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TAUBERIAN THEOREMS FOR GENERALIZED SUMMABILITY METHODS IN BANACH SPACES

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Abstract. Tauberian theorems (T-theorems) for different generalized summability methods $\mathcal{A} = (A_{nk})$ and sequences of points in Banach spaces X are proved by a universal method. The operators $A_{nk} : X \to X$ are continuous and linear on X. T-theorems with o-conditions for generalized Riesz and Euler–Knopp methods of $c_X \to c_X$ type are presented (c_X being the space of convergent X-valued sequences). The applications of general results to scalar matrix methods are discussed. Several classical T-theorems arise from the results of the study as conclusions.

Key words: Banach spaces, operators, generalized summability methods, Tauberian theorems, Tauberian conditions, Mercer's theorems.

1. INTRODUCTION AND PRELIMINARIES

The summability theory of divergent series involves an extensive branch known as Tauberian theory. Generally, in a Tauberian theorem (shortly, T-theorem) the summability of a series $\sum_k x_k$ by a summability method $\mathcal{A}$, together with a restriction on (x_k), implies the convergence of $\sum_k x_k$. Our T-theorems are given for X-valued sequences $\chi = (x_k)$, and the Tauberian conditions (shortly, T-conditions) are o-conditions for the differences $\overline{\Delta} x_k = x_k - x_{k-1}$ $(k \in \mathbf{N})$. Here and below X will be a Banach space (shortly, B-space) over the field $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. A comprehensive account of the classical Tauberian theory can be found in [1].

In the present work some of the classical T-theorems, known for number sequences and scalar matrix methods, are extended to point sequences in B-spaces and generalized triangular summability methods (see [2–7]). Our main T-theorems for generalized Riesz and Euler–Knopp methods (see [5–7]) are given in Sections 4 and 6. The conclusions of these results for the corresponding scalar matrix methods and sequences in B-spaces are presented in Section 7, where also the respective T-theorems in a classical form and content are considered. Section 2 describes, in a sense, universal method that can be used to prove various T-theorems. Sections 3 and 5 include some auxiliary results for generalized Riesz and Euler–Knopp methods (see Lemmas 1,2 and 3,4) considerably simplifying the proofs of our main theorems.

We denote the most important sequence spaces by $m_X = \{(x_k) : x_k \in X;\ \sup_k \|x_k\| < \infty\}$, $c_X = \{(x_k) : x_k \in X;\ \exists \lim_k x_k\}$, and $nc_X \subset c_X$, where $\lim_k x_k = \theta$. As is known, m_X, c_X, and nc_X are B-spaces with $\|\chi\| = \sup_k \|x_k\|$ (cf. [3,4]). Throughout the paper the convergence of (x_k) means the convergence in the norm of X. For any two spaces X and Y the notation $\mathcal{F} : X \to Y$ denotes that the operator $\mathcal{F}$ is of $X \to Y$ type. If X and Y are B-spaces, then the set $\mathcal{L}(X, Y)$ of all continuous linear operators from X into Y is a B-space [4]. Let further I and θ signify the identity and the zero operator on any B-space. The operator norm of $\mathcal{F} \in \mathcal{L}(X, Y)$ is, as usual, $\|\mathcal{F}\| = \sup_{\|\chi\| \le 1} \|\mathcal{F}\chi\|$.

In the sequel we deal with generalized triangular summability methods $\mathcal{A} = (A_{nk})$, where $A_{nk} \in \mathcal{L}(X, X)$ $(k, n \in \mathbf{N})$. More information about such methods can be found, e.g., in [6,7]. Recall that a sequence $\chi = (x_k)$ is said to be summable by a method $\mathcal{A} = (A_{nk})$, or $\mathcal{A}$-summable if the sequence

$$y_n = \sum_{k=0}^{n} A_{nk} x_k \tag{1}$$

is convergent. This definition is analogous to the well-known definition of scalar matrix methods (cf., e.g., [8,9]).

We shall use also the following notations, where s_X and s'_X are certain pairs of c_X and nc_X. Let the operator $\mathcal{A}_n : s_X \to X$ be defined by

$$\mathcal{A}_n \chi = \sum_{k=0}^{n} A_{nk} x_k \qquad (\chi \in s_X; n \in \mathbf{N}) \tag{2}$$

and the special case $\mathcal{A}_n : X \to X$ of it by

$$\mathcal{A}_n x = \sum_{k=0}^{n} A_{nk} x \qquad (x \in X; n \in \mathbf{N}). \tag{3}$$

The operator $\mathcal{A} : s_X \to s'_X$ can be determined by

$$\eta = \mathcal{A}\chi, \tag{4}$$

where $\eta = (y_n)$ and $y_n = \mathcal{A}_n \chi$ are fixed by (1), (2). Hence,

$$\eta = \mathcal{A}\chi = (y_n) = (\mathcal{A}_n \chi) = \Big(\sum_{k=0}^{n} A_{nk} x_k\Big) \qquad (\chi \in s_X). \tag{5}$$

It is proved in [6] that the first operator $\mathcal{A}_n \in \mathcal{L}(s_X, X)$, the second operator $\mathcal{A}_n \in \mathcal{L}(X, X)$, and the third operator $\mathcal{A} \in \mathcal{L}(s_X, s'_X)$. In what follows $\mathcal{A}: c_X \to c_X$ and $\mathcal{A}: nc_X \to nc_X$ means that the method $\mathcal{A}$ is conservative and null-regular, respectively.

