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On the linear spline collocation for pseudodifferential equations on the torus

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Abstract. We examine the spline collocation method for a class of pseudodifferential equations on a two-dimensional torus. In the analysis, we assume nonuniform mesh, continuous piecewise linear splines, and nodal point collocation. By employing the "Arnold–Wendland trick" we are able to carry out the stability and convergence analysis. The results show quasi-optimal order estimates for the convergence of the collocation solution.

Key words: boundary elements, collocation.

1. INTRODUCTION

In the spline collocation method, an approximate solution for the equation $\mathcal{A}u = f$ is searched from a finite dimensional spline space assuming that the given equation is valid in collocation points. The boundary element spline collocation solution is widely studied in the literature, and fundamental results of [1−5] cover the analysis if $\mathcal{A}$ is a strongly elliptic operator on a one-dimensional smooth boundary curve.

If a quasi-uniform mesh is assumed, the stability and convergence analysis can be done by adopting the approach of [1]. The basic idea is to reduce the collocation problem to an equivalent Galerkin problem by means of the now well known "Arnold–Wendland trick". The same method is applied by Arnold and Saranen [3] in the analysis of periodic problems in the case of several dimensions. The analysis considers partial differential equations of the second order, and it can be quite easily extended to the case of more general periodic problems if one assumes that the principal symbol has constant coefficients. Unfortunately, this is not the case in real applications arising, for example, if the boundary integral solution of elliptic problems in several dimensions is considered.

A more general approach in the analysis can be adopted by assuming that $\mathcal{A}$ is a strongly elliptic pseudodifferential operator. This class covers the classical boundary integral operators, such as the single layer and hypersingular operators, for example. The extensions of such problems to several dimensions is studied in [6–10]. Especially, in [6,8–10] the collocation problem on a multi-dimensional torus has been considered. While the analysis in [8] is restricted to pseudodifferential operators of order zero and analysis of [10] is valid only for a certain modified collocation, the analysis carried out in [6,9] covers also a more general case. However, a uniform mesh has been assumed in [6–9] and the case of a nonuniform mesh is open up to now.

In this work we analyse the spline collocation method assuming that $\mathcal{A}$ is in a class of strongly elliptic pseudodifferential operators on a two-dimensional torus. As trial functions we use the tensor products of continuous piecewise linear splines with collocation at nodal points. A nonuniform mesh is assumed so that the "Arnold–Wendland trick" can be employed. This leads to a certain nonstandard Galerkin method which is equivalent to the collocation problem. The obtained results show quasi-optimal order estimates for the convergence of the collocation solution. Detailed proofs for the results of this paper are given in [11]. Also, the trial space is expanded such that spline functions of arbitrary high odd degree are employed.

2. PRELIMINARIES

Our aim is to analyse the spline collocation method corresponding to the equation

$$\mathcal{A}u(x,y) = f(x,y), \qquad (x,y) \in \mathbb{T}^2,$$

where $\mathbb{T}^2$ is a two-dimensional torus and thus, the variables x and y are both one-periodic. The operator $\mathcal{A}$ is a strongly elliptic classical pseudodifferential operator of degree α. Below we will impose some additional properties for the operator $\mathcal{A}$. These conditions are needed in the analysis of the collocation problem and are known to be true for classical boundary integral operators.

Let $\hat{u}(m,n)$ denote the complex Fourier coefficient of $u(x,y)$. Then the Sobolev space H^s, $s \in \mathbb{R}$, of biperiodic distributions is defined through the norm

$$\|u\|_s = \Big\{ \sum_{n,m\in\mathbb{Z}} |\hat{u}(m,n)|^2 (\underline{m}^2 + \underline{n}^2)^s \Big\}^{1/2}, \qquad \underline{m} = \max\{1, 2\pi|m|\}.$$

We note that the embedding $H^s \subset C_{1,1}(\mathbb{R}^2)$ is continuous if $s > 1$. Here $C_{1,1}(\mathbb{R}^2)$ refers to the space of continuous biperiodic functions. Furthermore, we use the notation $\mathcal{L}(H^s, H^{s'})$ for the space of all bounded linear operators $A : H^s \to H^{s'}$. Now we set the following conditions **(a)**–**(c)** for the operator $\mathcal{A}$.

