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PSEUDOSYMMETRIC CONTACT METRIC MANIFOLDS IN THE SENSE OF M. C. CHAKI

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Abstract. We consider pseudosymmetric and pseudo Ricci symmetric manifolds in the sense of M. C. Chaki. The case M is assumed to be a contact metric manifold with ξ belonging to (k, μ)-nullity distribution.

Key words: contact manifolds, Einstein (η-Einstein) manifolds, $(k,\ \mu)$-nullity distribution, pseudosymmetric manifolds of Chaki type.

1. INTRODUCTION

Throughout this paper we use the notations and terminology of [1,2]. Let M be a $(2n+1)$-dimensional Riemannian C^∞ manifold. M^{2n+1} is said to be a *contact manifold* if it admits a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field satisfying

$$\eta(\xi) = 1, \quad d\eta(\xi, X) = 0 \tag{1}$$

for any vector field X. It is well known that there exists a Riemannian metric g and a (1,1)-tensor field φ such that

$$\eta(X) = g(X,\xi),\ d\eta(X,Y) = g(X,\varphi Y),\ \text{and}\ \varphi^2 X = -X + \eta(X)\xi, \tag{2}$$

where X and Y are vector fields on M. From (2) it follows that

$$\varphi\xi = 0,\ \eta \circ \varphi = 0,\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{3}$$

A Riemannian manifold M, equipped with structure tensors (φ, ξ, η, g) satisfying (2), is said to be a *contact metric manifold* and is denoted by $M = (M^{2n+1}, \varphi, \xi, \eta, g)$.

Given a contact metric manifold M, we can define a $(1,1)$-tensor field h by $h = \frac{1}{2}L_\xi\varphi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \text{ and } h\varphi = -\varphi h, \tag{4}$$

$$\nabla_X\xi = -\varphi X - \varphi hX, \tag{5}$$

where ∇ is the Levi–Civita connection [[2]].

We denote by R the *Riemannian curvature tensor field* defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \tag{6}$$

for all vector fields X, Y, Z.

For a contact metric manifold M one may define naturally an almost complex structure on $M \times \mathbb{R}$. If this almost complex structure is integrable, M is said to be a *Sasakian manifold*. A Sasakian manifold is characterized by the condition

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(X)Y \tag{7}$$

for all vector fields X and Y on the manifold [[1]].

Let M be a contact metric manifold. It is well known that M is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{8}$$

for all vector fields X and Y [[1]].

A contact metric manifold M is said to be *η-Einstein* if

$$Q = aI_d + b\eta \otimes \xi, \tag{9}$$

where Q is the Ricci operator and a, b are smooth functions on M [[2]].

2. KNOWN RESULTS

In this section we give some well-known results.

Let M be a contact metric manifold. The (k, μ)-*nullity distribution* of M for the pair (k, μ) is a distribution

$$N(k, \mu) : p \to N_p(k, \mu) = \{Z \in T_pM \mid R(X, Y, Z) = k[g(Y, Z)X - g(X, Z)Y] \\ + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \tag{10}$$

where $k, \mu \in \mathbb{R}$ and $k \leq 1$ (see [3,4]). If $k = 1$, then $h = 0$ and M is a Sasakian manifold [2]. So if the characteristic vector field ξ belongs to the (k, μ)-nullity distribution, we have

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{11}$$

Lemma 2.1 (*see* [2]). *Let M be a contact metric manifold with ξ belonging to the (k, μ)-nullity distribution. Then*

(i) $(\nabla_X h)Y = [(1-k)g(X, \varphi Y) - g(X, h\varphi Y)]\,\xi + \eta(Y)h(\varphi X + \varphi hX) - \mu\eta(X)\varphi hY$,

(ii) $h^2 = (k-1)\varphi^2$, $k \leq 1$, *and* $h = 1$ *iff* M *is Sasakian,*

(iii) $R(\xi, X)Y = k(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX)$,

(iv) $Q\xi = 2nk\xi$,

where X and Y are any vector fields of M and $k, \mu \in \mathbb{R}$.

Lemma 2.2 (*see* [2]). *Let M^{2n+1} $(n \geq 1)$ be a contact metric manifold with ξ belonging to the (k, μ)-nullity distribution $(k < 1)$. For any vector field X, the Ricci operator Q is given by*

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + [2(1-n) + n(2k+\mu)]\eta(X)\xi. \tag{12}$$

Using Lemma 2.2, we obtain the following result.