Remark 1. We know [2,3,7] that the method $\mathcal{A} = (A_{nk})$ with $A_{nk} \in \mathcal{L}(X, X)$ is of $c_X \to c_X$ type if and only if

$$1^0 \qquad \text{there exists } \lim_n A_{nk}x = A_k x \qquad (x \in X; k \in \mathbf{N}),$$

$$2^0 \qquad \text{there exists } \lim_n \sum_{k=0}^{n} A_{nk}x = Ax \qquad (x \in X),$$

$$3^0 \qquad \sup_{\|\chi\| \le 1} \Big\| \sum_{k=0}^{n} A_{nk}x_k \Big\| = O(1).$$

Furthermore, $\mathcal{A}$ is regular if and only if the conditions $1^0 - 3^0$ with $A = I$ and $A_k = \theta \ (k \in \mathbf{N})$ are satisfied.

2. THE PROOF METHOD EMPLOYED

In this section we describe the method which can be used to prove different generalized T-theorems (see also [10]).

Suppose that a sequence $\chi = (x_k) \ (x_k \in X)$ is $\mathcal{A}$-summable to $y^* \in X$. Consequently, $\mathcal{A}\chi = (y_n) \in c_X$, where $\mathcal{A}\chi$ is given by (5), and

$$\lim_n y_n = y^*. \tag{6}$$

We shall give the T-condition in terms of

$$\tau_k \overline{\Delta} x_k = o(1) \tag{7}$$

with a fixed sequence (τ_k). The T-condition (7) may have various forms. This depends essentially, but not only, on the method $\mathcal{A}$. We shall see below that (τ_k) can be a certain operator sequence (see Sections 3 and 4) or a fixed positive number sequence (see Sections 5 and 6).

Note. In this work we consider only the little o T-conditions, but our method can be employed to T-theorems with big O T-conditions too (see [10]).

In accordance with our proof method, one has to give (y_n) the shape which makes it possible to use a Mercer's theorem to reach the fundamental conclusion of a T-theorem, i.e. $(x_k) \in c_X$. To this end, let (y_n), given by (1), be rewritten in the form

$$\frac{1}{\alpha} \sum_{k=0}^{n} A_{nk}x_k = \Big[x_n - q \sum_{k=0}^{n} B_{nk}x_k \Big] - \frac{1}{\alpha}\sigma_n \qquad (n \in \mathbf{N}) \tag{8}$$

for any $\alpha > 0, \alpha \neq 1$, and $q = \frac{\alpha-1}{\alpha} \in \mathbf{R}$. Here (σ_n) is a $\Phi = (F_{nk})$ transformation of $(\tau_k \overline{\Delta} x_k)$ related to $\mathcal{A} = (A_{nk})$, and the summability method $\mathcal{B} = (B_{nk})$ can be uniquely determined by (8) and the methods $\mathcal{A}, \Phi$. The method Φ can be taken rather freely but certainly so that $(\sigma_n) = \Phi(\tau_k \overline{\Delta} x_k) \in c_X$ or, even better, that $(\sigma_n) \in nc_X$. Hence, the existence of $\lim_n [x_n - q\mathcal{B}_n \chi]$ follows from (8) by $\mathcal{A}\chi \in c_X$ and $(\sigma_n) \in c_X$. Then we use a Mercer's theorem to complete the proof.

3. SOME AUXILIARY RESULTS FOR RIESZ SUMMABILITY METHODS

Let $\mathcal{A}$ be a generalized Riesz method $\Re = (\Re, P_n) = (R_{nk})$, given by (see [5,7])

$$R_{nk} = \begin{cases} \Re_n P_k & (k = 0, 1, \dots, n), \\ \theta & (k > n) \end{cases} \tag{9}$$

with $\Re_n, P_k \in \mathcal{L}(X, X)$ such that

$$\Re_n \sum_{k=0}^{n} P_k x = x \qquad (x \in X; n \in \mathbf{N}). \tag{10}$$

As we know (see [7], Theorem 4), the method $(\Re, P_n)$ is conservative if and only if

$$\lim_n \Re_n x = \mathcal{R}^* x \qquad (x \in X) \tag{11}$$

and

$$\sup_{\|\chi\| \leq 1} \left\| \Re_n \sum_{k=0}^{n} P_k x_k \right\| = O(1). \tag{12}$$

Furthermore, $(\Re, P_n)$ is regular if and only if (11) with $\mathcal{R}^* = \theta$ and (12) are valid.

