A method to solve ordinary differential equations

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Abstract. Algebraic transformations for nonlinear and nonhomogeneous ordinary differential equations are introduced that yield, step by step, quasilinear forms and analytic solutions on small subintervals. Usually in the relevant physical example these analytical solutions allow us to obtain the time evolution operator, for any initial conditions.

Key words: differential equations, Lyapunov stability, Bessel's equation.

1. INTRODUCTION

Many problems in physics, chemistry, engineering, and operative research are mathematically formulated using ordinary differential equations. These equations are neither linear nor homogeneous and they are endowed with variable coefficients. In such cases the analytic solutions are difficult to obtain or they are generated by doubtful approaches. Many calculation methods exist [1,2], a good summary of which can be found in [3,4].

In this work we show that it is possible to obtain numeric solutions using the theory of linear equations. We propose a calculation method step by step. These steps represent the solutions in the basis associated to the linear form of the problem.

2. THE PROBLEM

Let us consider the differential equation in a general form

$$F(x(t), x^{(1)}(t), \ldots, x^{(n)}(t), t) = 0, \tag{2.1}$$

$$X^{(j)}(t_0) = x_0^{(j)}, \quad j = 0, \ n-1,$$

or, when possible, as the equivalent system of equations:

$$\dot{y}_k = f_k(y_1, \ldots, y_n, t), \tag{2.2}$$

$$y_k(t_0) = y_{k0}, \quad k = 1, \ n.$$

Let us further suppose that Eq. (2.1) may be nonlinear or linear with variable coefficients, but it can be reduced to the quasilinear form:

$$\sum_{j=0}^{n} a_j x^{(j)}(t) = f, \tag{2.3}$$

where

$$a_j = a_j(x, x^{(1)}, \ldots, x^{(n-1)}, t), \tag{2.4}$$

$$f = f(x, x^{(1)}, \ldots, x^{(n-1)}, t). \tag{2.5}$$

We have written the same equation in different forms in order to use some properties of the linear systems.

3. LINEARIZATION AND EVOLUTION OPERATOR

We add the condition that both a_j and f would be nonsingular functions to integrate Eq. (2.3). More precisely, these functions must be continuous with respect to t and to the set of variables $x, x^{(1)}, \ldots, x^{(n-1)}$, in a certain domain of t. As in other classical methods the derivability condition of a_j and f is not necessary. Let us decompose the t domain in intervals I_i, defined as

$$T_i \leq t \leq t_{i+1}, \quad t_{i+1} = t_i + h_i, \tag{3.1}$$

where $h_i > 0$ is the width of the interval I_i and $i = 0, 1, \ldots n$. We will consider two cases.

3.1. Case $f = f(t)$

In this case we consider Eq. (2.3) in each interval I_i as

$$\sum_{j=0}^{n} a_{ij} z_i^{(j)}(t) = f(t), \quad t \in I_i, \tag{3.2}$$

where $z_i(t)$ is an estimated function of $x(t)$ in the interval I_i, and

$$a_{ij} = a_j(z_i(t), z_i^{(1)}(t_i), \ldots, z_i^{(n-1)}(t_i), t_i). \tag{3.3}$$

Now we solve the linear equation (3.2) in the interval I_i beginning with the initial condition at $t = 0$, then integrating in I_0, obtaining the initial condition at t_1, integrating in $I_1, \ldots$, etc. For each interval I_i we obtain the estimated solution,

$$z_i(t) = \sum_{k=1}^{n} c_{ik} \phi_{ik}(t) + z_{ip}(t), \tag{3.4}$$

where $\{\phi_{ik}(t)\}$ is a set of N linearly independent solutions of the homogeneous version of Eq. (3.2) and $z_{ip}(t)$ is a particular solution of its nonhomogeneous version. Constant c_{ik} can be computed using the boundary conditions at $t = t_1$, more precisely, by solving the system

$$z_i^{(1)}(t_i) = \sum_{k=1}^{n} c_{ik} \phi_{ik}^{(1)}(t_i) + z_{ip}^{(1)}(t_i), \quad l = 0, \ldots, n-1,$$
$$z_0^{(1)}(t_0) = x^{(1)}(t_0). \tag{3.5}$$

This system can be solved since $\{\phi_{ik}(t)\}$ is a linearly independent set of solutions. In this way we have obtained an analytic approximation of $x(t)$ in the ith interval given by Eq. (3.4). In order to compute the solution in these intervals I_{i+1}, the initial conditions $Z_{i+1}^{(l)}(t)$, $l = 0, \ldots, n-1$ at t_{i+1} are determined with their derivatives. Formally we can say that

$$Z(t_{i+1}) = G(z_i(t_i), z_I^{(1)}(t_i), \ldots, z_i^{(n-1)}(t_i)), \tag{3.6}$$

or, written in a more compact way,

$$z_{i+1} = P_i z_i, \tag{3.7}$$

where P is the evolution operator in the interval I_i and z_i is the estimated value of x in t_i. So we can determine the solution of Eq. (2.1) in an approximative way, with continuity up to the $(n-1)$th derivative.

