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SEMIPARALLEL SUBMANIFOLDS WITH PLANE GENERATORS OF CODIMENSION TWO IN A EUCLIDEAN SPACE

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Abstract. A submanifold generated by plane leaves of codimension two in a Euclidean space is, in general, intrinsically a Riemannian manifold of conullity two. All such manifolds have been classified into four classes: planar, hyperbolic, parabolic, and elliptic, i.e. having, respectively, infinitely many, two, one, or no real intrinsically asymptotic distributions. It is proved that if such a submanifold is semiparallel and intrinsically a manifold of conullity two, then it must be planar. This verifies, for the case considered here, a conjecture that a semiparallel submanifold, which is intrinsically of conullity two, must be planar. Validity of this conjecture has been established previously by the author for the three-dimensional semiparallel submanifolds.

Key words: Riemannian manifolds of conullity two, asymptotic foliations, semiparallel submanifolds.

1. INTRODUCTION

Let E^n be an n-dimensional Euclidean space and M^m an m-dimensional C^∞ submanifold in E^n, generated by $(m-2)$-dimensional planes of E^n. Intrinsically this M^m is a Riemannian manifold of conullity two (in the sense of [1]), i.e. foliated by Euclidean leaves of codimension two. (These leaves are, of course, the generator $(m-2)$-planes of the considered submanifold.) The Riemannian manifolds of conullity two constitute a particular class of semisymmetric Riemannian manifolds characterized by the condition $R(X,Y)\circ R = 0$ and classified in [2]; here R is the curvature tensor of the manifold and $R(X,Y)$ is the corresponding curvature operator for arbitrary two vector fields X and Y acting on this tensor.

In the geometry of submanifolds in E^n there exists a class of semiparallel submanifolds characterized by the condition $\bar{R}(X,Y) \circ h = 0$, where $\bar{R}$ is the curvature tensor of the van der Waerden–Bortolotti connection $\bar{\nabla}$ (the pair consisting of the Levi–Civita connection ∇ and normal connection $\nabla^{\perp}$) and h is the second fundamental form. It is known that every semiparallel submanifold is intrinsically a semisymmetric Riemannian manifold (see [3,4]), but there exist intrinsically semisymmetric not-semiparallel submanifolds.

The aim of the present paper is to investigate the submanifolds M^m generated by $(m-2)$-dimensional planes of E^n, which are semiparallel at the same time. In [5] the following conjecture is formulated: *If a semiparallel submanifold M^m in E^n is intrinsically a Riemannian manifold of conullity two, then it can be only planar* (according to the classification given in [1,6,7]). This conjecture arose in the study of the three-dimensional semiparallel submanifolds M^3 in E^n and was confirmed for this case of $m = 3$ and arbitrary n in [5].

Below (Theorem 3) it will be shown that this conjecture is true also for the semiparallel submanifolds M^m, generated by $(m-2)$-dimensional planes of E^n; here m and n can be arbitrary (of course, $n > m$).

2. SUBMANIFOLDS M^m WITH GENERATOR $(m-2)$-PLANES

If an m-dimensional Riemannian manifold M is immersed isometrically into a Euclidean space E^n as a submanifold M^m of E^n, then the derivation formulae

$$dx = e_I \omega^I, \quad de_I = e_J \omega_I^J, \quad \omega_I^J + \omega_J^I = 0$$

and structure equations

$$d\omega^I = \omega^J \wedge \omega_J^I, \quad d\omega_I^J = \omega_I^K \wedge \omega_K^J$$

for the bundle $O(E^n)$ of orthonormal frames $(x; e_1, ..., e_n)$ in E^n can be used for the subbundle $O(M^m, E^n)$ of frames adapted to M^m, so that $e_1, ..., e_m$ are tangent and $e_{m+1}, ..., e_n$ normal to M^m at $x \in M^m$, and imply

$$\omega^\alpha = 0, \quad \omega_i^\alpha = h_{ij}^\alpha \omega^j, \tag{1}$$

where $i, j, ...$ run over $\{1, ..., m\}$ and $\alpha, \beta, ...$ run over $\{m+1, ..., n\}$ (see, e.g., [4], Sections 1 and 2). Note that here x denotes both the point and its radius vector, and dx for this vector does not depend on the origin point, but h_{ij}^α are the components of the second fundamental (mixed) tensor, symmetric with respect to i, j. By means of h_{ij}^α the vector valued second fundamental tensor $h_{ij} = e_\alpha h_{ij}^\alpha$ can be introduced. For two tangent vectors $X = e_i X^i$ and $Y = e_j Y^j$ in $T_x M^m$ the second fundamental form h is determined by $h : (X,Y) \longmapsto h(X,Y) = h_{ij} X^i Y^j$.