Notes. 1. In what follows it is supposed that for $(\Re, P_n)$ methods there exist all inverse operators $P_k^{-1}, \Re_n^{-1} \in \mathcal{L}(X, X)$.

2. Let in the sequel all elements with negative indexes be null-elements. Thereby $\overline{\Delta} x_0 = x_0 - x_{-1} = x_0$.

Below, for $(\Re, P_n)$ of $c_X \to c_X$ type, we shall reach the T-condition

$$P_k^{-1} \Re_k^{-1} \overline{\Delta} x_k = o(1). \tag{13}$$

The method $\Phi = (F_{nk})$, which specifies σ_n in (8), is fixed by

$$\sigma_n = \sum_{k=1}^{n} F_{nk}(P_k^{-1}\Re_k^{-1}\overline{\Delta}x_k) \qquad (n \in \mathbf{N}), \tag{14}$$

where

$$F_{nk} = \begin{cases} \Re_n P_k & (k = 1, 2, \ldots, n), \\ \theta & (k = 0; k > n). \end{cases} \tag{15}$$

So $\sigma_n = \Re_n(\Re_n^{-1}x_n - P_n x_{n-1} - \ldots - P_2 x_1 - \Re_1^{-1}x_0)$ $(n \in \mathbf{N})$, which follows from (14), (15), $\Re_1^{-1}x_0 = P_0x_0 + P_1x_0$, and (10). The last expression of σ_n yields

$$\sigma_n = x_n - \Re_n \sum_{k=0}^{n} P_k \xi_k \qquad (n \in \mathbf{N}), \tag{16}$$

where $\zeta = (\xi_k)$ with $\xi_0 = x_0, \xi_k = x_{k-1}$ $(k = 1, 2, \ldots)$.

Thereafter $\mathcal{B} = (B_{nk})$ can be uniquely specified as a solution of the system (8) for $\mathcal{A} = (\Re, P_n)$, with the help of (9), (10), and (14)–(16). Thus we get for $\mathcal{B} = (B_{nk})$ the representation

$$B_{nk} = \begin{cases} \frac{1}{\alpha-1}\Re_n P_1 & (k = 0), \\ \frac{1}{\alpha-1}\Re_n(P_{k+1} - P_k) & (k = 1, 2, \ldots, n-1), \\ I - \frac{1}{\alpha-1}\Re_n P_n & (k = n), \\ \theta & (k > n). \end{cases} \tag{17}$$

To prove the T-theorems for a $(\Re, P_n)$ method, we shall need the following two lemmas.

Lemma 1. *The methods $\Phi = (F_{nk})$ and $\mathcal{B} = (B_{nk})$, defined by* (15) *and* (17), *respectively, are both regular or conservative if $(\Re, P_n)$ is regular or conservative, respectively.*

The validity of these assertions follows easily from Remark 1, based on (15), and

$$\mathcal{B}_n\chi = x_n - \frac{1}{\alpha - 1}\Re_n\Big(\sum_{k=0}^{n} P_k x_k - \sum_{k=0}^{n} P_k \xi_k\Big) \qquad (n \in \mathbf{N}), \tag{18}$$

derived from (8) with $\mathcal{A} = (\Re, P_n)$ and in view of (16).

In accordance with the method that we use to prove T-theorems, let us start from (8), rewritten for $\mathcal{A} = \Re$ and $q = \frac{\alpha-1}{\alpha}$ in the form

$$\frac{1}{\alpha}\Re_n \sum_{k=0}^{n} P_k x_k = \left(x_n - q\sum_{k=0}^{n} B_{nk}x_k\right) - \frac{1}{\alpha}\sigma_n \qquad (n \in \mathbf{N}), \tag{19}$$

where σ_n and $\mathcal{B} = (B_{nk})$ are fixed by $\{(14), (15)\}$ and (17), respectively.

An important part of the proof method is the application of a generalized Mercer's theorem to

$$\lim_n \left(x_n - q \sum_{k=0}^{n} B_{nk} x_k \right) = \frac{1}{\alpha} y^*. \tag{20}$$

For this purpose, we have to find such domains of α where

$$\left| \frac{\alpha - 1}{\alpha} \right| \|\mathcal{B}\| < 1 \tag{21}$$

is valid. Therefore we require the following lemma.

Lemma 2. *Let the method* $(\Re, P_n) = (R_{nk})$*, given by* (9) *and* (10)*, be regular or conservative. Suppose the method* $\mathcal{B} = (B_{nk})$ *is defined by* (17) *or, more generally, by* (18)*, where* $\alpha > 0$ *and* $\alpha \neq 1$*.*

Then $|\frac{\alpha-1}{\alpha}|\|\mathcal{B}\| < 1$ *if* $\alpha > 1$.