(a) Let $\alpha \in \mathbb{R}$ be given. For any $J \in \mathbb{N}$ there holds $\mathcal{A} = \sum_{j=0}^{J-1} A_j + A_R$, where $A_R \in \mathcal{L}(H^s, H^{s-\alpha+J})$, $s \in \mathbb{R}$, and the operators A_j, $0 \le j \le J-1$, have the form

$$A_j u(x,y) = \sum_{m,n\in\mathbb{Z}} a_j(x,y,m,n)\hat{u}(m,n)\,\mathrm{e}^{\mathrm{i}2\pi(mx+ny)}\,.$$

Here, the symbols $a_j(x,y,m,n)$, $1 \le j \le J-1$, are in $C^\infty_{1,1}(\mathbb{R}^2 \times \mathbb{Z}^2)$ and are bounded such that for all $\kappa, \lambda \in \mathbb{N}_0$ there holds

$$|\partial_x^\kappa \partial_y^\lambda a_j(x,y,m,n)| \le C_{j,\kappa,\lambda}(\underline{m}+\underline{n})^{\alpha-j}, \quad x,y \in \mathbb{R}, \quad m,n \in \mathbb{Z}. \quad (1)$$

(b) The principal symbol a_0 of the main part operator A_0 has the form

$$a_0(x,y,m,n) = a_{00}(\nu(x)m, \tau(y)n), \quad x,y \in \mathbb{R}, \quad m,n \in \mathbb{Z},$$

where ν and τ are positive one-periodic smooth functions, and $a_{00} \in C^\infty(\mathbb{R}^2)$ is a homogeneous function such that

$$a_{00}(\rho\xi, \rho\eta) = \rho^\alpha a_{00}(\xi,\eta), \qquad \rho \ge 1, \quad |\xi| + |\eta| \ge 1.$$

(c) $\mathcal{A} : H^s \to H^{s-\alpha}$ is an isomorphism for all $s \in \mathbb{R}$, and strongly elliptic such that there exists a constant $C > 0$ such that $\mathrm{Re}(\mathcal{A}u, u)_0 \ge C\,\|u\|^2_{\alpha/2}$ for any $u \in H^{\alpha/2}$. Here, $(\cdot,\cdot)_0$ is the L^2-inner product.

In order to give an example of an operator that has properties **(a)**–**(c)**, we recall from [6] the single layer boundary operator V, defined by

$$Vu(x_\Gamma, y_\Gamma) = \frac{1}{4\pi}\int_\Gamma \frac{u(\tilde{x}_\Gamma, \tilde{y}_\Gamma)\,\mathrm{d}s_\Gamma}{\big|(x_\Gamma, y_\Gamma) - (\tilde{x}_\Gamma, \tilde{y}_\Gamma)\big|}, \qquad (x_\Gamma, y_\Gamma) \in \Gamma,$$

where $\Gamma \subset \mathbb{R}^3$ is a torus with radii r and R, and $\mathrm{d}s_\Gamma$ is the element of the surface area on Γ. We note that, by using curvilinear coordinates, V can be viewed as an operator on $\mathbb{T}^2$. According to [6], the principal symbol of V attains the form

$$a_{V_0}(x,y,\xi,\eta) = \frac{1}{2}\nu(x)\big(\nu(x)^2\xi^2/r^2 + \eta^2\big)^{-1/2},$$

where $\nu(x) = R + r\cos 2\pi x$ is a positive smooth function. This principal symbol does not fulfil the requirements of **(b)**, but if we define $A = \nu(x)^{-1}V$, then the main part of A has the desired form with $\alpha = -1$. This modification is not crucial from the collocation point of view.

3. THE COLLOCATION PROBLEM

For any given positive integers M and N we define nonuniform periodic meshes $\Delta_x = \{x_m\}_{m\in\mathbb{Z}}$ and $\Delta_y = \{y_n\}_{n\in\mathbb{Z}}$ such that $x_m < x_{m+1}$, $x_{m+M} = x_m + 1$ for all $m \in \mathbb{Z}$, and $y_n < y_{n+1}$, $y_{n+N} = y_n + 1$ for all $n \in \mathbb{Z}$. The corresponding mesh parameters are defined by $h_{\Delta_x} = \max_{m\in\mathbb{Z}} |x_{m+1} - x_m|$ and $h_{\Delta_y} = \max_{n\in\mathbb{Z}} |y_{n+1} - y_n|$, and we denote $h_\Delta = \max\{h_{\Delta_x}, h_{\Delta_y}\}$. Let $S^1_{\Delta_x}$ and $S^1_{\Delta_y}$ be the spaces of all one-periodic continuous piecewise linear splines with respect to the meshes Δ_x and Δ_y. In the collocation method we apply the tensor product trial space $\mathcal{M}^1_\Delta = S^1_{\Delta_x} \otimes S^1_{\Delta_y}$ and nodal point collocation as follows:

$$\text{Find } u_\Delta \in \mathcal{M}^1_\Delta : \ \mathcal{A}u_\Delta(x_m, y_n) = \mathcal{A}u(x_m, y_n), \quad 1 \le m \le M, \quad 1 \le n \le N. \tag{2}$$