Due to (1) $de_i = e_j\omega_i^j + h_{ij}\omega^j$, where ω_i^j are the connection 1-forms of ∇. For a tangent vector field Y from here $dY = e_j\nabla Y^j + h_{jk}Y^j\omega^k$ with $\nabla Y^j = dY^j + Y^i\omega_i^j$. For dx, collinear to a tangent vector field X when ω^k are proportional to X^k, this gives the Gauss formula (see, e.g., [8])

$$d_XY = \nabla_XY + h(X,Y). \tag{2}$$

In the extrinsic geometry of a submanifold M^m in E^n two tangent directions at $x \in M^m$ determined by X and Y are said to be *conjugate* if $h(X,Y) = 0$. Two vector subspaces Δ_1 and Δ_2 of T_xM^m are said to be conjugate if each direction of the first subspace is conjugate to each direction of the second subspace, i.e. if $h(X,Y) = 0$ for every $X \in \Delta_1$ and $Y \in \Delta_2$. A vector subspace Δ in T_xM^m is said to be *asymptotic* (extrinsically) if it is self-conjugate, i.e. if $h(X,Y) = 0$ for every two $X, Y \in \Delta$ (see, e.g., [9,10]).

Let the submanifold M^m in E^n be generated by $(m-2)$-planes. Let the frame from $O(M^m, E^n)$ be adapted further so that e_u $(u, v, ... = 3, ..., m)$ belong to the $(m-2)$-plane through $x \in M^m$. Then these planes are the leaves of the foliation determined by the differential system $\omega^a = 0$ $(a, b, ... = 1, 2)$. Therefore

$$de_u = e_a\omega_u^a + e_v\omega_u^v + h_{ua}\omega^a + h_{uv}\omega^v, \tag{3}$$

considered by mod$\{\omega^1, \omega^2\}$, must be expressed only by $e_3, ..., e_m$, thus

$$\omega_u^a = A_{ub}^a\omega^b, \quad h_{uv} = 0. \tag{4}$$

Here the equalities $h_{uv} = 0$ show that every generating $(m-2)$-plane has the asymptotic direction.

Let $G_{m-2}(E^n)$ be the Grassmann manifold of all $(m-2)$-dimensional planes in E^n. If a submanifold M^m in E^n is generated by $(m-2)$-planes, then it can be considered as an image in E^n of a two-dimensional submanifold ${}_GM^2$ of $G_{m-2}(E^n)$. Every curve (i.e. one-dimensional submanifold) in ${}_GM^2$ determines a “ruled” submanifold M^{m-1} of M^m, formed by $(m-2)$-plane generators of M^m. Among such “ruled” M^{m-1} there can be the “developable” ones, characterized by the property that the tangent $(m-1)$-plane of M^{m-1} at an arbitrary point x of an arbitrarily fixed generator $(m-2)$-plane, spanned by x and T_xM^{m-1}, is the same for all these points x.

Let us consider a “ruled” M^{m-1} and let its tangent $(m-1)$-plane T_xM^{m-1} be spanned by the point x and the unit vectors $e_3, ..., e_m$, $e = e_1\cos\varphi + e_2\sin\varphi$. Along this M^{m-1}, $dx = e_1\omega^1 + e_2\omega^2 + e_u\omega^u$ must be expressed only by e and all e_u, therefore the vectors $e_1\omega^1 + e_2\omega^3$ and e must be collinear. Thus there exists a non-vanishing 1-form θ, so that $\omega^1 = \theta\cos\varphi$, $\omega^2 = \theta\sin\varphi$, and hence $dx = e\theta + e_u\omega^u$. Let us introduce the other unit vector $e^\perp = -e_1\sin\varphi + e_2\cos\varphi$, orthogonal to e. For this M^{m-1}, due to (2) and (3),