Proof. To prove this assertion, we proceed as follows.

1) By applying (18) in the form $\mathcal{B}\chi = \chi - \frac{1}{\alpha-1}(\Re\chi - \Re\zeta)$, the well-known two-sided inequalities or formulas for $\|a \pm b\|$ or $\sup(a \pm b)$, and the sense of $\|\mathcal{B}\|$, we get for all mentioned α that $\|\mathcal{B}\| \leq 1 + \frac{2\|\Re\|}{|\alpha-1|}$. To get the opposite inequalities, we will start from $\|\mathcal{B}\| \geq \sup_{\|\chi\|\leq 1} \||\chi\| - \frac{1}{|\alpha-1|}\|\Re\chi - \Re\zeta\||$. Here $\|\mathcal{B}\| \geq \sup_{\|\chi\|\leq 1}(\|\chi\| - \frac{1}{|\alpha-1|}\|\Re\chi - \Re\zeta\|) = 1$ if $\alpha \geq 1 + \|\Re\|$, and $\|\mathcal{B}\| \geq \sup_{\|\chi\|\leq 1}[\frac{1}{|\alpha-1|}(\pm\|\Re\chi\| \mp \|\Re\zeta\|) - \|\chi\|] = \frac{\|\Re\|}{|\alpha-1|}$ if $1 < \alpha < 1 + \|\Re\|$, whereas $\frac{\|\Re\|}{|\alpha-1|} \geq 1$.

From the above two-sided inequalities we get

$$1 + \frac{2\|\Re\|}{\alpha - 1} \geq \|\mathcal{B}\| \geq 1 \quad (\alpha > 1). \tag{22}$$

2) By (22) it is possible that $\|\mathcal{B}\| = 1$; then (21) holds for all $\alpha > 1$. Let now $\|\mathcal{B}\| > 1$ in (22). Clearly, for $\alpha > 1$ the condition (21) is valid if $\alpha < \frac{\|\mathcal{B}\|}{\|\mathcal{B}\|-1} = 1 + \frac{1}{\rho} < \infty$, where $\rho = \|\mathcal{B}\| - 1 > 0$. Here (21) holds if $\alpha > 1$ and $\alpha < \infty$ simultaneously, i.e., if $\alpha > 1$. This completes the proof.

4. TAUBERIAN THEOREMS FOR GENERALIZED RIESZ METHODS

By applying the auxiliary summability methods given in Section 3, we can obtain two Tauberian theorems for generalized Riesz summability methods.

Theorem 1. *Let the method* $\Re = (\Re, P_n)$*, given by* (9) *and* (10) *with* $\Re_n$, $P_k \in \mathcal{L}(X, X)$*, be regular and let there exist all inverse operators* $P_k^{-1}, \Re_n^{-1} \in \mathcal{L}(X, X)$*.*

If $\chi = (x_k)$ is $\Re$-summable to y^* and if (13)*, i.e.* $P_k^{-1}\Re_k^{-1}\overline{\Delta} x_k = o(1)$*, holds, then $\chi = (x_k)$ converges to y^*.*

Proof. Let χ be $\Re$-summable to y^* and let (13) hold. Then, in accordance with our proof method described in Section 2, the sequence $\Re\chi = (\Re_n \sum_{k=0}^{n} P_k x_k)$ can be written in the form (19). In this case the sequence $(P_k^{-1}\Re_k^{-1}\overline{\Delta} x_k)$ is Φ-summable to θ because (13) is assumed and $\Phi = (F_{nk})$ is regular by Lemma 1.

Thereafter the relation (20), i.e. $\lim_n (x_n - q\sum_{k=0}^{n} B_{nk} x_k) = \frac{1}{\alpha} y^*$, follows from (19) if we take into account that $\chi = (x_k)$ is $\Re$-summable to y^* by the assumption. The method $\mathcal{B} = (B_{nk})$ in (20) is regular in view of Lemma 1. Then, by Lemma 2 there exist such domains of α, where the inequality (21), i.e. $|q|\|\mathcal{B}\| < 1$, holds. Therefore, the existence of $\lim_n x_n = y^*$ can be inferred from (20) with the help of a generalized Mercer's theorem (see [7], Theorem 1, Lemma 1). This completes the proof.

The next theorem regarding the conservative summability method $(\Re, P_n)$ can be proved exactly in the same manner as Theorem 1.

