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TAUBERIAN REMAINDER THEOREMS FOR TWO FAMILIES OF SUMMABILITY METHODS

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Abstract. Some Tauberian remainder theorems are proved for two families of triangular matrix methods. Using these theorems, several special cases are studied. The discussed method gives an additional way to get Tauberian remainder theorems for some normal methods of summability not belonging to these families, but which are close to these families in some respect.

Key words: Tauberian remainder theorems.

Let

$$\hat{K}(z) = \int_{-\infty}^{+\infty} K(t) \exp(-2\pi i z t) dt$$

be the Fourier transform of the function $K(t) \in L(\mathbf{R})$ and

$$(K * f)(t) = \int_{-\infty}^{+\infty} K(t-\tau) f(\tau) d\tau.$$

Let $\Re z$ and $\Im z$ be the real and the imaginary part of a complex variable z.

Beurling (see [1]) proved in 1938 the basic Tauberian remainder theorem, which we present as the Lemma.

Lemma. *Let* $K(t) \in L(\mathbf{R})$, $\rho > 0.5$, $a > s > 0$, $c > 0$, *and suppose that there is a function* $g(z)$, *analytic in the strip* $0 < \Im z < a$ *and*

$$\lim_{\Im z \to 0+} g(z) \hat{K}(\Re z) = 1,$$

$$\left|g'(z)\right| \le c\,(1+|z|)^{\rho-1} \quad (0<\Im z<a)\,. \tag{1}$$

Let $f(t)\in L(\mathbf{R})$ satisfy the left-handed Tauberian condition

$$f(u)-f(t)=O_L(\exp(-st/\,(\rho+1))) \quad (t\to+\infty) \tag{2}$$

for all

$$t_0<t<u<t+\exp\left(-st/\,(\rho+1)\right).$$

Then

$$(K*f)\,(t)=O\left(\exp\left(-st\right)\right) \quad (t\to+\infty) \tag{3}$$

implies

$$f(t)=O\left(\exp\left(-st/\,(\rho+1)\right)\right) \quad (t\to+\infty)\,. \tag{4}$$

A sequence $x=\{\xi_k\}$ is called λ-bounded if

$$\lim_k \xi_k=\xi \wedge \lambda_k\left(\xi_k-\xi\right)=O(1),$$

whereas $0<\lambda_k\nearrow$. Let us denote by m^λ the set of all λ-bounded real sequences $x=\{\xi_k\}$, and by m_0^λ the subset of m^λ with $\xi=0$. We define (see [2]) the set (A,m_0^λ) by

$$x\in(A,m_0^\lambda)\Leftrightarrow Ax\in m_0^\lambda,$$

where the method of summability is determined by the matrix A.

Let us define a family of the kernels

$$K(t)=\sum_{\nu=1}^{m} c_\nu\left(1-\exp(-t)\right)^{\nu-1}\exp\left(-t\right)\mathbf{1}(t) \quad (c_1,\ldots,c_m\in\mathbf{R})\,, \tag{5}$$

where $\mathbf{1}(t)$ is the Heaviside function. According to Ganelius [3], these kernels are "related to Cesàro and Riesz summation". The corresponding family of the triangular methods of summability is given by lower triangular matrices $A=(a_{nk})$ with

$$a_{nk}=\sum_{\nu=1}^{m}\frac{c_\nu\left((n+1-k)^\nu-(n-k)^\nu\right)}{\nu\,(n+1)^\nu} \quad (k\le n)\,. \tag{6}$$

Let us denote

$$\varsigma_n=\sum_{k=0}^{n} a_{nk}\xi_k.$$

Theorem 1. *Let* $0.5 < \rho,\ 0 < s < 1/(2\pi),\ \lambda_n = (n+1)^s,$ *and* $\mu_n = (n+1)^{s/(1+\rho)}$. *Let* $x = \{\xi_k\}$ *be a bounded sequence satisfying the left-handed Tauberian condition*

$$\mu_n\,(\xi_m - \xi_n) = O_L(1)\ \ (n_0 < n < m < n\exp(1/\mu_n)\,,\ n \to +\infty). \tag{7}$$