$$de_u = e_v\omega_u^v + eB_u\theta + e^{\perp}C_u\theta + \theta(h_{u1}\cos\varphi + h_{u2}\sin\varphi),$$

$$de = -\sum_u e_uB_u\theta + e^{\perp}(\omega_1^2 + d\varphi) + (h_{11}\cos^2\varphi + 2h_{12}\cos\varphi\sin\varphi + h_{22}\sin^2\varphi)\theta$$
$$+\sum_u(h_{u1}\cos\varphi + h_{u2}\sin\varphi)\omega^u,$$

where
$$B_u = A^1_{u1}\cos^2\varphi + (A^1_{u2} + A^2_{u1})\cos\varphi\sin\varphi + A^2_{u2}\sin^2\varphi$$
and
$$C_u = A^2_{u1}\cos^2\varphi + (A^2_{u2} - A^1_{u1})\cos\varphi\sin\varphi - A^1_{u2}\sin^2\varphi.$$

Let us fix the point $x \in M^{m-1}$. Then $\theta = \omega^u = 0$ for all values of u, but de_u and de must be then some linear combinations of only e_v and e. This leads to $\omega_1^2 + d\varphi = \gamma\theta + \gamma_u\omega^u$.

Let the "ruled" M^{m-1} be a "developable" one. Then T_xM^{m-1} must be invariant along every generator $(m-2)$-plane determined by the equation $\theta = 0$. This equation yields $de = e^{\perp}\gamma_u\omega^u + \sum_u(h_{u1}\cos\varphi + h_{u2}\sin\varphi)\omega^u$, so the invariance above is equivalent to $\gamma_u = 0$ and $h(e_u, e) \equiv h_{u1}\cos\varphi + h_{u2}\sin\varphi = 0$. Here the last relation shows that the $(m-2)$-direction of the plane generator and orthogonal to it 1-direction on this "developable" M^{m-1} are conjugate with respect to the considered M^m with generator $(m-2)$-planes.

Intrinsically this M^m with generator $(m-2)$-planes is a Riemannian manifold of conullity two and these generators are its locally Euclidean leaves, but ∇ is the Levi–Civita connection of this manifold. A "ruled" M^{m-1}, whose T_xM^{m-1} is parallel along M^{m-1} with respect to ∇, is said to be *asymptotic* (intr.) in the inner geometry of such a M^m (see [1,6,7]). Since

$$\nabla e_u = e_v\omega_u^v + eB_u\theta + e^{\perp}C_u\theta, \quad \nabla e = -\sum_u e_uB_u\theta + e^{\perp}(\gamma\theta + \gamma_u\omega^u),$$

due to the Gauss formula (2), a "ruled" M^{m-1} is asymptotic (intr.) if and only if $C_u = 0$ and $\gamma = \gamma_u = 0$. Here the first condition can be represented as

$$A^2_{u1}\cos^2\varphi + (A^2_{u2} - A^1_{u1})\cos\varphi\sin\varphi - A^1_{u2}\sin^2\varphi = 0$$

or, equivalently, as

$$A^2_{u1}(\omega^1)^2 + (A^2_{u2} - A^1_{u1})\omega^1\omega^2 - A^1_{u2}(\omega^2)^2 = 0, \tag{5}$$

but the other conditions imply $\omega_1^2 + d\varphi = 0$.

Note that Eqs. (4) and (5) differ from the corresponding equations in [1] only by denotations: in [1] instead of $A^1_{u1}, A^1_{u2}, A^2_{u1}, A^2_{u2}$ there are used a_u, b_u, c_u, e_u.

Moreover, the addition “(intr.)” is not used in [1]; here it is needed to avoid confusing with *asymptotic* (extr.), explained above.