We note that $\mathcal{A}u$ is continuous if $u \in H^s$, $s > \alpha + 1$. Moreover, $\mathcal{M}^1_\Delta$ is a subspace of H^s if $s < 3/2$, and therefore, $\mathcal{A}u_\Delta$ is continuous for $\alpha \le 0$.

Let us define the operators $\underline{\partial}_x = \partial_x + J_x$ and $\underline{\partial}_y = \partial_y + J_y$, where

$$J_x u(x,y) = \int_0^1 u(z,y)\,\mathrm{d}z, \qquad J_y u(x,y) = \int_0^1 u(x,z)\,\mathrm{d}z.$$

Let us note that $\underline{\partial}_x, \underline{\partial}_y \in \mathcal{L}(H^s, H^{s-1})$ and $\underline{\partial}_x^{-1}, \underline{\partial}_y^{-1} \in \mathcal{L}(H^s, H^s)$. Let us also note that $\underline{\partial}_x\underline{\partial}_y = \underline{\partial}_y\underline{\partial}_x$ and $\underline{\partial}_x^{-1}\underline{\partial}_y^{-1} \in \mathcal{L}(H^s, H^{s+1})$. Moreover, we define the discretized counterpart of $\underline{\partial}_x$ by $\underline{\partial}_{\Delta_x} = \partial_x + J_{\Delta_x}$, where

$$J_{\Delta_x} u(x,y) = \sum_{m=0}^{M-1} \frac{x_{m+1} - x_{m-1}}{2} u(x_m, y)$$

is the trapezoidal rule approximation. The operator $\underline{\partial}_{\Delta_y}$ is defined analogously. Let us now define the operator $\tilde{\mathcal{A}}_\Delta = \underline{\partial}_{\Delta_x}\underline{\partial}_{\Delta_y}\mathcal{A}\underline{\partial}_x^{-1}\underline{\partial}_y^{-1}$.

Let u be the solution of $\mathcal{A}u = f$. Let us consider the following Galerkin problem:

$$\text{Find } u_\Delta \in \mathcal{M}^1_\Delta : \ \langle \tilde{\mathcal{A}}_\Delta \underline{\partial}_x\underline{\partial}_y u_\Delta, \underline{\partial}_x\underline{\partial}_y v\rangle = \langle \tilde{\mathcal{A}}_\Delta \underline{\partial}_x\underline{\partial}_y u, \underline{\partial}_x\underline{\partial}_y v\rangle, \qquad v \in \mathcal{M}^1_\Delta. \tag{3}$$

Here $\langle\cdot,\cdot\rangle$ refers to the duality brackets and is an extension of the L^2-inner product. There holds

Lemma 1. *The collocation problem* (2) *and the Galerkin problem* (3) *are equivalent.*

4. MAPPING PROPERTIES OF THE STUDIED OPERATORS

By the formula $\tilde{A} = \underline{\partial}_x \underline{\partial}_y A \underline{\partial}_x^{-1} \underline{\partial}_y^{-1}$ we define a modified operator $\tilde{A}$ corresponding to any bounded operator A. In what follows we need the operator

$$\tilde{\mathcal{A}} = \underline{\partial}_x \underline{\partial}_y \mathcal{A} \underline{\partial}_x^{-1} \underline{\partial}_y^{-1} = \sum_{j=0}^{J-1} \tilde{A}_j + \tilde{A}_R.$$

Lemma 2. *Let* $(\mathbf{a})$–$(\mathbf{c})$ *be valid and assume that* $u \in H^{\alpha/2+2}$ *and* $\alpha \leq 0$. *Then there exists a constant* $C > 0$ *such that*

$$\|(\tilde{\mathcal{A}} - \tilde{\mathcal{A}}_\Delta)\underline{\partial}_x \underline{\partial}_y u\|_{-\alpha/2} \leq C h_\Delta \|\underline{\partial}_x \underline{\partial}_y u\|_{\alpha/2}.$$