If in the strip $0 < \Im z < 1/(2\pi)$ *we have*

$$\left| \frac{d}{dz}\left(\sum_{\nu=1}^{m} \frac{(\nu-1)!c_\nu}{\prod_{\tau=1}^{\nu}(\tau + 2\pi i z)} \right)^{-1} \right| \le c\,(1+|z|)^{\rho-1}\,, \tag{8}$$

then

$$x \in (A, m_0^\lambda) \tag{9}$$

implies

$$x \in m_0^\mu. \tag{10}$$

Proof. Let us apply Lemma with the kernel $K(t)$ defined by (5). As the Fourier transform $\hat{K}(z)$ of this kernel may be represented in the form

$$\begin{aligned}
\hat{K}(z) &= \int_{-\infty}^{+\infty} \sum_{\nu=1}^{m} c_\nu\,(1-\exp(-t))^{\nu-1}\exp(-t - 2\pi i z t)\,\mathbf{1}(t)dt \\
&= \sum_{\nu=1}^{m} c_\nu \int_0^{+\infty} \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} (-1)^j \exp(-jt - t - 2\pi i z t)dt \\
&= \sum_{\nu=1}^{m} c_\nu \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \int_0^{+\infty} \exp(-t - jt - 2\pi i z t)dt \\
&= \sum_{\nu=1}^{m} c_\nu \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \\
&\quad \times \lim_{A\to+\infty} \frac{1 - \exp((-1 - j - 2\pi i \Re z + 2\pi \Im z)\,A)}{1 + j + 2\pi i z},
\end{aligned}$$

in the half-plane $\Im z < 1/(2\pi)$ we have

$$\hat{K}(z) = \sum_{\nu=1}^{m} c_\nu \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \frac{1}{1+j+2\pi i z} = \sum_{\nu=1}^{m} \frac{(\nu-1)!c_\nu}{\prod_{\tau=1}^{\nu}(\tau + 2\pi i z)}.$$

We take

$$g(z)=\frac{1}{\hat{K}\left(z\right)}=\left(\sum_{\nu=1}^{m}\frac{(\nu-1)!c_{\nu}}{\prod_{\tau=1}^{\nu}\left(\tau+2\pi iz\right)}\right)^{-1}.$$

Considering the condition (8) in the strip $0<\Im z<1/(2\pi)$, we get

$$\left|g'(z)\right|\leq c\left(1+|z|\right)^{\rho-1}.$$

The condition (1) of the Lemma is satisfied if we use the selection $a=1/(2\pi)$. Choosing $f(t)=F(\exp(t))$, while

$$F(t)=\xi_{n}\quad(n\leq t<n+1),$$

we may represent (2) in the following form:

$$\begin{aligned}
&f(u)-f(t)=O_{L}(\exp(-st/\left(\rho+1\right)))\quad(t\to+\infty)\\
&\qquad t_{0}<t<u<t+\exp\left(-st/\left(\rho+1\right)\right)\\
&\Leftrightarrow F\left(\exp(u)\right)-F\left(\exp(t)\right)=O_{L}(\exp(-st/\left(\rho+1\right)))\ (t\to+\infty)\\
&\qquad \exp\left(t_{0}\right)<\exp\left(t\right)<\exp\left(u\right)<\exp\left(t+\exp\left(-st/\left(\rho+1\right)\right)\right)\\
&\Leftrightarrow \xi_{m}-\xi_{n}=O_{L}\left(n^{-s/(\rho+1)}\right)\\
&\qquad n\to+\infty,\ n_{0}<n<m<n\exp\left(1/\mu_{n}\right)\\
&\Leftrightarrow \mu_{n}\left(\xi_{m}-\xi_{n}\right)=O_{L}\left(1\right)\\
&\qquad n\to+\infty,\ n_{0}<n<m<n\exp\left(1/\mu_{n}\right).
\end{aligned}$$