In [1,6,7] the Riemannian manifolds of conullity two are divided into three classes according to the number of solutions of Eq. (5). If Eq. (5) has infinitely many, two, one, or no real solutions $\omega^1 : \omega^2$, this manifold is, respectively, of the *planar*, *hyperbolic*, *parabolic*, or *elliptic* type. For instance, the planar type is characterized by

$$A^2_{u1} = A^1_{u2} = 0, \quad A^1_{u1} = A^2_{u2}. \tag{6}$$

3. ADDITION OF THE SEMIPARALLELITY CONDITION

For a general submanifold M^m in E^n the curvature 2-forms of ∇ and $\nabla^\perp$ are determined, respectively, by $\Omega_{ij} = -R_{ij,kl}\omega^k \wedge \omega^l$ and $\Omega^{\alpha\beta} = -R^{\alpha\beta}_{kl}\omega^k \wedge \omega^l$, where $R_{ij,kl} = \langle h_{i[k}, h_{l]j}\rangle$ and $R^{\alpha\beta}_{kl} = \sum_i h^\alpha_{i[k} h^\beta_{l]i}$ are the curvature tensors of ∇ and $\nabla^\perp$, respectively.

For a M^m in E^n the semiparallelity condition $\bar{R}(X,Y) \circ h = 0$ in a more explicit form is

$$\sum_p (\Omega_{ip} h^\alpha_{pj} + \Omega_{jp} h^\alpha_{ip}) - \sum_\beta \Omega^{\alpha\beta} h^\beta_{ij} = 0, \tag{7}$$

which after substitutions reduces to

$$\sum_p (H_{i[k,l]p} h_{pj} + H_{j[k,l]p} h_{ip} - H_{ij,p[k} h_{l]p}) = 0, \tag{8}$$

where $H_{ik,lj} = \langle h_{ik}, h_{lj}\rangle$ (see [4]).

For the considered M^m with generator $(m-2)$-planes in E^n the condition (8) by (k,l) = (a,u) reduces to

$$\sum_p [(H_{ia,up} - H_{iu,ap})h_{pj} + (H_{ja,up} - H_{ju,ap})h_{ip} - H_{ij,pa}h_{up} + H_{ij,pu}h_{ap}] = 0,$$

and this by (i,j) = (v,w) gives, due to (4),

$$\sum_b (H_{va,ub} h_{wb} + H_{wa,ub} h_{vb}) = 0.$$

Using the last condition by $u = v = w$ leads to the system of two equations

$$\langle h_{u1}, h_{u1}\rangle h_{u1} + \langle h_{u1}, h_{u2}\rangle h_{u2} = 0, \tag{9}$$

$$\langle h_{u2}, h_{u1}\rangle h_{u1} + \langle h_{u2}, h_{u2}\rangle h_{u2} = 0. \tag{10}$$

Here the following lemma can be used.

Lemma 1. *If in a real Euclidean vector space some two vectors p and q satisfy simultaneously $\langle p,p\rangle p+\langle p,q\rangle q=0$ and $\langle p,q\rangle p+\langle q,q\rangle q=0$, then $p=q=0$.*

Proof. Every two vectors p and q lie in a two-dimensional vector subspace. The orthonormal basis in this subspace can be chosen so that $p=(p_1,0),\ q=(q_1,q_2)$. The two conditions above are

$$p_1^2(p_1,0)+p_1q_1(q_1,q_2)=0, \quad p_1q_1(p_1,0)+(q_1^2+q_2^2)(q_1,q_2)=0.$$

For the second coordinates this means that $p_1q_1q_2=(q_1^2+q_2^2)q_2=0$ and leads to $q_2=0$, but for the first coordinates then $(p_1^2+q_1^2)p_1=(p_1^2+q_1^2)q_1=0$, therefore $p_1=q_1=0$.

Theorem 2. *If a submanifold M^m with generator $(m-2)$-planes in E^n is semiparallel, then its tangent m-planes along each of its $(m-2)$-plane generators coincide, so that the tangent plane of this M^m depends on no more than two parameters.*

Proof. Indeed, then the system of Eqs. (9) and (10) must be satisfied, but this due to Lemma 1 leads to $h_{ua}=0$. Now

$$de_a=-\sum_u A^a_{ub}\omega^b e_u+\omega_a^b e_b+h_{ab}\omega^b, \tag{11}$$

$$de_u=e_v\omega_u^v+A^a_{ub}\omega^b e_a; \tag{12}$$

the latter due to (3) and (4). This shows that both subspaces of T_xM^m, spanned on e_a ($a,b,...$ run over $\{1,2\}$) and on e_u ($u,v,...$ run over $\{3,...,m\}$) are invariant along each of the generator $(m-2)$-planes, which are determined by $\omega^b=0$.