Lemma 2.3. *Let M be a contact metric manifold. If ξ belongs to the (k, μ)-nullity distribution $(k < 1)$, then*

$$\begin{aligned}(\nabla_X S)(Y, Z) =& [2(n-1) + \mu]g(\nabla_X h)(Y, Z) \\ &+ [2(1-n) + n(2k+\mu)]\left\{g(Y, \nabla_X \xi)\eta(Z) + g(Z, \nabla_X \xi)\eta(Y)\right\}.\end{aligned} \tag{13}$$

Proof. By the covariant differentiation of S with respect to X we obtain

$$(\nabla_X S)(Y, Z) = \nabla_X S(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{14}$$

Using the fact that $S(Y, Z) = g(QY, Z)$ and differentiating this with respect to X and using (12), we get

$$\begin{aligned}\nabla_X S(Y, Z) =& [2(n-1) - n\mu]\,[g(\nabla_X Y, Z) + g(Y, \nabla_X Z)] \\ &+ [2(n-1) + \mu]\,[g(\nabla_X (hY), Z) + g(hY, \nabla_X Z)] \\ &+ [2(1-n) + n(2k+\mu)]\,[g(\nabla_X Y, \xi) + g(Y, \nabla_X \xi)]\,\eta(Z) \\ &+ [2(1-n) + n(2k+\mu)]\,[g(\nabla_X Z, \xi) + g(Z, \nabla_X \xi)]\,\eta(Y).\end{aligned} \tag{15}$$

In virtue of (12) we obtain

$$
\begin{aligned}
-S(\nabla_X Y, Z) =& - g(Q(\nabla_X Y), Z) \\
=& - [2(n-1) - n\mu] g(\nabla_X Y, Z) \\
& - [2(n-1) + \mu] g(h\nabla_X Y, Z) \\
& - [2(1-n) + n(2k+\mu)] \eta(\nabla_X Y)\eta(Z)
\end{aligned} \tag{16}
$$

and, similarly,

$$
\begin{aligned}
-S(Y, \nabla_X Z) =& - [2(n-1) - n\mu] g(Y, \nabla_X Z) \\
& - [2(n-1) + \mu] g(hY, \nabla_X Z) \\
& - [2(1-n) + n(2k+\mu)] \eta(Y)\eta(\nabla_X Z).
\end{aligned} \tag{17}
$$

Hence, substituting (15)–(17) into (14), we obtain (13), which completes the proof.

3. PSEUDOSYMMETRIC CONTACT MANIFOLDS OF CHAKI TYPE

The notion of pseudosymmetric manifolds was introduced by M. C. Chaki.

A non-flat Riemannian manifold (M^{2n+1}, g) is called *pseudosymmetric* of Chaki type if its curvature tensor satisfies

$$
\begin{aligned}
(\nabla_X R)(Y, Z, W) = 2\alpha(X)R(Y,Z)W + \alpha(Y)R(X,Z)W + \alpha(Z)R(Y,X)W \\
+ \alpha(W)R(Y,Z,X) + g(R(Y,Z)W, X)A,
\end{aligned} \tag{18}
$$

where α is a non-zero 1-form, called the associated 1-form, and

$$
g(X, A) = \alpha(X) \tag{19}
$$

for any vector field X [5]; see also [6].

We have the following result.

Theorem 3.1. *Let M be a $(2n+1)$-dimensional contact manifold with ξ belonging to a (k,μ)-nullity distribution. If M is pseudosymmetric of Chaki type, then*

(i) *M is locally isometric to the product $\mathbb{E}^{n+1} \times S^n(4)$, or*

(ii) *M has vanishing scalar curvature, or*

(iii) *M is a μ-Einstein manifold, or*

(iv) *M is a (k,μ)-contact manifold with $\mu = \mp \frac{k(2n-1)}{\sqrt{1-k}}$, where $k \neq 1$.*

Proof. Since M is a contact manifold with ξ belonging to a (k,μ)-nullity distribution, making use of (11) we get

$$
\begin{aligned}
\alpha(R(X,Y)\xi) &= g(R(X,Y)\xi, A) \\
&= k\,[\alpha(X)\eta(Y) - \alpha(Y)\eta(X)] + \mu\,[\alpha(hX)\eta(Y) - \alpha(hY)\eta(X)]
\end{aligned} \tag{20}
$$

and, similarly,

$$\begin{aligned}\alpha(R(X,\xi)Y) &= g(R(X,\xi)Y,A)\\ &= k\left[\eta(Y)\alpha(X)-g(X,Y)\alpha(\xi)\right]\\ &\quad+\mu\left[\eta(Y)\alpha(hX)-g(hX,Y)\eta(A)\right], \end{aligned}\tag{21}$$

where $\eta(\xi)=1$ and $\eta(Y)=g(Y,\xi)$.