Using (7), we find that the last condition of this chain is satisfied. The condition (3) may be represented in the form (9):

$$\begin{aligned}
&\left(K*f\right)(t)=O\left(\exp\left(-st\right)\right)\quad(t\to+\infty)\\
&\Leftrightarrow\int_{-\infty}^{t}\sum_{\nu=1}^{m}\frac{c_{\nu}\left(1-\exp(-t+\tau)\right)^{\nu-1}}{\exp\left(t-st-\tau\right)}F\left(\exp\left(\tau\right)\right)d\tau=O(1)\\
&\Leftrightarrow\int_{0}^{\exp(t)}\sum_{\nu=1}^{m}\frac{c_{\nu}\left(1-u\exp(-t)\right)^{\nu-1}}{\exp\left(t-st\right)}F\left(u\right)du=O(1)\\
&\Leftrightarrow n^{s-1}\int_{0}^{n}\sum_{\nu=1}^{m}c_{\nu}\left(1-\frac{u}{n}\right)^{\nu-1}F\left(u\right)du=O(1)
\end{aligned}$$

$$\Leftrightarrow n^{s-1} \sum_{\nu=1}^{m} c_\nu n^{1-\nu} \int_0^n (n-u)^{\nu-1} F(u)\, du = O(1)$$

$$\Leftrightarrow n^{s} \sum_{\nu=1}^{m} \frac{c_\nu}{n^\nu} \left[\sum_{j=0}^{n-1} \int_j^{j+1} (n-u)^{\nu-1} \xi_j du \right] = O(1)$$

$$\Leftrightarrow n^{s} \sum_{\nu=1}^{m} \frac{c_\nu}{\nu} \left[\sum_{j=0}^{n-1} \frac{(n-j)^\nu - (n-1-j)^\nu}{n^\nu} \xi_j \right] = O(1)$$

$$\Leftrightarrow n^s \varsigma_{n-1} = O(1) \Leftrightarrow (n+1)^s \varsigma_n = O(1)$$

$$\Leftrightarrow x \in \left(A, m_0^\lambda\right).$$

As the requirements of the Lemma are satisfied, the assertion (4) of the Lemma is valid and we can represent it in the form (10):

$$f(t) = O\left(\exp\left(-st/(\rho+1)\right)\right) \ (t \to \infty)$$

$$\Leftrightarrow \exp\left(st/(\rho+1)\right) f(t) = O(1) \ (t \to \infty)$$

$$\Leftrightarrow (n+1)^{s/(\rho+1)} \xi_n = O(1) \ (n \to \infty)$$

$$\Leftrightarrow x \in m_0^\mu.$$

Thus Theorem 1 is proved.

As a particular case of Theorem 1 we get Corollary 1 for the Woronoi–Nörlund method (WN, p_n) with $p_n = (n+1)^p - n^p \ (p \in \mathbf{N})$.

Corollary 1. *Let* $0 < s < 1/(2\pi)$, $\lambda_n = (n+1)^s$, *and* $\mu_n = (n+1)^{s/(1+p)}$. *Let* $x = \{\xi_k\}$ *be a bounded sequence satisfying the left-handed Tauberian condition (7). Then* $x \in \left(\left(WN, (n+1)^p - n^p\right), m_0^\lambda\right)$ *implies* $x \in m_0^\mu$.

Proof. Let us apply Theorem 1 with $c_\nu = p\delta_{\nu,p} \ (\nu = 1, \ldots, m)$, where $\delta_{\nu,p}$ is the Kronecker delta and $1 \le p \le m$. Considering this additional assumption, we get

$$\left| \frac{d}{dz} \frac{\prod_{\tau=1}^{p} (\tau + 2\pi i z)}{p!} \right| = \left| \frac{2\pi i}{p!} \sum_{\tau=1}^{p} \prod_{\substack{j=1 \\ j \neq \tau}}^{p} (j + 2\pi i z) \right| \le c\,(1+|z|)^{p-1}$$

and the condition (8) is satisfied if we choose $\rho = p$. It means that the assertion of Corollary 1 is valid.

If we fix $p = 1$, Corollary 1 gives us the result for the method of arithmetical means (see also [4]).

Using a family of the kernels

$$K(t) = \sum_{\nu=1}^{m} c_\nu \exp(-\nu t)\mathbf{1}(t) \quad (c_1, \ldots, c_m \in \mathbf{R}), \tag{11}$$

where $\mathbf{1}(t)$ is the Heaviside function, we get the family of the triangular methods of summability given by lower triangular matrices $B = (b_{nk})$ with

$$b_{nk} = \sum_{\nu=1}^{m} \frac{c_\nu}{\nu (n+1)^\nu} \left((k+1)^\nu - k^\nu\right) \quad (k \le n). \tag{12}$$

Let us denote $\omega = \min_{c_\nu \neq 0} \nu$.