Note. The equality $h_{ua}=0$ shows that the last two subspaces, one tangent to the generator $(m-2)$-plane, the other orthogonal to it in the tangent vector space T_xM^m of the submanifold M^m considered in Theorem 2, have conjugate directions.

The main result of the present paper is the following statement.

Theorem 3. *If a semiparallel submanifold M^m with generator $(m-2)$-planes in E^n is intrinsically a Riemannian manifold of conullity two, then it is of the planar type.*

Proof. Let us use exterior differentiation in (1). This yields

$$(dh^\alpha_{ij}-h^\alpha_{kj}\omega_i^k-h^\alpha_{ik}\omega_j^k+h^\beta_{ij}\omega^\alpha_\beta)\wedge\omega^j=0$$

and thus, due to Cartan's lemma,

$$dh^\alpha_{ij}-h^\alpha_{kj}\omega_i^k-h^\alpha_{ik}\omega_j^k+h^\beta_{ij}\omega^\alpha_\beta=h^\alpha_{ijk}\omega^k,$$

where h^{α}_{ijk} are symmetric with respect to i, j, k. (The last statement is the famous Peterson–Codazzi identity; see [11,12].)

For $h_{ij} = e_\alpha h^{\alpha}_{ij}$ and $h_{ijk} = e_\alpha h^{\alpha}_{ijk}$ one obtains

$$dh_{ij} = -\sum_k e_k \langle h_{ij}, h_{kl} \rangle \omega^l + h_{kj}\omega^k_i + h_{ik}\omega^k_j + h_{ijk}\omega^k.$$

Since $h_{uv} = h_{ua} = 0$ for the considered here submanifold M^m, this gives by $(i, j) = (u, v)$ and by $(i, j) = (u, a)$, respectively, $h_{uvw} = h_{uva} = 0$ and $-h_{ac}\omega^c_u = h_{uab}\omega^b$. Thus $h_{uab} = -h_{ac}A^c_{ub}$, due to (4), and from here, due to symmetry, $h_{ac}A^c_{ub} = h_{bc}A^c_{ua}$, where a, b, c run $\{1, 2\}$. Therefore

$$h_{11}A^1_{u2} + h_{12}(A^2_{u2} - A^1_{u1}) - h_{22}A^2_{u1} = 0. \tag{13}$$

Suppose that span$\{h_{11}, h_{12}, h_{22}\}$ has the maximal possible dimension 3 at every point $x \in M^m$. Then (13) yields (6), and thus M^m is of the planar type, indeed. Therefore only the cases when this span has the dimension ≤ 2 need further analysis.

If this dimension is 0, the submanifold M^m is totally geodesic. Thus it is an open part of an m-dimensional plane and not of conullity two.

Let this dimension be 1. Then each of the vectors h_{ab} has only one coordinate and the symmetric matrix of these coordinates can be diagonalized by a suitable orthogonal transformation of $\{e_1, e_2\}$. (Note that the relations (4) are invariant with respect to this transformation; this is seen also from the fact that these relations have pure geometric meaning.) After that Eqs. (1) are

$$\omega^\alpha = 0, \qquad \omega^{m+1}_1 = \kappa_1\omega^1, \qquad \omega^{m+1}_2 = \kappa_2\omega^2, \qquad \omega^\xi_a = \omega^\alpha_u = 0,$$

where ξ runs over $\{m + 2, ..., n\}$. By exterior differentiation from here

$$(d\kappa_1 + \kappa_1 A^1_{u1}\omega^u) \wedge \omega^1 + [(\kappa_1 - \kappa_2)\omega^2_1 + \kappa_1 A^1_{u2}\omega^u] \wedge \omega^2 = 0,$$

$$[(\kappa_1 - \kappa_2)\omega^2_1 + \kappa_2 A^2_{u1}\omega^u] \wedge \omega^1 + (d\kappa_2 + \kappa_2 A^2_{u2}\omega^u) \wedge \omega^2 = 0.$$