Hereby, the difference $\tilde{\mathcal{A}} - \tilde{\mathcal{A}}_\Delta$ is "small" and the properties of the operator $\tilde{\mathcal{A}}$ instead of $\tilde{\mathcal{A}}_\Delta$ can be applied. The quasi-optimal approximation result can be proved by employing the decomposition $\tilde{\mathcal{A}} = \mathcal{A} + (\tilde{\mathcal{A}} - \mathcal{A})$. Here it is crucial that $\mathcal{A} \in \mathcal{L}(H^s, H^{s-\alpha})$ is strongly elliptic as stated in (**c**), and that $\tilde{\mathcal{A}} - \mathcal{A}$ is a compact mapping of H^s into $H^{s-\alpha}$ as will be shown next.

Theorem 3. *Let assumptions* $(\mathbf{a})$ *and* $(\mathbf{b})$ *be valid, and let* $\tilde{\mathcal{A}} = \underline{\partial}_x \underline{\partial}_y \mathcal{A} \underline{\partial}_x^{-1} \underline{\partial}_y^{-1}$. *Then* $\tilde{\mathcal{A}} - \mathcal{A} \in \mathcal{L}(H^s, H^{s-\alpha+1})$.

Proof. There holds $\tilde{\mathcal{A}} - \mathcal{A} = \sum_{j=0}^{J-1}(\tilde{A}_j - A_j) + (\tilde{A}_R - A_R)$. First, $A_R \in \mathcal{L}(H^s, H^{s-\alpha+J})$ by definition and if $J \geq 2$, then $\tilde{A}_R \in \mathcal{L}(H^s, H^{s-\alpha+1})$. Moreover, the boundedness condition (1) imposed for symbols $a_j(x, y, m, n)$, $1 \leq j \leq J-1$, yields that the corresponding operators $A_j \in \mathcal{L}(H^s, H^{s-\alpha+j})$. A similar result in the one-periodic case is given, for example, in [12], Theorem 7.3.1.

Consider next the modified operators $\tilde{A}_j$, $1 \leq j \leq J-1$. The symbol of $\tilde{A}_j$ is

$$\underline{a}_j(x, y, m, n) = \sigma(m)^{-1}\sigma(n)^{-1}\underline{\partial}_x \underline{\partial}_y \big(a_j(x, y, m, n)\, \mathrm{e}^{\mathrm{i}2\pi(mx+ny)}\big)\, \mathrm{e}^{-\mathrm{i}2\pi(mx+ny)},$$

where $\sigma(m) = \mathrm{i}2\pi m$ if $m \neq 0$, and $\sigma(0) = 1$. Furthermore, the Fourier coefficients of $\underline{a}_j$ are bounded such that

$$|\hat{\underline{a}}_j(\hat{m}, \hat{n}, m, n)| \leq C_r(\underline{\hat{m}} + \underline{\hat{n}})^{-r}(\underline{m} + \underline{n})^{\alpha-j},$$

where r can be any non-negative integer. Therefore, it can be concluded that also $\tilde{A}_j \in \mathcal{L}(H^s, H^{s-\alpha+j})$, cf. the proof of Theorem 7.3.1 in [12].

Considering the main part operator, we employ the decomposition

$$\tilde{A}_0 - A_0 = \underline{\partial}_y B_{00} \underline{\partial}_y^{-1} + B_{01},$$

where

$$B_{00} = \underline{\partial}_x A_0 \underline{\partial}_x^{-1} - A_0, \qquad B_{01} = \underline{\partial}_y A_0 \underline{\partial}_y^{-1} - A_0.$$

The Fourier coefficients of the symbol of B_{00} are $\hat{b}_{00}(\hat{m},\hat{n},m,n) = (\hat{m}/m)\,\hat{a}_0(\hat{m},\hat{n},m,n)$. Therefore, with any $p,q\in\mathbb{N}$, there holds

$$\left|\partial_x^\kappa\partial_y^\lambda b_{00}(x,y,m,n)\right| \le C\sum_{\hat{m},\hat{n}\in\mathbb{Z}} \underline{\hat{m}}^{\kappa+1-p}\underline{\hat{n}}^{\lambda-q}|m|^{-1}|\widehat{\partial_x^p\partial_y^q a_0}(\hat{m},\hat{n},m,n)|,\ m\neq 0. \tag{4}$$

