & \text{if } k \ge 1. \end{cases}$$

After such manipulations the method (2.1) transforms to

$$x_{k+1} = x_k - [F'(\Theta_k)]^{-1}F(x_k), \tag{2.8}$$

which has the convergence order equal to $1+\sqrt{2}$.

Indeed,

$$\begin{aligned}\|I - A_kF'(x_k)\| &= \|[F'(\Theta_{k-1})]^{-1}(F'(\Theta_{k-1}) - F'(x_k))\| \le CK\|x_k - \Theta_{k-1}\| \\ &= CK\left\|x_k - x_{k-1} + \frac{1}{2}[F'(\Theta_{k-2})]^{-1}F(x_{k-1})\right\| \\ &\le CK(\|[F'(\Theta_{k-1})]^{-1}F(x_{k-1})\| + \frac{1}{2}\|[F'(\Theta_{k-2})]^{-1}F(x_{k-1})\|),\end{aligned}$$

i.e. $\gamma_{2k} = O(\|F(x_{k-1})\|)$, and thus the prevailing term in (2.6) is

$$O(\|F(x_{k-1})\|\|F(x_k)\|^2).$$

According to the results of [6] the convergence order can be determined as the smallest positive solution of the equation $z^2 - 2z - 1 = 0$, and it equals to $1 + \sqrt{2}$. Note that the convergence rate $1 + \sqrt{2}$ for the method (2.8) can also be proved without the assumption on twice differentiability of F [7–9].

If instead of $[F'(\Theta_k)]^{-1}$ its approximation W_k and $\|I - F'(\Theta_k)W_k\| = O(\|F(x_{k-1})\|\|F(x_k)\|)$ is used, then the prevailing terms in (2.6) are of order

$$O(\|F(x_{k-1})\|\|F(x_k)\|^2),$$

which implies the convergence order equal to $1 + \sqrt{2}$.

3. CONCLUDING REMARKS

Performance of the methods of type (1.5) is equivalent to either solving the associated linear equations or computing inverses with an error at every iteration step. We can use a strategy of problem solving, which, instead of finding the exact solution of a linear problem at every iteration step solves it intentionally inexactly. The iterative method saves computational work and is adaptive in the sense that it uses low accuracy numerical solutions at inner iterations when the solution is not reached and improves the accuracy as the solution is approached. In many cases iterative methods are more appropriate and economic for solving linear problems than direct ones. Besides, iterative methods are usually self-correcting, and hence they are not sensitive to computational errors.

ACKNOWLEDGEMENT

The support of the Estonian Science Foundation under grant No. 5006 is gratefully acknowledged.

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Koonduvusteoreem puutuvate hüperboolide aproksimatiivsete meetodite kohta

Indrek Kaldo ja Otu Vaarmann

Paljud ülesanded numbrilises analüüsis viivad mittelineaarse võrrandi

$$F(x) = 0, \tag{1.1}$$

lahendamisele, kus F on nõutav arv kordi diferentseeruv operaator ühest abstraktsest ruumist teise. Võrrandi (1.1) lahendamiseks Banachi ruumis on teoreetiliselt uuritud ja edasi arendatud klassikalise kuupkoonduvusega puutuvate hüperboolide meetodi aproksimatiivseid variante (modifikatsioone) eeldusel, et F on kaks korda pidevalt diferentseeruv ja tuletisel F' eksisteerib ühtlaselt tõkestatud pöördoperaator lahendi ümbruses. On tõestatud üks koonduvusteoreem selliste aproksimatiivsete meetodite kohta, mis võimaldab ühtlasi kindlaks teha ühe või teise modifikatsiooni koonduvuskiiruse järgu, mis võib olla ühest kolmeni. On analüüsitud vaatluse all olevate meetodite arvutuslikke aspekte ja efektiivsust.