If M is a pseudosymmetric manifold of Chaki type, then by (18) we get

$$\begin{aligned}(\nabla_X S)(Y,Z) = 2\alpha(X)S(Y,Z)+\alpha(Y)S(X,Z)+\alpha(Z)S(Y,X)\\ +\alpha(R(X,Y)Z)+\alpha(R(X,Z)Y).\end{aligned}\tag{22}$$

Replacing Z with ξ in Eq. (22), we have

$$\begin{aligned}(\nabla_X S)(Y,\xi) = 2\alpha(X)S(Y,\xi)+\alpha(Y)S(X,\xi)+\alpha(\xi)S(Y,X)\\ +\alpha(R(X,Y)\xi)+\alpha(R(X,\xi)Y).\end{aligned}\tag{23}$$

Substituting (20), (21), and (11) into (23) and using (19), we get

$$\begin{aligned}(\nabla_X S)(Y,\xi) &= 4nk\alpha(X)\eta(Y)+2nk\alpha(Y)\eta(X)+\alpha(\xi)S(Y,X)\\ &\quad+k\left[\eta(Y)\alpha(X)-\alpha(Y)\eta(X)\right]+\mu\left[\alpha(hX)\eta(Y)-\alpha(hY)\eta(X)\right]\\ &\quad+k\left[\eta(Y)\alpha(X)-g(X,Y)\alpha(\xi)\right]+\mu[\eta(Y)\alpha(hX)-g(hX,Y)\eta(A)].\end{aligned}\tag{24}$$

Replacing X with ξ, we get Eq. (24) as follows:

$$\begin{aligned}(\nabla_\xi S)(Y,\xi) &= 4nk\alpha(\xi)\eta(Y)+2nk\alpha(Y)\eta(\xi)+\alpha(\xi)S(Y,\xi)\\ &\quad+k\left[\eta(Y)\alpha(\xi)-\alpha(Y)\eta(\xi)\right]+\mu\left[-\alpha(hY)\eta(\xi)\right].\end{aligned}\tag{25}$$

On the other hand, replacing X and Z with ξ in Eq. (14), we get

$$(\nabla_\xi S)(Y,\xi)=\nabla_\xi S(Y,\xi)-S(\nabla_\xi Y,\xi)-S(Y,\nabla_\xi\xi).\tag{26}$$

By using the equation $Q\xi=2nk\xi$, after some computation Eq. (26) reduces to $(\nabla_\xi S)(Y,\xi)=0$. Therefore Eq. (25) becomes

$$6nk\alpha(\xi)\eta(Y)+k(2n-1)\alpha(Y)+k\eta(Y)\alpha(\xi)-\mu\alpha(hY)=0.\tag{27}$$

Substituting Z with ξ in (27), we get

$$8nk\alpha(\xi)=0.\tag{28}$$

So we have the following possible cases:

Case I. $\alpha(\xi) = 0$; $k \neq 0$,
Case II. $\alpha(\xi) \neq 0$; $k = 0$,
Case III. $\alpha(\xi) = 0$; $k = 0$.

Let us consider these in turn.

Case I. If $\alpha(\xi) = 0$; $k \neq 0$, then by (27) we have

$$k(2n-1)\alpha(Y) - \mu\alpha(hY) = 0. \tag{29}$$

Replacing Y with hY in Eq. (29), we obtain

$$k(2n-1)\alpha(hY) - \mu\alpha(h^2Y) = 0. \tag{30}$$

On the other hand, substituting the equations $h^2Y = (k-1)\varphi^2Y$, $k \leq 1$, and $\varphi^2Y = -Y + \eta(Y)\xi$ into (30), we get

$$k(2n-1)\alpha(hY) + \mu(k-1)\alpha(Y) = 0. \tag{31}$$

Using (29) and (31), we also get

$$\left[k^2(2n-1)^2 + \mu^2(k-1)\right]\alpha(Y) = 0. \tag{32}$$

However, $\alpha(Y) = 0$ is inadmissible. Therefore

$$k^2(2n-1)^2 - \mu^2(1-k) = 0. \tag{33}$$

If $k = 1$, then by (33) $n = \frac{1}{2}$, which contradicts the fact that $n \in \mathbb{Z}$. Thus $k \neq 1$ and hence $\mu = \mp\frac{k(2n-1)}{\sqrt{1-k}}$.