Theorem 2. *Let the method $\Re = (\Re, P_n)$, given by* (9) *and* (10) *with $\Re_n, P_k \in \mathcal{L}(X, X)$, be conservative and let there exist all inverse operators $P_k^{-1}, \Re_k^{-1} \in \mathcal{L}(X, X)$.*

If $\chi = (x_k)$ is $\Re$-summable and if (13)*, i.e.* $P_k^{-1}\Re_k^{-1}\overline{\Delta} x_k = o(1)$*, holds, then $\chi = (x_k)$ is convergent.*

Remark 2. The relation between $\lim_k x_k = x^*$ and $\lim_n \Re_n \sum_{k=0}^{n} P_k x_k = y^*$ can be determined by the general formula for conservative summability methods (see [2], Satz 1; [3], Theorem 1).

5. SOME AUXILIARY PARTICULARS FOR EULER–KNOPP SUMMABILITY METHODS

Let now $\mathcal{A}$ be a generalized Euler–Knopp method $\mathcal{E} = (\mathcal{E}, \Lambda) = (E_{nk})$, given by (see [5,7])

$$E_{nk} = \begin{cases} \binom{n}{k}\Lambda^k (I-\Lambda)^{n-k} & (k = 0, 1, \ldots, n), \\ \theta & (k > n), \end{cases} \tag{23}$$

where $\Lambda \in \mathcal{L}(X, X)$ and $\Lambda^0 = I$. As we know [5,7],

$$\mathcal{E}_n x = x \qquad (x \in X; n \in \mathbf{N}), \tag{24}$$

where $\mathcal{E}_n \chi = \sum_{k=0}^{n} E_{nk} x_k$ $(\chi \in s_X; n \in \mathbf{N})$ is valid for every $(\mathcal{E}, \Lambda)$. We recall [7] that $(\mathcal{E}, \Lambda)$ is conservative if and only if

$$\|\Lambda\| + \|I - \Lambda\| = 1 \tag{25}$$

or regular if and only if (25) and

$$\|I - \Lambda\| < 1 \tag{26}$$

hold.

The purpose of Section 6 is to prove that

$$\sqrt{k}\ \overline{\Delta} x_k = o(1) \tag{27}$$

is a T-condition for a $(\mathcal{E}, \Lambda)$ method of $c_X \to c_X$ type. For that we need two auxiliary methods, $\Phi = (F_{nk})$ and $\mathcal{B} = (B_{nk})$, like those in Section 3. Here Φ is fixed by

$$\sigma_n = \sum_{k=1}^{n} F_{nk}(\sqrt{k}\ \overline{\Delta} x_k) \qquad (n \in \mathbf{N}) \tag{28}$$

with

$$F_{nk} = \begin{cases} E_{nk} & (k = 1, 2, \ldots, n), \\ \theta & (k = 0; k > n). \end{cases} \tag{29}$$

We also notice that σ_n can be rewritten in the form

$$\sigma_n = \sum_{k=1}^{n} (\sqrt{k}\ E_{nk} - \sqrt{k+1}\ E_{n,k+1}) x_k - E_{n1} x_0 \qquad (n \in \mathbf{N}). \tag{30}$$

For convenience we use below the following notations: $\chi^* = (\xi_n) = (\sqrt{n}\ \overline{\Delta} x_n)$, $\Phi_n \chi^* = \sum_{k=1}^{n} F_{nk} \xi_k$. Then, from (8) with $\mathcal{A} = (\mathcal{E}, \Lambda)$ and in view of (23), (28), and (29), we get for $\mathcal{B} = (B_{nk})$ the following representation:

$$\mathcal{B}_n \chi = \frac{1}{\alpha - 1}(\alpha x_n - \mathcal{E}_n \chi - \Phi_n \chi^*) \qquad (n \in \mathbf{N}), \tag{31}$$

where $\mathcal{E}_n \chi$ and $\Phi_n \chi^*$ are fixed above. Thus, and owing to (23), (30), and (31), the method $\mathcal{B} = (B_{nk})$ is uniquely determined by

$$B_{nk} = \begin{cases} \frac{1}{\alpha-1}[\sqrt{k+1} E_{n,k+1} - (1 + \sqrt{k}) E_{nk}] & (k = 0, \ldots, n-1), \\ \frac{1}{\alpha-1}[\alpha I - (1 + \sqrt{n}) E_{nn}] & (k = n), \\ \theta & (k > n). \end{cases} \tag{32}$$

We shall need for (31) also a different kind of expression, namely

$$\mathcal{B}\chi = \frac{\alpha}{\alpha - 1}\chi - \frac{1}{\alpha - 1}(\mathcal{E}\chi + \mathcal{E}\chi^*), \tag{33}$$

because $\Phi\chi^* = \mathcal{E}\chi^*$ in view of $\Phi_n\chi^* = \sum_{k=1}^n E_{nk}\xi_k = \sum_{k=0}^n E_{nk}\xi_k = \mathcal{E}_n\chi^* \quad (n \in \mathbf{N})$, $E_{n0}\xi_0 = \theta$, and $\xi_0 = \theta$.

Next we propose analogues to both lemmas of Section 3.