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3.2. Case $f = f(x, x^{(1)}, \ldots, x^{(n-1)}, t)$

When, via factorization, this problem is written as in Eq. (2.3), it may happen that the function f will be such that the terms which eventually appear cannot be incorporated in the quasilinear form. Then in each element I_i we can define a number

$$f_i = f(x(t_i), \ldots, x^{(n-1)}(t_i), t_i).\qquad(3.8)$$

The nonhomogeneous terms are taken as constants and Eq. (2.3) reads

$$\sum_{j=0}^{n} a_{ij} z_i^{(j)}(t) = f_i.\qquad(3.9)$$

Then we can follow the procedure as in the previous subsection. We could also work with greater precision if we defined

$$f_i = f(x(t_i), \ldots, x^{(n-1)}(t_i), t) = g_i(t).\qquad(3.10)$$

In this case, however, some difficulties appear when we try to compute the solution of the particular inhomogeneous equation, so we will not follow this path.

4. LYAPUNOV STABILITY

One of the advantages of using the sectional continuous linear equation (obtained by reducing Eq. (2.3) to the quasilinear form and using Eqs. (3.3), (3.8)) is that in each interval we have an nth-order linear equation with a constant coefficient that allows us to study the stability of the solution using the Lyapunov criterion [5]. This can be done using the basis $\{\phi_{ik}\}$. Then, if in our nonlinear problem we have a sequence of intervals, where at least one root of the characteristic polynomial associated to the linear equations is positive, we can find a subset of the set solution, where these solutions evolve in an exponential way. Then this will be a critical consideration for the choice of h_i. In this way we have a quasiempirical method for finding transition or chaotic regions, as can be see in Section 9.

5. STRUCTURAL STABILITY

The evolution operators may create a problem if the roots of their characteristic polynomial of Eq. (3.2) vanish, or are very close to zero. In this case small fluctuations in the equation coefficients, produced by the numerical

truncation or by a systematical error, may cause changes of signs of the value of the roots, and therefore substantial modifications in the form of the solution.

If at least one of the roots persists to be near zero, the efficiency of the method can be in danger. Nevertheless, this problem is also present in usual methods as, e.g., the Runge–Kutta method for different orders [6,7].

6. NUMERICAL CONVERGENCE

We do not have an analytic result to obtain a convergence condition, but it is clear that when $h \to 0$, the numerical results tend to exact solutions, but it is not a useful criterion. We can use a pragmatic criterion to determine the "reliability" of the sectional analytic solutions. Since these solutions depend on the value of h, we must study the sensibility of the solution by varying h and determine an appropriate step as a function of the desired degree of precision.

When we can find at least a constant of motion, this constant must not be used to reduce the order of the equation, but must be taken as a control variable. Then the selection of the integration step can be determined by the maximum variation of the motion constants, according to the degree of precision required by the problem.

7. A SIMPLE MODEL

As the first example of the nonlinear system the ecological model of Volterra [8] is studied. It is defined by the system

$$
\begin{aligned}
\frac{d}{dt} x &= x(a - by), \\
\frac{d}{dt} y &= -y(c - dx),
\end{aligned}
\tag{7.1}
$$

where x is the population of the vegetarian species and y is the population of the predator of the species x and a, b, c, d are constants of the systems. The first integral of Eq. (7.1) is

$$
\ln(y^a x^c) - by - dx = C.
\tag{7.2}
$$

Using the calculation technique developed in Section 3, we get the iterative solution

$$
\begin{aligned}
x_{i+1} &= x_i \exp(\alpha_i h), \\
y_{i+1} &= y_i \exp(\beta_i h),
\end{aligned}
\tag{7.3}
$$

where

$$\begin{aligned} \alpha_i &= a - by, \\ \beta_i &= -c + dx_i, \end{aligned}$$
(7.4)

and h is the time increment. The evolution of the motion constant (7.2) is

$$C_{i+1} = C_i + E_i,$$
(7.5)

where E_i is the error of the method,

$$E_i = -(dx_i\alpha_i^2 + by_i\beta_i^2)\frac{h^2}{2}$$
(7.6)

and

$$E_i = 0, \forall i,$$
(7.7)

when the initial conditions of Eqs. (7.1) are

$$\begin{aligned} x_0 &= c/d, \\ y_0 &= a/b. \end{aligned}$$
(7.8)