Theorem 2. *Let* $0.5 < \rho,\ 0 < s < \omega/(2\pi),\ \lambda_n = (n+1)^s$, *and* $\mu_n = (n+1)^{s/(1+\rho)}$. *Let* $x = \{\xi_k\}$ *be a bounded sequence satisfying the left-handed Tauberian condition* (7). *If*

$$\left| \frac{d}{dz} \frac{\prod_{\nu=1}^{m} (\nu + 2\pi i z)}{\sum_{\nu=1}^{m} c_\nu \prod_{\substack{j=1 \\ j \neq \nu}}^{m} (j + 2\pi i z)} \right| \le c\,(1 + |z|)^{\rho - 1} \quad (0 < \Im z < \omega/(2\pi)), \tag{13}$$

then

$$x \in (B, m_0^\lambda) \tag{14}$$

implies $x \in m_0^\mu$.

The proof of Theorem 2 is similar to the proof of Theorem 1. As a particular case of Theorem 2 we get Corollary 2 for the Zygmund method $Z = (Z, p)$ with $p_n = (n+1)^p - n^p$ (see [5]).

Corollary 2. *Let* $p \in \mathbf{N},\ 0 < s < p/(2\pi),\ \lambda_n = (n+1)^s$, *and* $\mu_n = \sqrt{\lambda_n}$. *If* $x = \{\xi_k\}$ *is a bounded sequence satisfying the left-handed Tauberian condition* (7) *and* $x \in ((Z,p), m_0^\lambda)$, *then* $x \in m_0^\mu$.

Proof. Let us apply Theorem 2 with $c_\nu = p\delta_{\nu,p}$ $(\nu = 1, \ldots, m)$ and $1 \le p \le m$. As $\omega = p$ and

$$\left| \frac{d}{dz} \frac{\prod_{\nu=1}^{p} (\tau + 2\pi i z)}{p \prod_{j=1}^{p-1} (j + 2\pi i z)} \right| = \left| \frac{d}{dz} \frac{1}{p} (p + 2\pi i z) \right| = \left| \frac{2\pi i}{p} \right| = \frac{2\pi}{p},$$

the condition (13) is satisfied if we choose $\rho = 1$. It means that the assertion of Corollary 2 is valid.

For a normal method of summability which belongs neither to the family (5) nor (11), but is in some respect close to these families, Tauberian remainder theorems can be obtained in the following way.

Remark 1. If a normal matrix method D satisfies the condition

$$\left(AD^{-1}\right) m_0^\lambda \subset m_0^\lambda,$$

then the assertion of Theorem 1 stays valid if the condition (9) is substituted by $x \in (D, m_0^\lambda)$.

Remark 2. If a normal matrix method D satisfies the condition

$$\left(BD^{-1}\right) m_0^\lambda \subset m_0^\lambda,$$

then the assertion of Theorem 2 stays valid if the condition (14) is substituted by $x \in (D, m_0^\lambda)$.

REFERENCES

1. Subkhankulov, M. A. *Tauberian Remainder Theorems*. Nauka, Moscow, 1976 (in Russian).
2. Kangro, G. A Tauberian remainder theorem for the Riesz method, II. *Acta Comment. Univ. Tartuensis*, 1972, 305, 156–166 (in Russian).
3. Ganelius, T. *Tauberian Remainder Theorems*. Springer-Verlag, Berlin, 1971.
4. Tammeraid, I. Wiener's Tauberian remainder theorems and summability with rapidity. *Trans. Tallinn Tech. Univ.*, 1994, 738, 45–50.
5. Tammeraid, I. On the comparison of Tauberian remainder theorems. *Proc. Estonian Acad. Sci. Phys. Math.*, 1996, **45**, 234–241.

JÄÄKLIIKMEGA TAUBERI TEOREEMID KAHE SUMMEERIMISMENETLUSTE PERE JAOKS

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On uuritud jääkliikmega Tauberi teoreeme kahe maatriksmenetluste pere korral, lähtudes A. Beurlingi tõestatud teoreemist. On käsitletud ka mõningaid erijuhte. On esitatud võte, kuidas lisatingimustel leida jääkliikmega Tauberi teoreeme normaalse maatriksmenetluse korral, mis ei kuulu kummassegi neist peredest.