Case II. If $k = 0$, then by (27) we have $\mu\alpha(hY) = 0$. So we have the following subcases:
(a) $\mu = 0$, or
(b) $\alpha(hY) = 0$, or
(c) $\mu = 0$ and $\alpha(hY) = 0$.

Let us consider these in turn.

Case II(a). If $k = 0$ and $\mu = 0$, then $R(X,Y)\xi = 0$. Therefore by Theorem 2.1 in [1] M is locally isometric to the product $\mathbb{E}^{n+1} \times S^n(4)$.

Case II(b). If $k = 0$ and $\alpha(hY) = 0$, then, replacing X with ξ and after some calculation we have Eqs. (13) and (22) in the form

$$(\nabla_\xi S)(Y,Z) = \mu\left[2(n-1) + \mu\right] g(hY, \varphi Z), \tag{34}$$

$$(\nabla_\xi S)(Y,Z) = 2\alpha(\xi)S(Y,Z). \tag{35}$$

The left-hand sides of Eqs. (34) and (35) are equal, so

$$\mu\,[2(n-1)+\mu]\,g(hY,\varphi Z)=2\alpha(\xi)S(Y,Z). \tag{36}$$

Replacing Z with Y in Eq. (36), we get

$$2\alpha(\xi)S(Y,Y)=\mu\,[2(n-1)+\mu]\,g(hY,\varphi Y). \tag{37}$$

Further, let us replace Y with φY and use $\varphi^2Y=-Y+\eta(Y)\xi$. Hence Eq. (37) takes the form

$$2\alpha(\xi)S(\varphi Y,\varphi Y)=-\mu\,[2(n-1)+\mu]\,g(\varphi Y,hY). \tag{38}$$

Now, using (37) and (38), we obtain

$$2\alpha(\xi)\,[S(Y,Y)+S(\varphi Y,\varphi Y)]=0. \tag{39}$$

Since $\alpha(\xi)\neq 0$ and $k=0$, we get

$$S(Y,Y)+S(\varphi Y,\varphi Y)=0, \tag{40}$$

$$S(\xi,\xi)=0. \tag{41}$$

So, by the definition of scalar curvature (see [7], p. 445) M has vanishing scalar curvature, i.e. $\tau=0$.

Case II(c). If $k=\mu=0$ and $\alpha(hY)=0$, again M is locally isometric to the product $\mathbb{E}^{n+1}\times S^n(4)$.

Case III. If $\alpha(\xi)=0$ and $k=0$, then by (27) we have $\mu\alpha(hY)=0$. So we come back to Case II. This completes the proof of the theorem.

4. PSEUDO RICCI SYMMETRIC MANIFOLDS OF CHAKI TYPE

In this section we consider pseudo Ricci symmetric manifolds which were introduced by M. C. Chaki.

A non-flat Riemannian manifold (M^{2n+1},g) is called *pseudo Ricci symmetric* of Chaki type if its Ricci tensor S is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z)=2\alpha(X)S(Y,Z)+\alpha(Y)S(X,Z)+\alpha(Z)S(Y,X), \tag{42}$$

where α is a non-singular 1-form defined as in (19) (see [8]).