Lemma 3. *The methods $\Phi = (F_{nk})$ and $\mathcal{B} = (B_{nk})$, defined by* (29) *and* (32), *are both regular or conservative if $(\mathcal{E}, \Lambda)$ is regular or conservative, respectively.*

The validity of these assertions can be shown with the help of Remark 1 and (28)–(33).

In order to employ our proof method in Section 6, let us rewrite (8) with $\mathcal{A} = (\mathcal{E}, \Lambda)$ in the following form:

$$\frac{1}{\alpha}\mathcal{E}_n\chi = \left(x_n - q\sum_{k=0}^{n} B_{nk}x_k\right) - \frac{1}{\alpha}\sum_{k=1}^{n} F_{nk}(\sqrt{k}\ \overline{\Delta}x_k) \qquad (n \in \mathbf{N}) \tag{34}$$

with $\Phi = (F_{nk})$ and $\mathcal{B} = (B_{nk})$ given by (29) and (32), respectively.

Lemma 4. *Let the method $(\mathcal{E}, \Lambda) = (E_{nk})$, given by* (23), *be regular. Suppose the method $\mathcal{B} = (B_{nk})$ is defined by* (32) *or, more generally, by* (31) *with $\alpha > 0$ and $\alpha \neq 1$.*

Then $|\frac{\alpha-1}{\alpha}|\|\mathcal{B}\| < 1$ if $\alpha > 1$.

Proof. As in Lemma 2, we confine ourselves only to some essential notes needed for the proof.

1) By applying (33), $\sup_{\|\chi^*\|\leq 1}\|\mathcal{E}\chi^*\| \leq \sup_{\|\chi\|\leq 1}\|\mathcal{E}\chi\| = \|\mathcal{E}\| = 1$ (see [6], Corollary 3.2), the well-known inequalities or formulas about $\|a\pm b\|$ or $\sup(a\pm b)$, we get $\|\mathcal{B}\| \leq \frac{\alpha+2}{\alpha-1}$ for all mentioned α. For the opposite inequalities we first obtain $\|\mathcal{B}\| \geq \frac{1}{|\alpha-1|}\sup_{\|\chi\|\leq 1}|\alpha\|\chi\| - \|\mathcal{E}\chi + \mathcal{E}\chi^*\||$; thereafter it is easy to check that $\|\mathcal{B}\| \geq \frac{1}{|\alpha-1|}\sup_{\|\chi\|\leq 1}(\alpha\|\chi\| - \|\mathcal{E}\chi + \mathcal{E}\chi^*\|) = \frac{\alpha}{\alpha-1}$ for $\alpha > 1$. Here $\frac{\alpha}{\alpha-1} > 2$ if $1 < \alpha < 2$ and $\frac{\alpha}{\alpha-1} > 1$ if $\alpha \geq 2$. Then, owing to the three last inequalities for $\|\mathcal{B}\|$, we can propose the following two-sided bonds:

$$\frac{\alpha+2}{\alpha-1} \geq \|\mathcal{B}\| \begin{cases} > 2, & 1 < \alpha < 2, \qquad \text{(I)} \\ > 1, & \alpha \geq 2. \qquad \text{(II)} \end{cases} \tag{35}$$

Consequently, $\|\mathcal{B}\| > 1$ for all fixed feasible values of α. However, it is also possible that $\|\mathcal{B}\| = 1$, since α is unlimited in (35), (II).

2) Now we are able to state such domains of α where (21), i.e. $|\frac{\alpha-1}{\alpha}|\|\mathcal{B}\| < 1$, holds.

a) Let $\|\mathcal{B}\| = 1$. Then (21) is obviously satisfied for all $\alpha > \frac{1}{2}$, but owing to (35), (II), for $\alpha \geq 2$. b) In the general case of (35), (II), where $\alpha \geq 2$ and $\|\mathcal{B}\| > 1$, the condition (21) holds if $\alpha < 1 + \frac{1}{\|\mathcal{B}\|-1} < \infty$, thus, if $\alpha \geq 2$.

For (35), (I) with $1 < \alpha < 2$ and $\|\mathcal{B}\| > 2$, we get that (21) holds if $\alpha < 1 + \frac{1}{\|\mathcal{B}\|-1} < 2$, hence if $1 < \alpha < 2$. The proof is complete.

6. TAUBERIAN THEOREMS FOR GENERALIZED EULER–KNOPP METHODS

Now we can prove two generalized Tauberian theorems for $(\mathcal{E},\Lambda)$ methods.

Theorem 3. *Let the method $\mathcal{E} = (\mathcal{E},\Lambda)$, given by (23) with $\Lambda \in \mathcal{L}(X,X)$, be regular.*

If $\chi = (x_k)$ is $\mathcal{E}$-summable to y^* and if $\sqrt{k}\ \overline{\Delta} x_k = o(1)$, then $\chi = (x_k)$ converges to y^*.