Recall the properties of the symbol a_0 given in (**b**). There exist positive constants c_ν and c_τ such that $\nu(x)\ge c_\nu$ and $\tau(x)\ge c_\tau$. Let $c_{\nu,\tau}=\min\{c_\nu,c_\tau\}/2\pi$. Then we can choose $\rho=c_{\nu,\tau}(\underline{m}+\underline{n})$ and have

$$a_{00}(\nu(x)m,\tau(y)n)=\rho^\alpha a_{00}\Big(\frac{\nu(x)m}{\rho},\frac{\tau(y)n}{\rho}\Big),\qquad |m|+|n|\ge\big(\min\{c_\nu,c_\tau\}\big)^{-1}.$$

By applying formula (0.430) in [13] for the general order derivative of composite function we find that

$$\begin{aligned}&\partial_x^p\partial_y^q a_{00}(\nu(x)m,\tau(y)n)\\&\qquad=\sum_{k=1}^{p}\sum_{l=1}^{q}\rho^{\alpha-k-l}m^k n^l R_\nu^{k,p}(x)R_\tau^{l,q}(y)a_{00}^{(k,l)}\Big(\frac{\nu(x)m}{\rho},\frac{\tau(y)n}{\rho}\Big).\end{aligned}$$

Here the superscript notation in a_{00} indicates the partial derivatives with respect to the first and the second argument, and

$$R_\nu^{k,p}(x)=\frac{1}{k!}\sum_{k'=0}^{k-1}\binom{k}{k'}(-1)^{k'}\nu(x)^{k'}D_x^p\big(\nu(x)^{k-k'}\big).$$

The function $R_\tau^{l,q}(y)$ is defined analogously. Hence we find that

$$\left|\partial_x^p\partial_y^q a_{00}(\nu(x)m,\tau(y)n)\right|\le C_{p,q}|m|\cdot(\underline{m}+\underline{n})^{\alpha-1}. \tag{5}$$

Next we combine (4) and (5) and choose p and q such that the series on the right-hand side is convergent. Then we have

$$\left|\partial_x^\kappa\partial_y^\lambda b_{00}(x,y,m,n)\right|\le C_{\kappa,\lambda}(\underline{m}+\underline{n})^{\alpha-1}. \tag{6}$$

Therefore, $B_{00}\in\mathcal{L}(H^s,H^{s-\alpha+1})$. Also, the operator B_{01} has the same property based on symmetry. It remains to consider the operator $\underline{\partial}_y B_{00}\underline{\partial}_y^{-1}$. Since (6) is valid, we can proceed in a similar way as in the case of the operators $\tilde{A}_j$, $1\le j\le J-1$, earlier in the proof. As an outcome we find that $\underline{\partial}_y B_{00}\underline{\partial}_y^{-1}$ is also in $\mathcal{L}(H^s,H^{s-\alpha+1})$. ☐

Based on Theorem 3 and the Fredholm properties of $\tilde{\mathcal{A}}$ we can prove

Theorem 4. *Let assumptions* (**a**)–(**c**) *be valid. Then the operator $\tilde{\mathcal{A}}$ is an isomorphism of H^s onto $H^{s-\alpha}$ for any $s\in\mathbb{R}$.*

5. STABILITY AND CONVERGENCE RESULTS

We note that $\underline{\partial}_x\underline{\partial}_y$ is an isomorphism of $\mathcal{M}^1_\Delta$ onto $\mathcal{M}^0_\Delta = S^0_{\Delta_x} \otimes S^0_{\Delta_y}$, which is the space of tensor products of the one-dimensional piecewise constant splines. Moreover, $\mathcal{M}^0_\Delta$ is a subspace of $H^{\alpha/2}$ if $\alpha \leq 0$. Therefore, we can employ in the space $\mathcal{M}^1_\Delta$ the norm

$$|\!|\!| u |\!|\!| = \|\underline{\partial}_x\underline{\partial}_y u\|_{\alpha/2}, \qquad u \in \mathcal{M}^1_\Delta.$$

Based on the equivalence of the collocation problem (2) and the Galerkin problem (3), Theorem 4, and the Babuska–Aziz infsup-condition (see [14]), we can prove