Theorem 4.1. *Let M be a $(2n+1)$-dimensional contact manifold with ξ belonging to a (k,μ)-nullity distribution. If M is pseudo Ricci symmetric of Chaki type, then*

(i) *M is locally isometric to the product $\mathbb{E}^{n+1}\times S^n(4)$, or*

(ii) *M has vanishing scalar curvature (i.e., $\tau=0$) with $\mu=2(\frac{n-1}{n})$, or*

(iii) *M is a μ-Einstein manifold.*

Proof. If M is pseudo Ricci symmetric of Chaki type, by the use of (41) we get

$$(\nabla_X S)(Y,\xi) = 2\alpha(X)S(Y,\xi) + \alpha(Y)S(X,\xi) + \alpha(\xi)S(Y,X). \qquad (43)$$

Substituting the equation $Q\xi = 2nk\xi$ into (43), we get

$$(\nabla_X S)(Y,\xi) = 4nk\alpha(X)\eta(Y) + 2nk\alpha(Y)\eta(X) + \alpha(\xi)S(Y,X). \qquad (44)$$

Further, let us replace X with ξ and use the relations $\eta(\xi) = 1$, $Q\xi = 2nk\xi$. Then Eq. (42) becomes

$$6nk\alpha(\xi)\eta(Y) + 2nk\alpha(Y) = 0. \qquad (45)$$

Now we shall replace Y with ξ, and Eq. (45) becomes $8nk\alpha(\xi) = 0$. So we have the following possible cases:

Case I. $\alpha(\xi) = 0$; $k \neq 0$, or
Case II. $\alpha(\xi) \neq 0$; $k = 0$, or
Case III. $\alpha(\xi) = 0$; $k = 0$.

Let us consider these in turn.

Case I. If $\alpha(\xi) = 0$, then by (45) we have $\alpha(Y) = 0$, which is inadmissible. So this case does not occur.

Case II. If $k = 0$, then by Eqs. (42) and (13) we get

$$(\nabla_\xi S)(Y,Z) = 2\alpha(\xi)S(Y,Z), \qquad (46)$$

$$(\nabla_\xi S)(Y,Z) = [2(n-1)+\mu]g((\nabla_\xi h)Y,Z). \qquad (47)$$

By the use of Lemma 2.1, Eq. (47) turns into

$$(\nabla_\xi S)(Y,Z) = -\mu[2(n-1)+\mu]g(\varphi hY,Z). \qquad (48)$$

From the right-hand sides of (46) and (47) we obtain

$$2\alpha(\xi)S(Y,Z) = \mu\,[2(n-1)+\mu]\,g(hY,\varphi Z). \qquad (49)$$

From the discussion given in the proof of Theorem 3.1 Case II(b) we can conclude that $\tau = 0$.

By Theorem 2 of [5] we have $\tau = 2n\,(2(n-1)+k-n\mu)$. Since $\tau = 0$, we get $\mu = 2\left(\frac{n-1}{n}\right)$.

Case III. If $k = \alpha(\xi) = 0$, then by the use of (49) one gets $\mu[2(n-1) + \mu]g(\varphi hY,Z) = 0$. So we have the following subcases:
(a) $\mu = 0$, or
(b) $2(n-1)+\mu = 0$, or
(c) $g(hY,\varphi Z) = 0$.
Let us consider these in turn.

Case III(a). If $k = \mu = 0$, then M is locally isometric to the product $\mathbb{E}^{n+1} \times S^n(4)$.

Case III(b). If $k = \alpha(\xi) = 0$ and $2(n-1) + \mu = 0$, then by (12) $QX = 2[(n-1)(n+1)](X - \varphi(X)\xi)$. Therefore M is a η-Einstein manifold.

Case III(c). If $k = \alpha(\xi) = 0$ and $g(hY, \varphi(Z)) = 0$, then by (12)

$$g(QY, \varphi(Z)) = (2(n-1) - n\mu)g(Y, \varphi(Z)). \tag{50}$$

Replacing $\varphi(Z)$ with Z in (50) we can see after an easy calculation that M is η-Einstein. This completes the proof of the theorem.

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PSEUDOSÜMMEETRILISED KONTAKTSED MEETRILISED MUUTKONNAD M. C. CHAKI MÕTTES

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$(2n+1)$-mõõtmelisi kontaktseid meetrilisi muutkondi M, mis on pseudosümmeetrilised M. C. Chaki mõttes, on uuritud eeldusel, et ξ sisaldub (k, μ)-defektsuse alamruumiväljas. On tõestatud, et sel puhul M on kas mitte-Sasaki muutkond ja $\mu = \mp\frac{k(2n-1)}{\sqrt{1-k}}$ või isomeetriline korrutisega $\mathbb{E}^{n+1} \times S^n(4)$ või nulliga võrduva skalaarkõverusega. Kui pseudosümmeetrilisuse asemel on pseudo-Ricci sümmeetria nõue, siis on võimalik ainult kolmas variant.