Proof. As in the proof of Theorem 1, we get for $\mathcal{E}_n\chi$ the representation (34). The methods $\Phi = (F_{nk})$ and $\mathcal{B} = (B_{nk})$ are now fixed by (29) and (32), respectively. As (27), i.e. $\sqrt{k}\ \overline{\Delta} x_k = o(1)$, is assumed and Φ is regular by Lemma 3, then $(\sqrt{k}\ \overline{\Delta} x_k)$ is Φ-summable to θ. Hence, by (34) and the assumption $\lim_n \mathcal{E}_n\chi = y^*$, we have

$$\lim_{n\to\infty}\left(x_n - \frac{\alpha-1}{\alpha}\sum_{k=0}^{n} B_{nk}x_k\right) = \frac{1}{\alpha}y^* \tag{36}$$

with $\mathcal{B} = (B_{nk})$ being regular because of Lemma 3. Then it follows in view of Lemma 4 that (21), i.e. $|\frac{\alpha-1}{\alpha}|\|\mathcal{B}\| < 1$, is valid for $\alpha > 1$.

Our assertion, namely the existence of $\lim_n x_n = y^*$, follows now from (36) by a generalized Mercer's theorem (see [7], Theorem 1 and Lemma 1). Hence the result.

For the non-regular conservative $(\mathcal{E},\Lambda)$ method, i.e. for $(\mathcal{E},\theta)$, we propose the following analogue of Theorem 3.

Theorem 4. *If $\chi = (x_k)$ is $(\mathcal{E},\theta)$-summable and if $\sqrt{k}\ \overline{\Delta} x_k = o(1)$, then $\chi = (x_k)$ is convergent.*

As the proof of this theorem is simple, we omit it.

7. CONCLUSIONS

The usefulness of scalar matrix summability methods in the Tauberian theory for series in abstract spaces is well expressed by Maddox ([4], Section 5, p. 66). For instance, the two well-known classical T-theorems, due to Littlewood and Hardy, were considered in a B-space context by Northcott [11] and Maddox [4], respectively.

As we know [5–7], every scalar matrix method $A = (a_{nk})$ can be treated also in an operator matrix form. To this end, instead of A we can use the method $\mathcal{A} = (A_{nk})$ with

$$A_{nk} = a_{nk}I,\quad \mathcal{A}_n\chi = \sum_{k=0}^{n} a_{nk}Ix_k,\quad \mathcal{A}\chi = (\mathcal{A}_n\chi) \qquad (n,k \in \mathrm{N}), \tag{37}$$

where $\chi = (x_k)$ and $x_k \in X$. Here X is a B-space.

Let, further, $\mathcal{R} = (\mathcal{R}, p_n) = (r_{nk})$ with $p_n \in \mathbf{K}$ be the classical Riesz method and $E = E_\lambda = (e_{nk})$ with $\lambda \in \mathbf{R}$ be the classical Euler–Knopp method. It is known [8,9] that these triangular methods, given by sequence-to-sequence transformations, are defined by

$$r_{nk} = \frac{p_k}{\mathcal{P}_n}, \ \mathcal{P}_n = \sum_{k=0}^{n} p_k \ \text{ and } e_{nk} = \tbinom{n}{k}\lambda^k(1-\lambda)^{n-k}, \tag{38}$$

respectively. Using these scalar summability methods in the form (37) for X-valued sequences and applying the suitable ones of Theorems 1–4, we get T-theorems similar to those treated in [4,11].

For the method $(\mathcal{R}, p_n)$, fixed by (37) and (38), there exist $P_k^{-1} = (p_k I)^{-1} = \frac{1}{p_k}$ and $\mathcal{R}_n^{-1} = (\frac{1}{\mathcal{P}_n} I)^{-1} = \mathcal{P}_n$. Consequently, (13) takes the form

$$\mathcal{P}_k \overline{\Delta} x_k = o(p_k). \tag{39}$$

Thus, immediately from Theorems 1 and 2 we can infer the next result which is somewhat more general than the one obtained in [10].

Theorem 5. *Let the method $\mathcal{R} = (\mathcal{R}, p_n)$, given by (38), be regular or conservative.*

If $\chi = (x_k)$ with $x_k \in X$ $(k \in \mathbf{N})$ is $\mathcal{R}$-summable to y^* and if (39) *holds, then $\chi = (x_k)$ converges to y^* if $\mathcal{R}$ is regular, or $\chi = (x_k)$ is convergent if $\mathcal{R}$ is conservative.*

Remark 3. We notice that if $X = \mathbf{K}$, then Theorem 5 turns into its analogue in the classical form.