8. BESSEL'S EQUATION

As the second example, a linear problem with variable coefficients is studied, more precisely the modified Bessel equation of integer order,

$$z^2\frac{d^2}{dz^2}w + z\frac{d}{dz}w - (z^2 + w^2)w = 0.$$
(8.1)

The analytical solutions are expressed as linear combinations of the modified Bessel functions of integer order [9],

$$I_v(z), \ K_v(z).$$

According to the structure of our calculation method, the solution of Eq. (8.1) is represented using a real exponential basis. In this way the discrete solutions to $w(z)$ and $w'(z)$ are expressed as

$$\begin{aligned} W_{n+1} &= C_n^+\exp(P_n^+h) + C_n^-\exp(P_n^-h), \\ W'_{n+1} &= P_n^+C_n^+\exp(P_n^+h) + P_n^-C_n^-\exp(P_n^-h), \end{aligned}$$
(8.2)

where

$$\begin{aligned} C_n^+ &= z_n(w'_n - w_nP_n^-), \\ C_n^- &= z_n(w_nP_n^+ - w'_n), \end{aligned}$$
(8.3)

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$$P_n^\pm = -\frac{1}{2z_n}\left(1 \pm \sqrt{1+4z_n^2}\right), \tag{8.4}$$

$$z_n = nh, \quad n \text{ is integer.} \tag{8.5}$$

For $v = 0$, $w(0) = 1$, and $w'(0) = 0$, the analytical solution of Eq. (8.1) is $w(z) = I_0(z)$ defined by

$$I_0(z) = \sum_{k=0}^{\infty}\left(\frac{z^2}{4}\right)^k K!^{-2}, \tag{8.6}$$

where we can see the divergence of $I_0(z)$ for $z \gg 1$. We regularize analytical and numerical solutions as follows

$$W(z) = \sqrt{z}\,\exp(-z)w(z), \tag{8.7}$$

$$W_n = \sqrt{z}\,\exp(-z_n)w_n. \tag{8.8}$$

9. QUANTUM OSCILLATOR

As the last example, an eigenvalue problem is studied, for the Schrödinger equation [10],

$$H\Psi = E\Psi, \tag{9.1}$$

more precisely, the one of a quantum oscillator. The associated equation is

$$\frac{d^2\phi}{dx^2} + (e - x^2)\phi = 0, \tag{9.2}$$

where ϕ is such that

$$\int_{-\infty}^{+\infty}\phi^*\phi\,dx = \text{finite.} \tag{9.3}$$

Here ϕ is the wave function and e is the eigenvalue. It is known that

$$e_k = 2k + 1 \tag{9.4}$$

and

$$\phi_k = \alpha_k\,\exp\left(-\frac{1}{2}x^2\right)H_k(x), \tag{9.5}$$

where $H_k(x)$ are the Hermite polynomials of degree k and α_k is the normalization constant. Moreover, being an eigenvalue problem, the differential

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equations are structurally unstable under the variations of e, making the obtained numerical solutions unstable for the big values of x. Our solution has better numerical properties with respect to other calculation methods, and also a smaller computation time. The region where the solution diverges from the exact one, corresponding to $x > x_c$, where $x_c = (e_k)^{1/2}$ is the classically allowed maximum width. The reason for this is that, in the classically allowed area, the equation associated with constant coefficients has an equilibrium point in the phase space: a centre, which gives the numerical stability and the solutions can be represented with a harmonic basis. In a similar way for $x > x_c$, the singular point is unstable and the basis of solutions contains a real exponential. Moreover, that is the reason why numerical fluctuation in the structure of the equation induces the loss of quality of the solution.

10. CONCLUSIONS

The advantage of the proposed method, which is in fact only an alternative to obtaining a solution of nonlinear differential equations or linear equations with variable coefficients, is that it is easier to implement as the roots of the characteristic polynomial are easy to obtain with low error. Moreover, in the set of problems presented in this paper, we obtain a smaller computation time with respect to the Runge–Kutta method for the 4th order. It is possible to use bigger steps of integration without losing a significative precision with respect to the other methods, because of the peculiar way the quasilinear form is generated. The method is useful for the study of the stability of the solution.

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Meetod harilike diferentsiaalvõrrandite lahendamiseks

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Cauchy ülesanne üldise mittelineaarse hariliku diferentsiaalvõrrandi jaoks on taandatud algebraliste teisenduste abil samm-sammult väikestes vahemikes kvaasilineaarsele kujule ja seejärel lahendatud analüütiliselt. Mitmetes füüsikalistes näidetes võimaldab see lähendada evolutsioonioperaatorit protsessi kulgemisest ajas.