The semiparallelity condition (8) reduces to $(\kappa_1 - \kappa_2)\kappa_1\kappa_2 = 0$. Here $\kappa_1\kappa_2 = 0$ leads to $\Omega_{12} = 0$; moreover, due to $h_{uv} = h_{ua} = 0$ also $\Omega_{uv} = \Omega_{ua} = 0$, so that $\Omega_{ij} = 0$ and thus M^m is intrinsically locally Euclidean and not of conullity two. Therefore $\kappa_1 = \kappa_2 = \kappa \neq 0$, and the exterior equations reduce to

$$(d\ln\kappa + A^1_{u1}\omega^u) \wedge \omega^1 + A^1_{u2}\omega^u \wedge \omega^2 = 0,$$

$$A^2_{u1}\omega^u \wedge \omega^1 + (d\ln\kappa + A^2_{u2}\omega^u) \wedge \omega^2 = 0.$$

From here

$$d\ln\kappa + A^1_{u1}\omega^u = P\omega^1, \quad A^1_{u2} = A^2_{u1} = 0, \quad d\ln\kappa + A^2_{u2}\omega^u = Q\omega^2.$$

Thus $A^1_{u1} - A^2_{u2} = P = Q = 0$, and comparison with (6) shows that M^m is intrinsically of conullity two of the planar type.

Let the dimension of span$\{h_{11}, h_{12}, h_{22}\}$ be 2. The orthonormal frame can be further adapted to M^m, taking e_{m+1} and e_{m+2} as belonging to this span. After that $h^\xi_{ij} = 0$ for $\xi \in \{m+3, ..., n\}$ and thus among $\Omega^{\alpha\beta}$ only $\Omega^{m+1,m+2} = \sum_i h^{m+1}_{i[k} h^{m+2}_{l]i} \omega^k \wedge \omega^l$ can be non-zero.

Summing in semiparallelity condition (7) by $i = j$ gives, due to symmetry of h_{ij} and antisymmetry of Ω_{ij}, $\sum_\beta \Omega^{\alpha\beta} H^\beta = 0$, where $H^\beta = \frac{1}{m}\sum_i h^\beta_{ii}$ are the components of the mean curvature vector H of M^m. For the considered case this, due to antisymmetry of $\Omega^{\alpha\beta}$, reduces to

$$\Omega^{m+1,m+2} H^{m+2} = 0, \quad \Omega^{m+1,m+2} H^{m+1} = 0.$$

The semiparallel submanifold in E^n is minimal (i.e. has $H = 0$) only if it is an open part of a plane and thus is not of conullity two (see [13] and [4], Section 8). Therefore here only the case when $\Omega^{m+1,m+2} = 0$ is possible. This leads to the consequence that the matrices $\|h^{m+1}_{ab}\|$ and $\|h^{m+2}_{ab}\|$ commute and therefore can be diagonalized simultaneously by a suitable orthogonal transformation of $\{e_1, e_2\}$. After that $h_{ab} = k_a \delta_{ab}$ and the semiparallelity condition (8) reduces to $(k_1 - k_2)\langle k_1, k_2\rangle = 0$. Here $k_1 - k_2 = 0$ is impossible for the considered case (because the dimension of span$\{k_1, k_2\}$ is 2), therefore $\langle k_1, k_2\rangle = 0$, so $\Omega_{12} = 0$. Moreover, $\Omega_{uv} = \Omega_{ua} = 0$ due to $h_{uv} = h_{ua} = 0$, so that the submanifold M^m is locally Euclidean and cannot be of conullity two.

Theorem 3 is proven.

This theorem confirms once more the conjecture formulated in the Introduction.

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2-KODIMENSIONAALSETE TASANDILISTE MOODUSTAJATEGA SEMIPARALLEELSED ALAMMUUTKONNAD EUKLEIDILISES RUUMIS

Ülo LUMISTE

On tõestatud, et kui 2-kodimensionaalsete tasandiliste moodustajatega alammuutkond eukleidilises ruumis on semiparalleelne ja sisegeomeetriliselt konullilisusega 2, siis ta on planaarne, s.t. tal on lõpmata palju sisegeomeetriliselt asümptootilisi foliatsioone.