Theorem 5. *Let $\mathcal{A}$ be defined by* (**a**)–(**c**) *and suppose that $u \in H^{\alpha/2+2}$ and $\alpha \leq 0$. Then there exists $h_0 > 0$ such that the collocation problem* (2) *has a unique solution provided that $0 < h_\Delta \leq h_0$. Moreover, we have the quasi-optimal approximation result*

$$|\!|\!| u - u_\Delta |\!|\!| \leq C \inf \left\{ |\!|\!| u - v |\!|\!| :\, v \in \mathcal{M}^1_\Delta \right\}.$$

In the convergence analysis we need the spaces $H^{s,r}$ defined through the norm

$$\|u\|_{s,r} = \Big\{ \sum_{m,n\in\mathbb{Z}} |\hat{u}(m,n)|^2 \underline{m}^{2s} \underline{n}^{2r} \Big\}^{1/2}, \qquad s, r \in \mathbb{R}.$$

The notations P_{Δ_x} and P_{Δ_y} correspond to the one-dimensional projections of L^2-functions to the appropriate spaces of splines. The one-dimensional approximation result (see [15]) implies that

$$\begin{aligned} \|u - P_{\Delta_x} u\|_{p,q} &\leq C h_{\Delta_x}^{s-p} \|u\|_{s,q}, \qquad && u \in H^{s,q}, \\ \|u - P_{\Delta_y} u\|_{p,q} &\leq C h_{\Delta_y}^{r-q} \|u\|_{p,r}, \qquad && u \in H^{p,r}, \end{aligned}$$

whenever $0 \leq p \leq s \leq 2$, $p < 3/2$ and $0 \leq q \leq r \leq 2$, $q < 3/2$. By $P_\Delta = P_{\Delta_x} P_{\Delta_y}$ we denote the projection of H^0 into $\mathcal{M}^1_\Delta$. By applying the decomposition $I - P_\Delta = I - P_{\Delta_x} + P_{\Delta_x}(I - P_{\Delta_y})$ and the norm

$$\|u\|'_{\alpha/2} = \Big\{ \inf \left\{ \|u_1\|^2_{\alpha/2,0} + \|u_2\|^2_{0,\alpha/2} : u = u_1 + u_2 \right\} \Big\}^{1/2}$$

we can prove the following approximation result.

Lemma 6. *If $\alpha \leq 0$, there holds*

$$|\!|\!| u - P_\Delta u |\!|\!| \leq C h_\Delta^{1-\alpha/2} \|u\|_{2,2}, \qquad u \in H^{2,2}.$$

By applying Lemma 6 and the quasi-optimality approximation result of Theorem 5 we obtain the following convergence results for the collocation solution.

Theorem 7. *Let $u \in H^{2,2}$ be the solution of the equation $\mathcal{A}u = f$, and let $\alpha \leq 0$. If $u_\Delta \in \mathcal{M}^1_\Delta$ is the solution of the collocation problem* (2), *then there exists a positive h_0 such that if $0 < h_\Delta \leq h_0$, there holds*

$$\begin{aligned} \|u - u_\Delta\| &\leq C h_\Delta^{1-\alpha/2} \|u\|_{2,2}, \\ \|u - u_\Delta\|_{1,1} &\leq C(h_{\Delta_x}, h_{\Delta_y}) h_\Delta \|u\|_{2,2}, \\ \max_{(x,y)\in\mathbb{R}^2} |(u - u_\Delta)(x,y)| &\leq C(h_{\Delta_x}, h_{\Delta_y}) h_\Delta \|u\|_{2,2}, \end{aligned}$$

where $C(h_{\Delta_x}, h_{\Delta_y}) = C\big(1 + \max\{(h_{\Delta_x}/h_{\Delta_y})^{\alpha/2}, (h_{\Delta_x}/h_{\Delta_y})^{-\alpha/2}\}\big)$.

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Lineaarsplainidega kollokatsioonimeetod pseudodiferentsiaalvõrrandite jaoks tooril

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On käsitletud kahemõõtmelisel tooril määratud pseudodiferentsiaalvõrrandite lahendamist kollokatsioonimeetodiga. On vaadeldud juhtu, kus koordinaatfunktsioonideks on tensorkorrutised pidevatest tükiti lineaarsetest splainidest ning kollokatsioonipunktid langevad kokku võrgu sõlmedega. On uuritud kollokatsioonimeetodi koonduvust ja koonduvuskiirust mitteühtlaste võrkude korral.