The generally known [8,9] special cases of $(\mathcal{R}, p_n)$ are as follows: the $(C, 1)$ method, the logarithmic method (ℓ), the Zygmund method (Z, ρ), and the $(\mathcal{R}, a^k)$ method. For these methods the T-conditions corresponding to (39) are $(k+1)\overline{\Delta} x_k = o(1)$, $(k+1)\ln(k+1)\overline{\Delta} x_k = o(1)$, $k\overline{\Delta} x_k = o(1)$, and $\overline{\Delta} x_k = o(1)$, respectively.

Remark 4. Clearly, the T-theorems for all above methods with the mentioned T-conditions and for different cases of the spaces X can be inferred from Theorem 5 as immediate corollaries.

All these results for $X = \mathbf{K}$ are known from [10,12], but the most significant T-theorems are proved by Hardy [1] for the $(C, 1)$ method and Ishiguro [13] for the (ℓ) method.

The method $E = E_\lambda$ is regular if and only if $0 < \lambda \leq 1$ or conservative if and only if $0 \leq \lambda \leq 1$ (see [8,9]). In this case and in view of (37), (38), we get directly from Theorem 3 the following result.

Theorem 6. *Let the method $E = E_\lambda$, given by* (38), *be regular.*

If $\chi = (x_k)$ with $x_k \in X$ $(k \in \mathbf{N})$ is E-summable to y^* and if (27) *holds, then $\chi = (x_k)$ converges to y^*.*

Recall that the non-regular conservative method $E = E_\lambda$ is this method with $\lambda = 0$, i.e. E_0. For this case the relevant T-theorem is Theorem 4 with E_0 instead of $(\mathcal{E}, \theta)$.

Remark 5. For $X = \mathbf{K}$, Theorem 6 has also an analogue in the classical form. Note that such T-theorem was proved by Knopp [14]. In [10] the T-theorem for regular E_λ is proved with the T-condition $\sqrt{k}\ \overline{\Delta} x_k = O(1)$.

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REFERENCES

1. Hardy, G. H. *Divergent Series*. Oxford, 1944.
2. Zeller, K. Verallgemeinerte Matrixtransformationen. *Math. Z.*, 1952, **56**, 18–20.
3. Kangro, G. Über Matrixtransformationen von Folgen in Banachschen Räumen. *ENSV TA Toim. Füüs.-Matem. Tehn. Tead. Seeria*, 1956, **5**, 108–128 (in Russian).
4. Maddox, I. J. *Infinite Matrices of Operators*. Springer, Berlin, 1980.
5. Nappus, A. and Sõrmus, T. Einige verallgemeinerte Matrixverfahren. *Proc. Estonian Acad. Sci. Phys. Math.*, 1996, **45**, 201–210.
6. Sõrmus, T. Some properties of generalized summability methods in Banach spaces. *Proc. Estonian Acad. Sci. Phys. Math.*, 1997, **46**, 171–186.
7. Sõrmus, T. Mercer's theorems for generalized summability methods in Banach spaces. *Proc. Estonian Acad. Sci. Phys. Math.*, 1998, **47**, 75–89.
8. Baron, S. *Einführung in die Limitierungstheorie*. 2nd edn. Valgus, Tallinn, 1977 (in Russian).
9. Zeller, K. and Beekmann, W. *Theorie der Limitierungsverfahren*. Springer, Berlin, 1970.
10. Sõrmus, T. Eine universale Beweismethode für Tauber-Sätze. *Acta Comment. Univ. Tartuensis*, 1993, 960, 105–116.
11. Northcott, D. G. Abstract Tauberian theorems with applications to power series and Hilbert series. *Duke Math. J.*, 1947, **14**, 483–502.
12. Tietz, H. and Zeller, K. Tauber-Sätze für bewichtete Mittel. *Arch. Math.*, 1997, **68**, 214–220.
13. Ishiguro, K. Tauberian theorems concerning the summability methods of logarithmic type. *Proc. Japan Acad.*, 1963, **39**, 156–159.
14. Knopp, K. Über das Eulersche Summierungsverfahren. *Math. Z.*, 1922, **15**, 226–253 and 1923, **18**, 125–156.

TAUBERI TEOREEMID ÜLDISTATUD SUMMEERIMISMENETLUSTELE BANACHI RUUMIDES

Tamara SÕRMUS

On üldistatud klassikalisest summeeruvusteooriast tuntud mitu Tauberi teoreemi (T-teoreemi) Banachi ruumi X jadade ja üldistatud summeerimismenetluste $\mathcal{A} = (A_{nk})$ jaoks. Menetluse $\mathcal{A}$ kõik elemendid $A_{nk} : X \to X$ on pidevad lineaarsed operaatorid. On esitatud üldistatud T-teoreemid Rieszi ja Euleri–Knoppi menetluste kohta. Töös on kasutatud autori küllaltki efektiivset üldist tõestusmeetodit. Saadud tulemustest järelduvad erijuhtudena mitmed tuntud klassikalised T-teoreemid.